NUMERICAL SOLUTIONS OF THE CURRENT DISTRIBUTION IN SUPERCONDUCTING CABLES

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Abstract

Superconducting cables are described by current sheets using the continuum model of W. Carr Jr. [1] and assuming an anisotropic conductivity. Two different situations are considered: a) finite length of cable in spatially independent magnetic field; b) infinitely long cable in a periodical magnetic field.

Introduction

The a.c. loss of long superconducting cables has been studied by G. Ries and S. Takács [2,3,4]. Their calculation is based on a localization of the Biot & Savart law. Representing the current density by a Taylor expansion, they derived a set of partial differential equations. In this way, finite cables in spatially independent magnetic field are treated. Their results were experimentally verified by K. Kwasnitza and B. Bruzzone [5]. A good qualitative agreement was found between measurements and the theory of G. Ries and S. Takács. In this paper we will show an alternative approach using Fourier series for representing both field and current density.

Theory

First we will consider the case of an infinitely long superconducting cable in a periodical magnetic field:

$$B_r^A = \hat{B}_r^A \sin\left(\frac{n\pi z}{L}\right) e^{i\omega t} \quad (1)$$

$2L$ = the period of the field along the axis of the cable.

This problem can be treated analytically. The corresponding solutions for j_z and j_ϕ are:

$$j_\phi(\phi, z, t) = \frac{L}{\mu_0 R \pi n} \left(a_{1,n} \cos\phi \sin\left(\frac{n\pi z}{L}\right) + a_{2,n} \sin\phi \cos\left(\frac{n\pi z}{L}\right) \right) e^{i\omega t} \quad (2)$$

$$j_z(\phi, z, t) = \frac{L^2}{\mu_0 R^2 \pi n^2} \left(-a_{1,n} \sin\phi \cos\left(\frac{n\pi z}{L}\right) - a_{2,n} \cos\phi \sin\left(\frac{n\pi z}{L}\right) \right) e^{i\omega t} \quad (3)$$

and for P the loss per cycle:

$$P = \frac{L^4}{2\mu_0 R^2 \pi n^3 \alpha} \left(\left(\frac{nL}{2L} + \frac{2L}{nL} \right) (|a_{1,n}|^2 + |a_{2,n}|^2) - (a_{1,n} \bar{a}_{2,n} + a_{2,n} \bar{a}_{1,n}) \right) \quad (4)$$

where

$$\alpha = \sigma(L_p^2 + (2\pi R)^2) / (2\pi R L_p)$$

L_p = twist length

R = Radius

σ = conductivity

$$a_{1,n} = \frac{-\frac{1}{4}\omega\tau_n \hat{B}_r^A \left(\frac{2L}{nL_p} + \frac{nL}{2L} \right) + i\omega\tau_n S_I S_K}{1 - \frac{1}{4} \left(\frac{2L}{nL_p} + \frac{nL}{2L} \right) + i\omega\tau_n S_I S_K}^2$$

$$a_{2,n} = \frac{\frac{1}{2}i\omega\tau_n \hat{B}_r^A}{1 - \frac{1}{4} \left(\frac{2L}{nL_p} + \frac{nL}{2L} \right) + i\omega\tau_n S_I S_K}^2$$

$$\tau_n = n\pi R^2 \alpha \mu_0 / L$$

$$S_I = I_0 \left(\frac{n\pi R}{L} \right) - \frac{L}{n\pi R} I_1 \left(\frac{n\pi R}{L} \right)$$

$$S_K = K_0 \left(\frac{n\pi R}{L} \right) + \frac{L}{n\pi R} K_1 \left(\frac{n\pi R}{L} \right)$$

with I_0, K_0, I_1 and K_1 modified Besselfunctions.

The situation $n = 0$, corresponding to a homogeneous magnetic field, can be treated separately. This results in the following expressions for j_z, j_ϕ and the loss per cycle, P_0 :

$$j_z = \frac{-(i\alpha L_p \omega \hat{B}_r^A / 2\pi) (1 - \mu_0 \alpha L_p \omega / 2\pi)}{1 + (\mu_0 \alpha L_p \omega / 2\pi)^2} \sin\phi e^{i\omega t} \quad (5)$$

$$j_\phi = 0 \quad (6)$$

$$P_0 = \frac{1}{2} \alpha \frac{L}{L_p} \left(\frac{\omega \hat{B}_r^A L_p R}{1 + \mu_0 \alpha L_p \omega / 4\pi} \right)^2 \quad (7)$$

The last expression corresponds exactly with the one obtained by G. Ries and S. Takács for the stationary situation.

Figure 1 shows that result (4) corresponds nicely with that of S. Takács except for the singularity near $2L/L_p = 1$, which does not occur in the analytical approach.

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The second problem, a finite cable in a magnetic field, can be solved by representing the current density and the applied field in a Fourier series.

$$j_z(r, \phi, t) = \sum_n j_{z,1}^n(r, t) \cos \phi \sin\left(\frac{n\pi z}{L}\right) + j_{z,2}^n(r, t) \sin \phi \cos\left(\frac{n\pi z}{L}\right) \quad (8)$$

$$j_\phi(r, \phi, t) = \sum_n \frac{-n\pi R}{L} (j_{z,1}^n(r, t) \sin \phi \cos\left(\frac{n\pi z}{L}\right) + j_{z,2}^n(r, t) \cos \phi \sin\left(\frac{n\pi z}{L}\right)) \quad (9)$$

$$B_r^A(\phi, z, t) = \sum_n \hat{B}_1^A(t) \cos \phi \cos\left(\frac{n\pi z}{L}\right) + \hat{B}_2^A(t) \sin \phi \sin\left(\frac{n\pi z}{L}\right) \quad (10)$$

Unlike the treatment of S. Takács, the Biot & Savart law is used on the whole cylinder to obtain a Fourier series for the induced field B_r^I . The main advantage of this approach is that a set of more coaxial cylinders may be treated.

To obtain a Fourier coefficient of the expansion of the induced field, a four-dimensional integral has to be solved. It is possible to reduce this to a two-dimensional integral. One gets the following matrix expression:

$$B_{r,n}^I = A_{n,n'} j_{n'} \quad (11)$$

which yields $B_{r,n}^I$ as a function of j_n as described by the Biot & Savart law.

$$\text{Solving } (\nabla \times E)_r = -\dot{B}_r \text{ into} \quad (12)$$

$$j_n = M_{nn'} \dot{B}_{r,n'} = M_{nn'} \dot{B}_{r,n'}^I + M_{nn'} \dot{B}_{r,n'}^A \quad (13)$$

a system of ordinary linear differential equations is derived:

$$j = N \frac{\partial j}{\partial t} + M \dot{B}^A. \quad (14)$$

We have solved this system for a particular case $2L/L_p = 2\frac{2}{3}$ for two different applied fields.

Results

Figure 2a...2d shows the current distribution of a finite superconducting cable with length $2L$ in a homogeneous magnetic field.

$$B^A(t) = \hat{B}^A t \quad (15)$$

Directly after applying the magnetic field on the virgin cable, the current distribution is that of a rectangular shape, in order to screen the applied field to the maximum. After a while, the twist of the cable becomes more visible in the current distribution because of a decrease of necessity for screening the field.

After some time it is even possible that before the current flow reaches the end of the cable it returns to its starting point. Finally all modes have been activated, resulting mainly in current flows with typical length L_p the twist length of the cable. Besides these

current flows based on the typical length L_p the original rectangular one still exists due to the end effects of the finite cable. Figure 2e is the analytical equivalence of 2d as time goes to infinity. For this limit the current distribution can be described analytically by:

$$j_z = \frac{-L}{2\pi} \alpha \hat{B}^A \left\{ \left(\cos\left(\frac{2\pi L}{L_p}\right) \sin\left(\frac{2\pi z}{L_p}\right) - \frac{z}{L} \sin\left(\frac{2\pi L}{L_p}\right) \cos\left(\frac{2\pi z}{L_p}\right) \right) \cos \phi + \right. \\ \left. \left(-\cos\left(\frac{2\pi L}{L_p}\right) \cos\left(\frac{2\pi z}{L_p}\right) - \frac{z}{L} \sin\left(\frac{2\pi L}{L_p}\right) \sin\left(\frac{2\pi z}{L_p}\right) + 1 \right) \sin \phi \right\} \quad (16)$$

$$j_\phi = R \alpha \hat{B}^A \left\{ \left(\cos\left(\frac{2\pi L}{L_p}\right) - \frac{L_p}{2\pi L} \sin\left(\frac{2\pi L}{L_p}\right) \cos\left(\frac{2\pi z}{L_p}\right) + \right. \right. \\ \left. \left. \frac{z}{L} \sin\left(\frac{2\pi L}{L_p}\right) \sin\left(\frac{2\pi z}{L_p}\right) \right) \sin \phi + \left(-\cos\left(\frac{2\pi L}{L_p}\right) + \right. \right. \\ \left. \left. \frac{L_p}{2\pi L} \sin\left(\frac{2\pi L}{L_p}\right) \sin\left(\frac{2\pi z}{L_p}\right) + \frac{z}{L} \sin\left(\frac{2\pi L}{L_p}\right) \cos\left(\frac{2\pi z}{L_p}\right) \right) \cos \phi \right\} \quad (17)$$

and the well known result of the loss P :

$$P = \frac{\alpha (L_p^2 + (2\pi R)^2) \hat{B}^A{}^2 R L}{2\pi} \left(1 - \left(\frac{L_p}{2\pi L} \right)^2 \sin^2\left(\frac{2\pi L}{L_p}\right) \right)$$

first given by G. Ries and S. Takács.

Figure 3 shows the momentary loss P as a function of time in a homogeneous magnetic field. Finally figure 4 shows the frequency dependence of the loss per cycle P for different cable length $2L$.

Conclusions

Calculations on the current distribution and loss in infinitely long cables with an arbitrary applied field can be done by representing the current density and the applied field in a Fourier series. For infinitely long cables with periodical applied field the problem can be treated totally analytically.

References

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Figure 1. The loss P in a sinusoidal ac field dependent on $2L/L_p$, the wavelength of the field $2L$ and the twistlength L_p , with respect to the loss in a homogeneous ac field.

Figure 2a. The current distribution of a finite length of cable ($2L$) in a homogeneous (linear with time increasing) field for several time-points. ($L_p/2\pi R = \infty$; the figures made by scaling R artificially) $t = 0.5\tau$.

Figure 2b. $t = 3\tau$.

Figure 2c. $t = 5\tau$.

Figure 2d. $t = 10\tau$.

Figure 2e. The analytical equivalence ($t = \infty$) of figure 2d.

Figure 4. The loss P per cycle in an ac field as a function of the frequency ω for different cable lengths (the value of $2L/L_p$ is given at the corresponding curves).

Figure 3. The momentary loss P as a function of time. P depends on a homogeneous (with time increasing) magnetic field. ($\tau = 2$ sec.).