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On the equations of motion for mixtures of liquid and gas bubbles

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On the basis of previous work by the author, equations are derived describing one-dimensional unsteady flow in bubble-fluid mixtures. Attention is subsequently focused on pressure waves of small and moderate amplitude propagating through the mixture. Four characteristic lengths occur, namely, wavelength, amplitude, bubble diameter and inter-bubble distance. The significance of their relative magnitudes for the theory is discussed. It appears that for high gas content the dispersion is weak and then the conservation of mass and momentum lead to equations similar to the Boussinesq equations, describing long dispersive waves of finite amplitude on a fluid of finite depth. For waves propagating in one direction only, the corresponding equation is similar to the Korteweg–de Vries equation.

It is shown that for mixtures of low gas content the frequency dispersion is in most cases not small. Finally, solutions of the Korteweg–de Vries equation representing cnoidal and solitary waves in a bubble-liquid mixture are given explicitly.

1. Introduction

A mixture of liquid and gas bubbles may be considered as a continuous medium if appreciable changes of quantities such as velocity and pressure occur over distances large with respect to the inter bubble distance. This medium owes mass density mainly to the liquid, compressibility mainly to the gas content. If in addition all involved frequencies are far below the lowest resonance frequency occurring in the bubble distribution, only the total gas content per unit of volume is important and not the distribution of this gas content over bubbles of specific size. The medium may be considered as a (fictitious) homogeneous one. Theories analogous to those for single phase compressible fluids can be constructed. Examples of these are given in Hsieh & Plesset (1961) and Campbell & Pitcher (1957). In the first work an expression for the sound velocity in the mixture is derived. The second reports a theoretical and experimental study on normal shock waves in a bubble water mixture.

When the above-mentioned condition on the frequencies is not fulfilled the interaction of the individual bubbles with the fluid and (through the fluid) mutually, has to be considered. A number of investigators have done this by taking in a linearized fashion the dynamic response of the individual bubbles into account (Carstensen & Foldy 1947; Meyer & Skudzrijk 1953).

The propagation of sound waves now depends on the frequency, i.e. the medium is dispersive. In Morse & Feshbach (1953) the multiple scattering of a sound wave in a bubble liquid mixture is considered. First, the scattering of a sound-wave incident on a single bubble is considered and subsequently by summation over the bubbles the velocity potential in the mixture is obtained. The results show the dispersion effect and, although obtained in a way different from those in the references mentioned earlier, take the same form. All this pertains to acoustic, i.e. small amplitude, waves and one may ask how to extend this to waves of finite amplitude. A theory both for linear and nonlinear dispersion in a bubble liquid mixture has been developed recently by the present author (van Wijngaarden 1964, 1966).

The essential point in this theory may be summarized as follows. In a mixture of liquid and bubbles we define a pressure p, which is the average over a region containing many bubbles. In the same way a velocity u is defined. A relation between the volume of a bubble and its pressure p_g is prescribed. The conservation of mass of the mixture is described in terms of the rate of change of the volume of the bubbles, whereas the conservation of momentum is expressed in terms of the average pressure p. The coupling between the average pressure p and the pressure p_g in the bubble is described by the full non-linear equation for the dynamic behaviour of the bubble. This was worked out in van Wijngaarden (1964 and 1966, henceforth denoted with I and II), under the assumption of a small volumetric gas ratio. Then (as will be shown in §2 below) among the nonlinear terms in the momentum equation those arising from the equation for the dynamic behaviour of the individual bubbles are dominant. In II the dispersion of small amplitude waves was discussed, which obey the dispersion equation

$$\omega = \frac{k}{(1+k^2)^{\frac{1}{2}}}, \tag{1.1}$$

where ω and k are the (dimensionless) frequency and wave number respectively.

In his written discussion on II Dr T. Brooke Benjamin noted that for long waves (1.1) is of the same form as the dispersion equation for long gravity waves on a fluid of finite depth. Dr Brooke Benjamin suggested that together with the amplitude dispersion occurring just as in gasdynamics in the homogeneous flow theory the dispersion of waves of finite amplitude propagating through a liquid-bubble mixture could be described by an equation of the type of the Korteweg–de Vries equation for long water waves. In the present paper this suggestion is followed.

Whereas in the theory of long water waves there are two parameters, one describing the amplitude dispersion and the other the frequency dispersion, the present investigation shows that in the case of liquid bubble mixtures there is also a third one, the ratio between volumetric liquid and gas fractions. It will be shown that in the case where this parameter is of order unity the two others being of the same order of smallness, equations similar to the Boussinesq equations for water waves hold.

2. The equations of motion

We consider a mixture consisting of gas bubbles with radius R and number density n in a fluid with density ρ_f. In the undisturbed state the pressure both in the fluid and in the bubbles is p_0. We make the simplifying assumption that in the undisturbed state the bubbles have all the same radius R_0. Because the density of the gas can reasonably be neglected we may write for the mass density ρ of the mixture

$$\rho = \rho_f(1-nV), \tag{2.1}$$

where $V = \frac{4}{3}\pi R^3$.

In a unit mass of the mixture the mass of gas is constant. To the same degree of approximation as used in (2.1) we have therefore, quantities in the undisturbed state being indicated with the subscript 0,

$$\frac{\rho_g n V}{1-nV} = \frac{\rho_{g0} n_0 V_0}{1-n_0 V_0}. \tag{2.2}$$

We focus attention on plane waves, so that all quantities depend on time t' and a distance x'. If the characteristic wavelength is denoted by λ we require that a region small in extent with respect to λ contains many bubbles, or

$$n_0^{-\frac{1}{3}} \ll \lambda. \tag{2.3}$$

The velocity in the mixture averaged over such a region is u, the pressure p. The liquid is regarded as incompressible which is correct as long as all involved velocities remain small with respect to the velocity of sound in the pure liquid: ρ depends therefore through nV on x' and t'.

The conservation of mass requires

$$\frac{\partial \rho}{\partial t'} + \frac{\partial}{\partial x'}(\rho u) = 0. \tag{2.4}$$

The momentum equation is, if we leave viscosity effects out of account,

$$\frac{\partial u}{\partial t'} + u\frac{\partial u}{\partial x'} = -\frac{1}{\rho}\frac{\partial p}{\partial x'}. \tag{2.5}$$

We cannot, as in ordinary gasdynamics, complete the set of equations directly by a relation between p and ρ. In the case of a liquid-bubble mixture we first prescribe the relation between the pressure p_g in a bubble and the density ρ_g. Plesset & Hsieh (1960) investigated bubble behaviour under oscillatory conditions and concluded that in a large range of frequencies, including the high frequency limit, the behaviour of the bubble is nearly isothermal. We shall therefore assume the isothermal relation

$$p_g/\rho_g = p_0/\rho_{g0}. \tag{2.6}$$

A relation between average pressure p, the pressure of the gas in the bubble p_g and the bubble volume is established as follows.

Consider a gas bubble in a fluid at rest. Let the pressure far from the bubble be p_∞. Then the compression or expansion of the bubble is governed by

$$\rho_f\left\{R\frac{d^2R}{dt'^2} + \frac{3}{2}\left(\frac{dR}{dt'}\right)^2\right\} = p_g - p_\infty, \tag{2.7}$$

when surface tension is left out of account. This relation is given in Lamb (1932, p. 122) and forms the basis of many studies on the behaviour of cavitation bubbles (see e.g. Plesset 1962). We suppose that in our liquid-bubble mixture the relative motion of bubbles and fluid is so small, that the relation (2.7) may be used with p, the average pressure in the mixture at the location x' of the bubble, in place of p_∞. The expression dR/dt' is the rate of change of the bubble radius as observed in a frame moving locally with the bubble.

Making use of (2.6) and the fact that the mass of one bubble, $V\rho_g$, is constant, we have

$$R = R_0\left(\frac{p_g}{p_0}\right)^{-\frac{1}{3}}. \tag{2.8}$$

The required relation between p and p_g is

$$p = p_g - \rho_f R_0^2\left[\left(\frac{p_g}{p_0}\right)^{-\frac{1}{3}}\frac{d^2}{dt'^2}\left(\frac{p_g}{p_0}\right)^{-\frac{1}{3}} - \frac{3}{2}\left\{\frac{d}{dt'}\left(\frac{p_g}{p_0}\right)^{-\frac{1}{3}}\right\}^2\right]. \tag{2.9}$$

The equations (2.1), (2.2), (2.4)–(2.6), (2.8) and (2.9) determine the unknown quantities p, u, ρ, ρ_g, p_g, n and R of one-dimensional unsteady flow in a bubble-liquid mixture.

3. Linear waves and waves of moderate strength

We choose from the unknown quantities p_g/p_0 as a central one. From (2.1), (2.2) and (2.6) we obtain

$$\rho = \rho_f\frac{p_g/p_0}{p_g/p_0 + n_0V_0/(1-n_0V_0)}. \tag{3.1}$$

We have therefore

$$\frac{dp_g}{d\rho} = \frac{p_0}{\rho_f}\frac{1-n_0V_0}{n_0V_0}\left(\frac{p_g}{p_0} + \frac{n_0V_0}{1-n_0V_0}\right)^2. \tag{3.2}$$

We attempt to construct a theory for linear waves and waves of moderate strength and therefore write

$$p_g/p_0 = 1 + \epsilon\xi, \tag{3.3}$$

where ϵ is a small number. Inserting this in (3.2) yields

$$\frac{dp_g}{d\rho} = \frac{p_0}{\rho_f n_0V_0(1-n_0V_0)}\{1 + 2\epsilon\xi(1-n_0V_0) + O(\epsilon^2)\}. \tag{3.4}$$

When at constant gas fraction the bubble size becomes very small, $R_0 \to 0$, it follows from (2.9) that $p_g \to p$. Under these circumstances (3.4) represents the square of the sound velocity of the mixture. If we denote this velocity by c, we have in the lowest approximation

$$c^2 = c_0^2 = \frac{p_0}{\rho_f n_0V_0(1-n_0V_0)}. \tag{3.5}$$

For the following it is convenient to render the independent variables dimensionless. For x' we choose a typical wavelength λ, for t' the time λ/c_0. Hence

$$t' = \frac{\lambda}{c_0}t; \quad x' = x\lambda. \tag{3.6}$$

From (3.1) it follows that the magnitude of the variations of ρ, besides depending on ϵ, also depend on the magnitude of the volumetric gas fraction $n_0 V_0$. With a view on mass conservation this holds also for u. We therefore render u dimensionless by

$$u = \epsilon c_0 n_0 V_0 v. \tag{3.7}$$

To express the mass conservations in terms of v and ξ we write (2.4) as

$$\frac{\partial p_g}{\partial t'} + u\frac{\partial p_g}{\partial x'} + \rho\left(\frac{\partial p_g}{\partial \rho}\right)\frac{\partial u}{\partial x'} = 0.$$

Using (3.1) and (3.3)–(3.7) this equation reduces, when terms of order ϵ^2 and lower order are retained, to

$$\frac{\partial \xi}{\partial t} + \epsilon n_0 V_0 v\frac{\partial \xi}{\partial x} + \{1 + (2 - n_0 V_0)\epsilon\xi\}\frac{\partial v}{\partial x} = 0. \tag{3.8}$$

Next we also introduce the dimensionless variables in (2.5). For p we insert the righthand side of (2.9). Using again (3.1) and (3.3)–(3.7) we obtain

$$\epsilon\left\{\frac{\partial v}{\partial t} + \epsilon n_0 V_0 v\frac{\partial v}{\partial x} + (1 - n_0 V_0 \epsilon\xi)\frac{\partial \xi}{\partial x}\right\} - \frac{(1 - n_0 V_0 \epsilon\xi)R_0^2}{\lambda^2 n_0 V_0 (1 - n_0 V_0)}\frac{\partial}{\partial x}\left[(1+\epsilon\xi)^{-\frac{1}{3}}\frac{d^2}{dt^2}(1+\epsilon\xi)^{-\frac{1}{3}} + \left\{\frac{d}{dt}(1+\epsilon\xi)^{-\frac{1}{3}}\right\}^2\right] = 0. \tag{3.9}$$

The non-linear acceleration term of lowest order appears to be of order $\epsilon^2 n_0 V_0$, as the convective term in (3.8). The terms between the square brackets in (2.9), associated with the inertia of the fluid displaced by an expanding or compressing bubble, are in the dimensionless form preceded by the factor

$$\sigma = \frac{R_0^2}{\lambda^2(1 - n_0 V_0)n_0 V_0}. \tag{3.10}$$

This parameter occurs also in I and II. There $(\lambda/R_0)(n_0 V_0)^{\frac{1}{2}}$ is used as parameter and is shown to govern the dispersion of pressure waves.

Before expanding the expression between square brackets in (3.9) in terms of ϵ, we consider the magnitude of σ. For this purpose we introduce the ratio between bubble radius and wavelength as

$$\alpha_1 = R_0/\lambda \tag{3.11}$$

and the ratio between inter-bubble distance $n_0^{-\frac{1}{3}}$ and wavelength as

$$\alpha_2 = n_0^{-\frac{1}{3}}/\lambda.$$

The parameters $n_0 V_0$ and σ can be expressed in terms of α_1 and α_2:

$$n_0 V_0 = \left(\frac{\alpha_1}{\alpha_2}\right)^3, \tag{3.12}$$

$$\sigma = \left(1 - \left(\frac{\alpha_1}{\alpha_2}\right)^3\right)^{-1}\frac{\alpha_2^3}{\alpha_1}. \tag{3.13}$$

Because for our theory to be valid the wavelength must be large with respect to the inter-bubble distance, we assume α_2 to be small (cf. 2.3). From (3.12) it follows that the ratio α_2/α_1 between bubble distance and bubble dimension is of order unity for high gas content and large for low gas content. We shall discuss first the case of high gas content, $\alpha_1/\alpha_2 = O(1)$. Then it follows from the fact that α_2 is small that the dispersion parameter σ is also small. Therefore we assume $\sigma = O(\epsilon)$.

We proceed to expand the terms associated with σ in (3.10) in terms of ϵ, stopping at one approximation beyond the linear approximation. This yields

$$\frac{\partial v}{\partial t}+\epsilon n_0 V_0 v\frac{\partial v}{\partial x}+(1-n_0V_0\epsilon\xi)\frac{\partial \xi}{\partial x}+\frac{\sigma}{3}\frac{\partial^3\xi}{\partial x\,\partial t^2}=0. \qquad (3.14)$$

The equations (3.8) and (3.14) for v and ξ are very similar to the Boussinesq equations for long water waves (for a recent discussion on these equations see Whitham 1965*b*). In that case there are three lengths, namely amplitude, depth and wavelength. The Boussinesq equations hold if the ratio between amplitude and depth and the square of the ratio between depth and wavelength are moderately small and of comparable magnitude.

In our case there are four lengths, viz. wavelength, amplitude, bubble diameter and inter-bubble distance. We have shown that if the ratio between the last and the one but last is of order one, the dispersion is weak and that wave propagation is governed under these circumstances by equations very similar to the Boussinesq equations.

In many practical circumstances mixtures are encountered for which the inter-bubble distance is appreciably larger than the bubble radii. If for example n_0V_0 is $O(\epsilon^3)$ then $R_0/n_0^{-\frac{1}{3}} = \alpha_1/\alpha_2 = O(\epsilon^{-1})$. In such a sparse mixture the dispersion parameter σ can be of order one rather than small. With $\alpha_2 = O(\epsilon^{\frac{1}{2}})$, for example, and $\alpha_1/\alpha_2 = O(\epsilon^{-1})$, we have (cf. 3.13) $\sigma = O(1)$. Note that this cannot occur in a dense mixture since there α_1/α_2 is of unit order and α_2 must be small for the theory to be valid.

We conclude that for sparse mixtures higher order terms of the 'bubble equation' (2.9), from which the terms in the square brackets in (3.9) are deduced, must be retained in many cases. This is in fact done in I and II where the full equations, with however the neglect of n_0V_0 with respect to unity, are used to analyse the dispersion of a finite pressure pulse, in particular (see I) in a cloud of cavitation bubbles in water. In the context of the present investigation we restrict ourselves to the case where σ is small of order ϵ. For a sparse mixture, $n_0V_0 = O(\epsilon^k)$ with $k > 1$, this implies the condition $R_0/\lambda = O(\epsilon^{\frac{1}{2}(k+1)})$. For $\sigma = O(\epsilon)$ the equations pertaining to a sparse mixture follow from (3.8) and (3.14) by simply neglecting the terms with the factor n_0V_0.

4. Some special cases

When at constant gas content $R \to 0$ and $n \to \infty$, then $p = p_g$ (cf. 2.9). The fluid is homogeneous and compressible. For linear waves, $\epsilon = 0$, (3.8) and (3.14) reduce with $\sigma = 0$ to

$$\frac{\partial \xi}{\partial t}+\frac{\partial v}{\partial x}=0, \qquad (4.1)$$

$$\frac{\partial v}{\partial t}+\frac{\partial \xi}{\partial x}=0. \qquad (4.2)$$

These equations represent acoustic waves travelling through the mixture at unit velocity. In an approximation one order beyond the linear one, (3.8) and (3.14) can be written in characteristic form. From the theory of gasdynamics we know that the characteristics are given by

$$\frac{dx}{dt} = \epsilon n_0 V_0 v \pm \{1 + (1 - n_0 V_0)\epsilon\xi\}, \tag{4.3}$$

where use has been made of (3.4).

The Riemann invariants are in physical variables given by $u \pm U'$, where

$$U' = \int \frac{dp}{\rho c}.$$

Because $p_g = p$ we find, using the relations (3.1), (3.3) and (3.5) and introducing $U = U'/n_0 V_0 c_0$,

$$U = \epsilon\xi + O(\epsilon^2).$$

The equations for nondispersive waves ($\sigma = 0$) are therefore in characteristic form

$$\left[\frac{\partial}{\partial t} + \{\epsilon n_0 V_0 v \pm (1 + (1 - n_0 V_0)\epsilon\xi)\}\frac{\partial}{\partial x}\right][v \pm \xi] = 0. \tag{4.4}$$

Consider a wave progressing to the right in an undisturbed mixture. Then $v = \xi$, since on the left characteristics $v - \xi = 0$. The right-going characteristics yield

$$\frac{\partial \xi}{\partial t} + (1 + \epsilon\xi)\frac{\partial \xi}{\partial x} = 0. \tag{4.5}$$

This relation shows the well-known steepening of a compressive wave sometimes described as amplitude dispersion. An equation of the type (4.5) was given also by Benjamin (1966) in his discussion of II. Note that due to the isothermal conditions (expressed by (3.4)) the terms with $n_0 V_0$ in (4.4) just cancel for the wave described by (4.5).

For linear waves, $\epsilon = 0$, the equations (3.8) and (3.14) reduce to

$$\frac{\partial \xi}{\partial t} + \frac{\partial v}{\partial x} = 0, \tag{4.6}$$

$$\frac{\partial v}{\partial t} + \frac{\partial \xi}{\partial x} + \frac{\sigma}{3}\frac{\partial^3 \xi}{\partial t^2 \partial x} = 0. \tag{4.7}$$

Elimination of v leads to

$$\frac{\partial^2 \xi}{\partial t^2} - \frac{\partial^2 \xi}{\partial x^2} - \frac{\sigma}{3}\frac{\partial^4 \xi}{\partial t^2 \partial x^2} = 0. \tag{4.8}$$

This equation was derived and discussed in II. The dispersion equation, obtained by seeking solutions of (4.8) of the form $\exp i(kx - \omega t)$, is

$$\omega = \frac{k}{(1 + \frac{1}{3}\sigma k^2)^{\frac{1}{2}}}. \tag{4.9}$$

The dispersion parameter σ can also be defined in terms of the bubble resonance frequency ω_B, which is, as follows from (2.9)

$$\omega_B = (3p_0/\rho_l R_0^2)^{\frac{1}{2}}.$$

From this relation together with (3.5) and (3.10) it follows that

$$\frac{\sigma}{3} = \frac{c_0^2}{\omega_B^2 \lambda^2}.$$

In the course of his recent work on linear and non-linear dispersive waves Whitham (1965*b*) discusses equation (4.8) from the point of view of his theory. The linearized Boussinesq equations lead to an equation of the form (4.8).

5. Waves propagating in one direction

We return to the full equations (3.8) and (3.14). It is known that in the case of long water waves the Boussinesq equations can for waves in only one direction be reduced to an equation known as the Korteweg–de Vries equation. This reduction can be carried out also for our equations. Using the method given by Broer (1964), which is based on the fact that $(\partial/\partial t)+(\partial/\partial x) = O(\epsilon)$ and also $v-\xi = O(\sigma) = O(\epsilon)$, we find for waves travelling to the right

$$\frac{\partial \xi}{\partial t}+\frac{\partial \xi}{\partial x}+\frac{\partial}{\partial x}(\tfrac{1}{2}\epsilon\xi^2)+\frac{\sigma}{6}\frac{\partial^3 \xi}{\partial x^3} = 0. \tag{5.1}$$

Because $p_g - p = O(\epsilon^2)$ and for right-going waves $(\partial/\partial t)+(\partial/\partial x) = O(\epsilon)$, (5.1) holds also for $(p-p_0)/p_0$. This equation has the form of the Korteweg–de Vries equation from which cnoidal wave solutions and solitary waves are derived in the theory of surface waves (Korteweg & de Vries 1895; see also Lamb 1932, §253). These waves have evidently their counterparts in bubble liquid mixtures. To obtain the solutions representing these waves we put

$$\epsilon\xi = P(x-Ct). \tag{5.2}$$

Introducing this in (5.1), integrating twice and taking $t = 0$, yields

$$\sigma\left(\frac{dP}{dx}\right)^2 = 2\{(a_1-P)(P-a_2)(P-a_3)\} \tag{5.3}$$

with

$$C = 1+\tfrac{1}{3}\{(a_1-a_2)-a_3\}. \tag{5.4}$$

The periodic solution of (7.2) is, a 'crest' of the wave being in $x = 0$,

$$P = (a_1-a_2)\,cn^2\left\{\left(\frac{a_1-a_3}{2\sigma}\right)^{\frac{1}{2}}x,\ \beta\right\}+a_2, \tag{5.5}$$

where *cn* is the Jacobian elliptic function and

$$\beta^2 = \frac{a_1-a_2}{a_1-a_3}. \tag{5.6}$$

The wavelength λ is implicitly (we recall that σ contains λ) given by

$$1 = 2\left(\frac{2\sigma}{a_1-a_3}\right)^{\frac{1}{2}}K(\beta). \tag{5.7}$$

$K(\beta)$ is the complete elliptic integral of the first kind. The mean value of P must be zero, which gives the relation

$$\frac{a_3}{a_1-a_3} = \frac{E(\beta)}{K(\beta)}, \tag{5.8}$$

where $E(\beta)$ is the complete elliptic integral of the second kind. For the parameters λ, C, β, a_1, a_2 and a_3 we have the four relationships (5.4) and (5.6)–(5.8). When, for instance, the maximum pressure a_1 in the wave and λ are chosen, the remaining parameters follow from these relations. The form of the wave is then determined by (7.4). A special case is $a_2 = a_3 = 0$, obtained when requiring in the integration of (5.1) that at infinity

$$P = \frac{dP}{dx} = \frac{d^2P}{dx^2} = 0.$$

From (5.6) it follows that $\beta = 1$ and, since $K(1) = \infty$, on account of (3.10) $\lambda = \infty$. The form assumed by (7.4) representing the solitary wave then is in dimensional variables

$$\frac{p-p_0}{p_0} = a\,\mathrm{sech}^2\left[\frac{x'}{R_0}\left\{\frac{an_0V_0(1-n_0V_0)}{2}\right\}^{\frac{1}{2}}\right]. \tag{5.9}$$

The wave velocity is by (5.4) given as $c_0(1+\frac{1}{3}a)$.

6. Discussion

The dynamics of bubble-liquid mixtures can under conditions given in the preceding sections be described by equations which are similar to the Boussinesq equations for water waves. The results given in the present paper open prospects in two ways. First, there is a firm body of knowledge regarding the properties of solutions of these equations in the theory of water waves. Cnoidal waves and solitary waves appear to have their counterparts in the dynamics of liquid-bubble mixtures. A powerful tool for the investigation of nonlinear dispersive waves is provided by Whitham in a recent series of papers (Whitham 1965*a*, *b*, 1967). In this work the Boussinesq and Korteweg–de Vries equations are among others dealt with.

Secondly, as Lighthill (1966) has pointed out, there is a need to verify results obtained from Whitham's theory experimentally. Lighthill suggested some experiments with water waves, including also the effect of surface tension.

The present work shows that experiments on liquid-bubble flow also may serve the purpose of experimental verification.

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