Scaling in hard turbulent Rayleigh-Bénard flow

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Rayleigh-Bénard flow for Rayleigh numbers $Ra = 10^8 - 10^{11}$ (hard turbulence regime) is studied solving the Boussinesq equations with the Fourier-Weierstrass ansatz introduced recently [Eggers and Grossmann, Phys. Fluids A **3**, 1958 (1991)]. The plumes and swirls detaching from the boundary layers are mimicked by volume stirring on all scales down to the Rayleigh-number-dependent scale of these thermals. Wave-number spectra, frequency spectra, and structure functions are presented both for velocity and temperature fluctuations. As the scale decreases the velocity-temperature cross correlation decreases much faster than both velocity and temperature autocorrelations. In the viscous subrange all wave-number spectra decay exponentially. Based on the experimental Ra dependence of the mean temperature fluctuations we can calculate the Ra dependence of the mean velocity fluctuations as well as of the mean temperature and velocity time derivatives. The inner length scale η is found to scale $\propto Ra^{-0.32 \pm 0.01}$.

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I. INTRODUCTION AND SUMMARY

Recent experiments by Libchaber and co-workers [1-3] on highly turbulent Rayleigh-Bénard (RB) convection in helium have revealed new scaling regimes. In particular, in the hard turbulence regime $Ra = 10^8 - 10^{11}$ the scaling of the temperature frequency spectrum $P(\omega)$ can be parametrized [4] by $P(\omega) \propto \omega^{-1.40 \pm 0.05}$, which is different from classical $-5/3$ scaling. In [5] $P(\omega) \propto \omega^{-1.30 \pm 0.10}$ is given. Henceforth, we refer to this scaling as $-7/5$ scaling [4], because the exponent $-7/5$ was commonly believed to be the correct one.

In this paper the dynamical equations for RB flow are analyzed. We restrict ourselves to the above Ra range as for $Ra > 10^{11}$ new physical effects seem to come in which result in different spectra and have to be treated separately [6]. The Fourier-Weierstrass expansion introduced recently [7] for the velocity field is extended to the temperature field. This offers us the opportunity to really solve the fluid equations for the bulk of turbulence for such high Rayleigh numbers as $Ra > 10^8$ numerically. In Ref. [8] we showed that a thermal forcing acting on the largest scale only results in classical $-5/3$ scaling for both velocity and temperature and thus does not explain the observed [4, 5] $-7/5$ -scaling.

It has been emphasized right from the beginning that *thermal plumes* play a prominent role in forcing the Rayleigh-Bénard system [9]. In [2] a model based on plumes being accelerated in a so-called mixing layer is given which successfully accounts for the Ra dependence of dynamical quantities such as the typical velocity fluctuations u_c and the typical temperature fluctuations Θ_c in the bulk and as the Nusselt number Nu . An alternative interpretation without introducing a mixing layer is given in [10].

In the following our central assumption also is that the plumes, swirls, and thermals (henceforth called "plumes" altogether) detaching from the boundary layers [11] are

the origin for the observed $-7/5$ scaling. We therefore *model* the plume forcing of the *bulk* of turbulence by appropriate deterministic volume forces. In simulations for lower Ra [12-14] the plumes come out as a *result* because boundary and mixing layers are considered explicitly. (See also the two-dimensional simulations of references [15-17]).

In [11] plumes of all kinds are visualized and their life cycle is documented. In this paper we assume that only the typical size of the plumes is important. The colored photographs in [11] impressively show that the typical size of the stem is of the order of the boundary-layer thickness λ_{BL} (which scales [2] like $\lambda_{BL}/L_{\text{expt}} \propto Ra^{-2/7}$, where L_{expt} is the height of the RB cell). We therefore choose to force all eddies of sizes $r \geq \lambda_{BL}$ by plumes directly.

Since we do not know how the plumes really force the bulk of turbulence, we represent their stirring in two alternative ways: We either ascribe to them *thermal* action only; in this case the velocity field is stirred only *indirectly* via buoyant coupling. Or we allow the plumes also to *directly* transfer *kinetic* energy to the velocity field. In both cases the strength of the plume stirring is adopted to the experimental temperature frequency spectrum, i.e., to the power-law exponent $-7/5$; all other spectra *result* from solving the dynamical equations. The exponent $-7/5$ corresponds to $-2/5$ of the discrete spectrum; and from now on we use discrete wave numbers only. The wave-number spectrum of the velocity scales slower than or at most with $-2/3$ (i.e., $-5/3$ in the continuous case) depending on whether one attributes also direct velocity stirring to the plumes or not. In any case, the decay of the spectrum is *never faster* than classical $-2/3$ in contrast to the claim of Bolgiano and Obukhov [18, 19] as we already showed in [8]. The velocity-temperature cross spectrum decays considerably faster with wave numbers than what one would naively guess from the decay of the temperature and velocity spectra, namely, with an

exponent of about -1.15 for the discrete spectrum (i.e., -2.15 in the continuous case). In the viscous subrange all wave-number spectra decay exponentially in agreement with the prediction in Ref. [20]. A measurement of the power-law exponents of the velocity spectrum and of the velocity-temperature cross spectrum would be very helpful to test our findings.

We shall present both wave-number and frequency spectra. The former are most easily accessible quantities from our calculation. It is not clear *a priori* that wave-number spectra scale with the same exponent as frequency spectra, which are obtained by fast Fourier transformation of the time-dependent signal. Taylor's frozen-flow hypothesis [21] is behind this assumption. Our numerical solution offers the opportunity to test this hypothesis and we find it fulfilled. Furthermore, structure functions are calculated that scale correspondingly.

Recently, Wu [22] published the values of the experimental control parameters of the experiments with a low-temperature helium cell of height $L_{\text{expt}} = 40$ cm and an aspect ratio of $\Gamma = 1/2$. The knowledge of these values will allow us to calculate even the absolute strength of the RB flow fluctuations. We fix our control parameters and units for the numerical calculation by comparison with the extension of the experimental scaling range and with the Ra dependence of the typical temperature fluctuations Θ_c in the bulk, $\Theta_c \propto \Delta_{\text{expt}} Ra^{-1/7}$, see Refs. [1, 2, 22], where Δ_{expt} is the temperature difference between the top and the bottom of the cell. We then can evaluate the typical velocity fluctuations u_c and find them $\propto Ra^{3/7}$ as in experiment [23]. For the normalized rms temperature time derivative Q_Θ (introduced and measured recently [5]) we obtain a Ra power-law exponent of about 0.58, which is only in rough agreement with the experimental value $Q_{\Theta, \text{expt}} \propto Ra^{0.67 \pm 0.04}$. For an analogously normalized rms velocity time derivative Q_u we predict a Ra scaling exponent of about 0.55.

The original Fourier-Weierstrass mode decomposition [7] has a shortcoming: It does not admit the possibility of spatial incoherence of eddies. In [24] the Fourier-Weierstrass mode analysis for Navier-Stokes turbulence was extended to a space resolving one which allows for spatial incoherence, and intermittency corrections could numerically be calculated. In that picture intermittency arises from competing offspring eddies extracting different amounts of energy from their parent eddy. As in experiment [25, 26] the corrections due to intermittency are quantitatively small and play a significant role for higher moments (and for the probability distribution function) only. So for the spectra—the main point of this paper—our non-space-resolving analysis according to [7] should be sufficient; intermittency effects within our Fourier-Weierstrass approach are discussed elsewhere [27].

Another characteristic feature of the hard-turbulence regime is the exponential-type probability distribution function (PDF) for the temperature [1, 2, 28–30]. Solomon and Gollub observed [28, 29] that coherent plumes which travel undisturbed in the bulk of turbulence, result in a Gaussian PDF there. Any kinds of perturbations destroying the coherence of the plumes, such as natural or artificial winds or large aspect ratios, lead to

PDF's with more or less stretched exponential tails [29]. Also the numerical simulations of Balachandar, Maxey, and Sirovich [12–14] give hints that the plumes are the origin of the exponential-type temperature PDF. Basing on this assumption, Yakhot [31] gives an analytical model for the temperature PDF's. For the purpose of this paper we concentrate on the kinetic and thermal input power provided by the plumes and disregard their phases and coherences.

Our paper is organized as follows: In Sec. II the Fourier-Weierstrass analysis for RB turbulence is introduced. In Sec. III we specify the plume stirring, in Sec. IV we point out how we fix the parameters for the numerical calculations. Sections V and IV contain our results about scale dependences of the wave-number spectra, the structure functions, and the frequency spectra, whereas Sec. VII is devoted to the Ra dependences of the quantities of interest.

II. THE FOURIER-WEIERSTRASS CASCADE ANALYSIS FOR THE BOUSSINESQ EQUATIONS

A. The Boussinesq equations

The equations of motion for the RB system are the Navier-Stokes and the heat-transfer equations in the Boussinesq approximation [32, 33] together with the incompressibility condition,

$$\partial_i u_i = -u_j \partial_j u_i - \partial_i P + \nu \partial_j \partial_j u_i + \beta g (\bar{\Theta} + \Theta) \delta_{i3}, \quad (2.1a)$$

$$\partial_i \Theta = -u_j \partial_j \Theta + \kappa \partial_j \partial_j \Theta - u_j \partial_j \bar{\Theta}, \quad (2.1b)$$

$$\partial_i u_i = 0. \quad (2.1c)$$

$u_i(\mathbf{x}, t)$, $i = 1, 2, 3$, $P(\mathbf{x}, t)$, and $\Theta(\mathbf{x}, t)$ are the velocity components, the kinematic pressure, and the temperature deviation from the mean temperature $\bar{\Theta}(z)$, respectively. ν , κ , β , and g are the viscosity, heat diffusivity, volume-expansion coefficient, and gravitational acceleration.

Equations (2.1) are completed by the boundary conditions $\bar{\Theta}(z = \pm L_{\text{expt}}/2) = \mp \Delta_{\text{expt}}/2$ and $u_i(\mathbf{x}) \equiv 0$, $\Theta(\mathbf{x}) \equiv 0$ at the walls. L_{expt} is the height of the RB cell and Δ_{expt} the measured temperature difference between the cold top and the hot bottom of the cell.

As the experiments show [2, 3, 23], the mean temperature $\bar{\Theta}(z)$ almost completely decreases within the boundary layers and stays constant in the bulk. Therefore, as we shall concentrate on the physics of the bulk, the terms with $\bar{\Theta}(z)$ in (2.1) can be skipped to a good approximation. For the same reason we substitute the turbulence driving effects of the boundary layers by adding volume force densities $\mathbf{f}_u(\mathbf{x}, t)$ and $f_\Theta(\mathbf{x}, t)$ to (2.1a) and (2.1b). These driving forces will be specified later; they have to represent the large scale character of the forcing as well as the effects of the plumes that permanently enter the bulk from the boundary layers [1–3, 11, 23, 28, 29]. We first examine a purely *thermal* forcing by the boundary

conditions as also assumed in [34], i.e., $\mathbf{f}_u = 0$. In this case the plumes do *not directly* supply the turbulence with kinetic energy, but only via buoyant coupling. We then add a stirring force $\mathbf{f}_u(\mathbf{x}, t)$ to Eq. (2.1a), i.e., the plumes are allowed to act also directly on the velocity field.

B. The Fourier-Weierstrass series

Assuming periodic-boundary conditions in a cubic volume of linear size $2\pi L$, the Boussinesq equations (2.1) may be Fourier transformed by the discrete transformations

$$u_i(\mathbf{x}, t) = \sum_{\mathbf{p}} u_i(\mathbf{p}, t) e^{i\mathbf{p} \cdot \mathbf{x}}, \quad (2.2a)$$

$$\Theta(\mathbf{x}, t) = \sum_{\mathbf{p}} \Theta(\mathbf{p}, t) e^{i\mathbf{p} \cdot \mathbf{x}}. \quad (2.2b)$$

Of course, $u_i(-\mathbf{p}) = u_i^*(\mathbf{p})$ and $\Theta(-\mathbf{p}) = \Theta^*(\mathbf{p})$. The connection of the periodicity length $2\pi L$ and the RB cell height L_{expt} will be obtained from comparison with experiment in Sec. IV.

The central idea of the Fourier-Weierstrass analysis [7] is to reasonably restrict the wave vectors $\mathbf{p}$ that contribute in (2.2) in order to keep even high Ra number turbulence feasible. Following [7, 24], we first define a basic set of 38 wave vectors $K_0 = \{\mathbf{p}_n^{(0)} | L\mathbf{p}_n^{(0)} = \pm(2, 2, 2), \pm(-1, 2, 2), \pm(2, -1, -1), \pm(1, 4, 4), \pm(4, 1, 1), \pm(1, -2, 4) + \text{permutations}\}$ describing the largest eddies (with wavelengths $\approx 2\pi L$). In addition, a geometrically scaling sequence of sets $K_l = \{\mathbf{p}_n^{(l)} | \mathbf{p}_n^{(l)} = 2^l \mathbf{p}_n^{(0)}, n = 1, 2, \dots, 38\}$, $l = 1, 2, \dots, N$ is introduced. The wave vectors $\mathbf{p}$ in (2.2) are restricted now to the wave vectors in $K = \bigcup_{l=0}^N K_l$. With the later choice $N = 11$, i.e., 12 levels l altogether, more than three decades of length scales are represented by only $19 \times 2 \times 3 \times 12 = 1368$ real modes [half the number of basic wave vectors times real plus imaginary part of the mode amplitudes times three independent fields u_1 , u_2 , and Θ (while u_3 is eliminated by incompressibility) times 12 levels]. This mode selection allows intralevel interactions as well as interlevel interactions between two neighboring levels, for details see [7].

Note that the wave-vector components chosen here differ by a factor of 2 from those introduced in [7]; the reason is that this guarantees orthogonality of even localized modes in all periodicity volumes $(2\pi L 2^{-l})^3$, cf. Ref. [24].

The Fourier representation (2.2) with the momenta restricted to K has the character of a typical Weierstrass series. (For references concerning the Weierstrass series, see e.g. [35].) Inserting it into the dynamical equations (2.1) only those interactions uu and $u\Theta$ are kept whose momenta add up again to an element of K . While these interactions are taken exactly as demanded by the equations of motion, all others are skipped completely.

C. The dynamical equations

The Boussinesq equations then read

$$\begin{aligned} \dot{u}_i(\mathbf{p}_n^{(l)}) = & -iM_{ijk}(\mathbf{p}_n^{(l)}) \sum_{\mathbf{q}}' u_j(\mathbf{q}) u_k(\mathbf{p}_n^{(l)} - \mathbf{q}) \\ & -\nu(\mathbf{p}_n^{(l)})^2 u_i(\mathbf{p}_n^{(l)}) \\ & +\beta g P_{i3}^\perp(\mathbf{p}_n^{(l)}) \Theta(\mathbf{p}_n^{(l)}) + \mathbf{f}_u(\mathbf{p}_n^{(l)}), \end{aligned} \quad (2.3a)$$

$$\begin{aligned} \dot{\Theta}(\mathbf{p}_n^{(l)}) = & -ip_{n,k}^{(l)} \sum_{\mathbf{q}}' u_k(\mathbf{q}) \Theta(\mathbf{p}_n^{(l)} - \mathbf{q}) \\ & -\kappa(\mathbf{p}_n^{(l)})^2 \Theta(\mathbf{p}_n^{(l)}) + f_\Theta(\mathbf{p}_n^{(l)}), \end{aligned} \quad (2.3b)$$

$$p_{n,k}^{(l)} u_k(\mathbf{p}_n^{(l)}) = 0. \quad (2.3c)$$

As usual, $M_{ijk}(\mathbf{p}) = [p_j P_{ik}^\perp(\mathbf{p}) + p_k P_{ij}^\perp(\mathbf{p})]/2$ with the orthogonal projector $P_{ij}^\perp(\mathbf{p}) = \delta_{ij} - p_i p_j / \mathbf{p}^2$. The dot denotes the time derivative and the prime on the sum indicates that only those terms are included for which $\mathbf{q}$ as well as $\mathbf{p}^{(l)} - \mathbf{q}$ belong to K . It is this set of ordinary differential equations (ODE's) whose numerical solution is the basis for all results reported in this paper. Its interaction structure, the chaotic character of the solutions, and various other properties have already been elaborated extensively for the case of pure fluid turbulence without a coupled temperature field in Ref. [7].

The average kinetic energy per unit mass contained in the velocity field of level l is

$$E_u^{(l)} = \frac{1}{2} \sum_{\mathbf{p} \in K_l} \langle [\mathbf{u}(\mathbf{p}, t)]^2 \rangle. \quad (2.4a)$$

The brackets $\langle \rangle$ indicate time averaging. We introduce an analogous positive definite quantity for the temperature fluctuations ('thermal intensity' or 'thermal spectrum')

$$E_\Theta^{(l)} = \frac{1}{2} \sum_{\mathbf{p} \in K_l} \langle [\Theta(\mathbf{p}, t)]^2 \rangle. \quad (2.4b)$$

The quantities $E_u^{(l)}$ and $E_\Theta^{(l)}$ represent the spectral power of the velocity and temperature fields on level l . Note that discrete spectra $E^{(l)} \propto (p^{(l)})^\zeta$ correspond to continuous spectra $E(p) \propto p^{\zeta-1}$. One obtains balance equations for the spectral powers by multiplying (2.3a) and (2.3b) with $u_i^*(\mathbf{p}_n^{(l)})$ and $\Theta^*(\mathbf{p}_n^{(l)})$, respectively, averaging, and summing over all level momenta labels n . On the time average they read:

$$0 = T_u^{l-1 \rightarrow l} - T_u^{l \rightarrow l+1} + e_u^{(l)} + e_g^{(l)} - \epsilon_u^{(l)}, \quad (2.5a)$$

$$0 = T_\Theta^{l-1 \rightarrow l} - T_\Theta^{l \rightarrow l+1} + e_\Theta^{(l)} - \epsilon_\Theta^{(l)}. \quad (2.5b)$$

Here, the transfer terms are defined as

$$T_u^{l \rightarrow l+1} \equiv \sum_{\mathbf{p}, \mathbf{q}, \mathbf{p}-\mathbf{q} \in K_l \cup K_{l+1}} i \langle M_{ijk}(\mathbf{p}) u_j(\mathbf{q}) u_k(\mathbf{p}-\mathbf{q}) u_i^*(\mathbf{p}) \rangle \quad (2.6a)$$

and

$$T_\Theta^{l \rightarrow l+1} \equiv \sum_{\mathbf{p}, \mathbf{q}, \mathbf{p}-\mathbf{q} \in K_l \cup K_{l+1}} i \langle p_k u_k(\mathbf{p}-\mathbf{q}) \Theta(\mathbf{q}) \Theta^*(\mathbf{p}) \rangle. \quad (2.6b)$$

The term

$$e_g^{(l)} \equiv \beta g \sum_{\mathbf{p} \in K_l} \langle u_3^*(\mathbf{p}) \Theta(\mathbf{p}) \rangle \quad (2.7a)$$

supplies the velocity field with energy via buoyant coupling to the temperature field, whereas the kinetic input

$$e_u^{(l)} \equiv \sum_{\mathbf{p} \in K_l} \langle \mathbf{f}_u(\mathbf{p}) \mathbf{u}^*(\mathbf{p}) \rangle \quad (2.7b)$$

and the thermal input

$$e_\Theta^{(l)} \equiv \sum_{\mathbf{p} \in K_l} \langle f_\Theta(\mathbf{p}) \Theta^*(\mathbf{p}) \rangle \quad (2.7c)$$

model the stirring of the system which is physically due to the boundary conditions.

Finally, there are dissipation and diffusion terms

$$\epsilon_u^{(l)} \equiv \nu \sum_{\mathbf{p} \in K_l} \langle \mathbf{p}^2 |\mathbf{u}(\mathbf{p})|^2 \rangle \quad (2.8a)$$

and

$$\epsilon_\Theta^{(l)} \equiv \kappa \sum_{\mathbf{p} \in K_l} \langle \mathbf{p}^2 |\Theta(\mathbf{p})|^2 \rangle. \quad (2.8b)$$

Note that these terms can be neglected for ranges above the viscous subrange (VSR).

Summing Eqs. (2.5) over all eddy scales l leads to the total kinetic energy and thermal intensity balance

$$\sum_l (e_u^{(l)} + e_g^{(l)}) = \sum_l \epsilon_u^{(l)} \equiv \epsilon_u, \quad (2.9a)$$

$$\sum_l e_\Theta^{(l)} = \sum_l \epsilon_\Theta^{(l)} \equiv \epsilon_\Theta, \quad (2.9b)$$

because the transfer terms cancel in the l sums. Note that the wave-vector restriction does not destroy the energy and thermal intensity budget.

The quantities ϵ_u and ϵ_Θ are the total viscous dissipation rate of the velocity field and its analogon of the temperature field. Equation (2.9b) expresses that the total heat diffusion loss has to be balanced by the thermal input due to forcing while Eq. (2.9a) shows that the viscous dissipation has to be provided from the thermal coupling $\propto \beta g$ and, if present, the kinetic input by plume forcing. If the latter one equals zero, the thermal driving force f_Θ alone has to cope with both losses ϵ_Θ and ϵ_u .

Scaling behavior is not only found in the wave-number spectra (2.4) but also in $\mathbf{r}$ space when looking at the structure functions

$$D_{uu}(\mathbf{r}) \equiv \langle |\mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(\mathbf{x})|^2 \rangle = 2 \sum_{\mathbf{p} \in K} \langle |\mathbf{u}(\mathbf{p})|^2 \rangle [1 - \cos(\mathbf{p} \cdot \mathbf{r})] \quad (2.10)$$

and corresponding expressions for $D_{\Theta\Theta}(\mathbf{r})$ and $D_{\Theta u_3}(\mathbf{r})$. We shall see that the influence of gravity is hardly visible anymore on smaller scales, so we do not keep track of the parallel and orthogonal projections with respect to $\mathbf{g}$ but average over *all* directions $\mathbf{r}/r$ to get

$$D_{uu}(r) = 2 \sum_{\mathbf{p} \in K} \langle |\mathbf{u}(\mathbf{p})|^2 \rangle \left[1 - \frac{\sin(pr)}{pr} \right] \quad (2.11)$$

and similar for $D_{\Theta\Theta}(r)$ and $D_{\Theta u_3}(r)$.

In the viscous subrange (VSR) the structure functions are $\propto r^2$, of course. The prefactors are directly related to the viscous dissipation ϵ_u and to the thermal dissipation ϵ_Θ : For $rp \ll 1$, we get from (2.8) and (2.11)

$$D_{uu}(r) = \frac{1}{3} \frac{\epsilon_u}{\nu} r^2 \quad (2.12a)$$

and

$$D_{\Theta\Theta}(r) = \frac{1}{3} \frac{\epsilon_\Theta}{\kappa} r^2. \quad (2.12b)$$

III. DRIVING TURBULENCE BY PLUMES: STIRRING SUBSTITUTES BOUNDARIES

A. Definition of the plume forcing

The volume forcing terms $\mathbf{f}_u$ and f_Θ in (2.3) are chosen to be

$$\mathbf{f}_u(\mathbf{p}_n^{(l)}, t) = \bar{e}_u^{(l)} \frac{\mathbf{u}(\mathbf{p}_n^{(l)}, t)}{\sum_{m=2}^7 |\mathbf{u}(\mathbf{p}_m^{(l)}, t)|^2} \quad (3.1a)$$

and analogously

$$f_\Theta(\mathbf{p}_n^{(l)}, t) = \bar{e}_\Theta^{(l)} \frac{\Theta(\mathbf{p}_n^{(l)}, t)}{\sum_{m=2}^7 |\Theta(\mathbf{p}_m^{(l)}, t)|^2} \quad (3.1b)$$

for $n = 2, 3, \dots, 7$, and vanishing forces for all other n . The choice of the selected modes in (3.1) is somewhat arbitrary, but this does not influence our results, because there is strong mixing between all wave-vector modes. The constants $\bar{e}_u^{(l)}$ and $\bar{e}_\Theta^{(l)}$ are input parameters whose choice completes our modeling of the boundary conditions by means of the volume forces $\mathbf{f}_u$ and f_Θ . In some sense this resembles the renormalization-group procedure [36, 37], where part of the physics also appears in the forcing terms. But note that our forcing (3.1) is purely deterministic. The simplest possibility would be $\bar{e}_\Theta^{(l)} = \delta_{l,0} \bar{e}_\Theta^{(0)}$ and $\bar{e}_u^{(l)} = 0$, meaning that there is thermal input on the largest scale only and no kinetic input at all [8]. The effect of the plumes [11], which statistically detach from the bottom and top boundary layers, can be dealt with by taking $\bar{e}_\Theta^{(l)}$ and possibly (if one attributes also a kinetic stirring effect to the plumes) also $\bar{e}_u^{(l)}$ nonzero on all scales $1/p^{(l)}$ that are actually excited by the plumes. With the definition (3.1) of the forcing, the thermal and the kinetic inputs $e_\Theta^{(l)}$ and $e_u^{(l)}$ of Eqs. (2.7c) and (2.7b)

are, of course, just the control parameters $\bar{e}_\Theta^{(l)}$ and $\bar{e}_u^{(l)}$ on each level l .

Let us also define the summed kinetic and thermal input

$$e_u^{(\text{sum}, l)} \equiv \sum_{l'=0}^l e_u^{(l')}, \quad (3.2a)$$

$$e_\Theta^{(\text{sum}, l)} \equiv \sum_{l'=0}^l e_\Theta^{(l')}, \quad (3.2b)$$

representing the total input on all levels ranging from the outer scale ($l = 0$) down to the scale corresponding to level l . As explained in Sec. I we interpret the plume picture of [11] such that the smallest scale on which the plumes feed in energy is of the order of the boundary layer thickness λ_{BL} . Although the height of the plumes is much larger and in fact defines the mixing layer thickness [2], we consider their smallest scale to be their stem, which is of the order of the boundary-layer thickness. Consequently, we have to force all levels with $l \leq l_{\text{BL}}$, where

$$l_{\text{BL}} = \log_2(L_{\text{expt}}/\lambda_{\text{BL}}). \quad (3.3)$$

But what exactly is the boundary-layer thickness? We adopt the definition of a mean λ_{BL} by making use of the heat-transfer properties. Within the two laminar boundary layers the heat flux H is due to diffusive heat transport, $H = \kappa \Delta_{\text{expt}}/(2\lambda_{\text{BL}})$. The boundary layers are the restricting elements for the total heat flux because the turbulent bulk does not contribute a substantial heat resistance. The experimental finding [1, 2] that the mean temperature profile $\bar{\Theta}(z)$ decreases nearly completely within the boundary layers and stays constant in the bulk verifies this statement. We used that already and omitted $\bar{\Theta}$ in (2.1). So the total heat transfer in terms of the molecular heat transfer, the Nusselt number Nu , is

$$Nu \equiv H L_{\text{expt}}/(\kappa \Delta_{\text{expt}}) = L_{\text{expt}}/(2\lambda_{\text{BL}}). \quad (3.4a)$$

We take this equation to *define* the mean boundary-layer thickness λ_{BL} . Nu is known experimentally [1, 2, 5, 22] to scale with the Rayleigh number Ra ,

$$Nu = 0.165 Ra^{2/7}. \quad (3.4b)$$

The prefactor depends on the aspect ratio Γ of the cell. Here and also for all further constants in Ra power laws we take the values for the prefactors which hold for an aspect ratio of $\Gamma = 1/2$. We found them in Ref. [22], which contains the completest collection of data.

From (3.4) and the definition (3.3) of l_{BL} we get

$$l_{\text{BL}} = -1.60 + \frac{2}{7} \log_2 Ra. \quad (3.5)$$

In the hard-turbulence regime ($Ra = 10^8 - 10^{11}$) we find $l_{\text{BL}} = 6$ to 9. The stirring subrange (SSR) from $l = 0$ to $l = l_{\text{BL}}$ thus is fairly extended in Rayleigh-Bénard turbulence due to the driving effect of the plumes.

B. Purely thermal plume forcing

We now examine the flow which is maintained by purely *thermal* forcing, i.e., no kinetic forcing, $\bar{e}_u^{(l)} = 0$. The strength of the thermal input depends on the “activity” of the boundary layers, namely, on the number and size of the detaching plumes, and is therefore not known *a priori*. We assume an exponential increase of $\bar{e}_\Theta^{(\text{sum}, l)}$ with decreasing scale, $\bar{e}_\Theta^{(\text{sum}, l)} = \bar{e}_\Theta^{(0)} 2^{\xi l}$, and adjust the parameter ξ such as to get the experimental Θ spectrum, namely, $\langle |\Theta(p^{(l)})|^2 \rangle \propto (p^{(l)})^{\zeta_\Theta}$ with [4] $\zeta_\Theta = -2/5$ in the range $l = 0$ to $l = l_{\text{BL}}$, the SSR. From numerical trials we find $\xi = 0.4$, i.e.,

$$\bar{e}_\Theta^{(\text{sum}, l)} = \bar{e}_\Theta^{(0)} 2^{0.4l}, \quad l = 0, 1, \dots, l_{\text{BL}}. \quad (3.6)$$

For $l > l_{\text{BL}}$ we put $\bar{e}_\Theta^{(l)} = 0$. Larger values of ξ would lead to less decrease in the spectrum, i.e., to $|\zeta_\Theta| < 2/5$. The agreement of $\xi = 0.4 = -\zeta_\Theta$ is of course incidental.

The numerical calculations show that the quantity which is most essential for the $-2/5$ scaling is the total sum $\bar{e}_\Theta^{(\text{sum}, l_{\text{BL}})}$, whereas the specific *form* (3.6) (exponential or not) of the input between $l = 0$ and $l = l_{\text{BL}}$ turns out to be secondary. Even if besides on scale $l = 0$ one *only* stirs on the boundary-layer scale λ_{BL} but not in between, the spectra visually appear to scale approximately with $-2/5$ in the SSR.

C. Thermal as well as kinetic plume forcing

As in the case of the $\bar{e}_\Theta^{(l)}$ we do not *a priori* know how the strength parameters $\bar{e}_u^{(l)}$ of the kinetic plume forcing scale with the level l and how big they are. We find the following argument plausible: The kinetic plume input should scale in the same way as the thermal plume input does. Since for all levels (except the largest one) according to (3.2) and (3.6) it is

$$\bar{e}_\Theta^{(l)} = \bar{e}_\Theta^{(0)} (1 - 2^{-0.4}) 2^{0.4l}, \quad 1 \leq l \leq l_{\text{BL}}, \quad (3.7a)$$

we choose

$$\bar{e}_u^{(l)} = \bar{e}_u^{(0)} (1 - 2^{-0.4}) 2^{0.4l}, \quad 1 \leq l \leq l_{\text{BL}}, \quad (3.7b)$$

for the kinetic plume input, too. The thermal stirring $\bar{e}_\Theta^{(0)}$ on the level $l = 0$ deserves special care. It does not represent the plumes (at least not mainly) but, instead, the external temperature difference Δ_{expt} between the top and the bottom of the RB cell. There is no such effect for the velocity other than via buoyancy which we already take care of explicitly in (2.3a). We therefore do not feed kinetic plume energy into the $l = 0$ level, i.e., we choose $\mathbf{f}_u(\mathbf{p}_n^{(0)}, t) \equiv 0$. The quantity $\bar{e}_u^{(0)}$ in (3.7b) is a parameter whose meaning is *not* the input on level $l = 0$, but a measure for the stirring strength on the levels $l > 0$.

Purely on dimensional reasons we can write

$$\bar{e}_u^{(l)} = c \bar{e}_\Theta^{(l)} (\beta g)^2 \tau^2, \quad l \geq 1. \quad (3.8)$$

Here τ is the typical time scale of the largest eddies, which will be precisely defined in the next section, and c

is a free constant. The *strength* of the kinetic plume input should be of a size comparable to the thermal plume input. Thus we may choose $c = 1$.

Having specified now the forcing we can numerically solve the equations of the fluid motion (2.3). We find that the choice (3.7) to (3.8) of the thermal and kinetic plume strength parameters yields the same experimentally observed $-2/5$ -scaling of the temperature spectrum as the previous choice (3.6) of purely thermal forcing did. The *velocity* spectrum changes. The question, what kind of stirring is the more realistic one, has to be decided by measuring also the velocity spectrum. Further numerical results will extensively be discussed in Sec. V.

Whether or not one regards a kinetic plume forcing in addition to the thermal plume forcing, there are, in principle, three ranges in the spectra altogether, some of which show scaling. The largest eddies between the external scale and λ_{BL} constitute the stirring subrange SSR, which is dominated by the plume stirring. Next the inertial subrange (ISR) would come, in which there is neither stirring nor viscous damping; but this is not developed in the RB flow considered here because λ_{BL} is smaller than the viscous crossover length 10η , see Fig. 8 in Sec. VII and the extended discussion in Ref. [6]. The final subrange is the VSR, in which viscosity pulls out the energy.

IV. THE PARAMETERS

The numerical solution of the ODE approximation (2.3) to the Boussinesq equations must be performed with dimensionless numbers. So we have to choose proper units in which we measure length, temperature, and time, denoted as L , Δ , and τ , respectively, where $2\pi L$ is the periodicity length. This, by the way, does not equal the RB cell height L_{expt} .

In the dimensionless form of the equations of motions (2.3), we choose the dimensionless coupling constant and the dimensionless thermal input on level $l = 0$ to be 1,

$$\tilde{\beta}g = \beta g \tau^2 \Delta / L = 1 \quad (4.1a)$$

and

$$\tilde{e}_{\Theta}^{(0)} = \tilde{e}_{\Theta}^{(0)} \tau / \Delta^2 = 1. \quad (4.1b)$$

(Henceforth, all dimensionless quantities will be labeled by a tilde.) From (3.8) and the choice $c = 1$ it follows that also $\tilde{e}_u^{(0)} = 1$. In addition the dimensionless equations of motion (2.3) contain the dimensionless conductivity $\tilde{\kappa} = \kappa \tau / L^2$ and the dimensionless viscosity $\tilde{\nu} = \nu \tau / L^2$, which is connected with $\tilde{\kappa}$ via the Prandtl number $Pr = \tilde{\nu} / \tilde{\kappa}$.

We now adjust $\tilde{\kappa}$ in such a way, that we get a scaling range of the same extension as in experiment [4, 5, 22]. This can best be determined from Fig. 4 of Ref. [5]. There the temperature frequency spectra are fitted by the function $\omega^{-\zeta} \exp(-\omega/\omega_d)$, where ω_d characterizes the frequency beyond which dissipation becomes important. Then the spectra are multiplied by $\exp(\omega/\omega_d)$. Thus—as VSR behavior is eliminated in this way—the scaling behavior can be identified more clearly in these plots. For

$Ra = \beta g L_{\text{expt}}^3 \Delta_{\text{expt}} / (\nu \kappa) = 7.3 \times 10^{10}$, a Rayleigh number in the hard-turbulence regime, the scaling range in this modified spectrum is 2.4 decades = 8.0 dyads. To achieve these 2.4 decades in our calculation, we have to choose

$$\tilde{\kappa} = \kappa \tau / L^2 = 1.5 \times 10^{-5}, \quad (4.1c)$$

compare also Sec. V B. The dimensionless viscosity $\tilde{\nu}$ follows from the corresponding [22] Prandtl number, $Pr = \nu / \kappa = \tilde{\nu} / \tilde{\kappa} = 0.656$. It turns out that 12 levels ($N = 11$) are sufficient for calculations with these parameters, because the energy of additional levels is practically zero. Already the last level $l = 11$ has only less than 1/100 of the energy of level $l = 10$. Since the amplitudes on sufficiently small scales can be considered to be zero, the changes of the fields on very small distance are smooth. The available range is $2^0 - 2^{11}$ corresponding to more than 3 decades.

With these settings the numerical calculations can be done. But how to relate the results to the experimental data? The model parameter $\tilde{e}_{\Theta}^{(0)}$ cannot be measured directly, since it is a theoretical substitute of the experimental boundary condition.

To fix the missing value of $\tilde{e}_{\Theta}^{(0)}$, we use another experimental information, namely, the strength of the typical temperature fluctuations in the bulk defined by $\Theta_c = \langle [\Theta(\mathbf{x})]^2 \rangle^{1/2}$, which is experimentally known [2, 22] to be

$$\Theta_{c,\text{expt}} = 0.46 \Delta_{\text{expt}} Ra^{-1/7}. \quad (4.2)$$

The prefactor again refers to an aspect ratio $\Gamma = 0.5$. We first use this experimental information to fix our temperature unit Δ , namely, by identifying the theoretical rms temperature fluctuation $\tilde{\Theta}_c \Delta$ with the experimentally measured $\Theta_{c,\text{expt}}$,

$$\Theta_{c,\text{expt}} = \tilde{\Theta}_c \Delta. \quad (4.1d)$$

From the four equations (4.1) we then can determine our units Δ , τ , and L , and the control parameter $\tilde{e}_{\Theta}^{(0)}$,

$$\Delta = \Theta_{c,\text{expt}} / \tilde{\Theta}_c, \quad (4.3a)$$

$$\tau = \left(\frac{\kappa}{\tilde{\kappa}} \right)^{1/3} (\beta g)^{-2/3} \Delta^{-2/3}, \quad (4.3b)$$

$$L = \left(\frac{\kappa}{\tilde{\kappa}} \right)^{2/3} (\beta g)^{-1/3} \Delta^{-1/3}, \quad (4.3c)$$

$$\tilde{e}_{\Theta}^{(0)} = \Delta^2 \tau^{-1}. \quad (4.3d)$$

From (3.8) $\tilde{e}_u^{(0)}$ follows to be

$$\tilde{e}_u^{(0)} = c \Delta^2 (\beta g)^2 \tau. \quad (4.4)$$

Taking now the values for the temperature drop Δ_{expt} , for κ , and for β , which are given for the Rayleigh number $Ra = 7.3 \times 10^{10}$ and the Prandtl number $Pr = 0.656$ in Ref. [22] together with the numerical result $\tilde{\Theta}_c = 6.1$, we determine Δ , τ , L , and $\tilde{e}_{\Theta}^{(0)}$, see Table I.

In Secs. V and VI we will relate the units Δ , τ , and L with the experimental temperatures, times, and lengths.

To obtain Ra power laws (see Sec. VII) from our numerical solutions of (2.3) we have to repeat the procedure of determining Δ , τ , L , and $\bar{\epsilon}_\Theta^{(0)}$ for further Ra numbers.

The quantity $\tilde{\kappa}$ is always based on the experimental extension of the scaling range of the modified temperature frequency spectra [5]. For a comparison of the *magnitude* of our numerical results with experiment the uncertainty in determining $\tilde{\kappa}$ has to be considered. This is done in

TABLE I. The parameters of some experimentally [5, 22] realized states. Ra , Pr , and ν , κ (in cm^2/s), β (in K^{-1}), and Δ_{expt} (in K) are taken from Table V of Ref. [22]. The smallest scale λ_{BL} which is plume fed is chosen according to (3.4) and the corresponding level l_{BL} from (3.5). The length of the scaling range of the temperature frequency spectra [multiplied with $\exp(\omega/\omega_d)$, see Sec. IV] is identified from Fig. 4 of Ref. [5]; it is given in decades as well as in dyads. $\tilde{\kappa}$ is chosen such that the numerical solution shows the same scaling range. The tildes always denote dimensionless quantities. $N+1$ is the number of levels we choose in our numerical calculation. Our temperature unit Δ (in mK), the time unit τ (in s), the length unit L (in cm), and the forcing strength $\bar{\epsilon}_\Theta^{(0)}$ (in $\text{K}^2 \text{s}^{-1}$) of the thermal plume input follow from (4.3); kinetic forcing is excluded. All further quantities are measured in multiples of Δ , τ , and L , respectively. $\tilde{\Delta}_{\text{expt}}$ and $\tilde{L}_{\text{expt}}$ follow from Δ_{expt} and $L_{\text{expt}} = 40$ cm. The solutions of (2.3) give $\tilde{\Theta}_c$, $\tilde{u}_c$, and the total dissipation rate $\tilde{\epsilon}_u$ is defined by (2.9a). These quantities only change moderately when $\tilde{\nu}$ and $\tilde{\kappa}$ are varied, i.e., they characterize predominantly the SSR, whereas the rms time derivatives $\partial_t \tilde{\Theta}_c$ and $\partial_t \tilde{u}_c$, of course, depend on $\tilde{\nu}$ and $\tilde{\kappa}$. Furthermore, we determine the Taylor lengths $\tilde{\lambda}_T^u$, $\tilde{\lambda}_T^\Theta$ and the Taylor-Reynolds numbers Re_λ^u , Re_λ^Θ , see Sec. VII A. The number of levels representing the SSR in the temperature wave-number spectrum is l_d . The number of data points we used when fitting with (5.1) equals the number of dyads which show scaling in the experiment, see row 10 in this table, but l_d hardly depends on that number. $l_{10\eta} = \log_2(L_{\text{expt}}/10\eta)$ is defined and discussed in Sec. VII B.

Ra	7.30×10^{10}	1.10×10^8	6.01×10^8	4.04×10^9	2.50×10^{10}
Pr	0.656	0.638	0.639	0.642	0.657
κ ($\text{cm}^2 \text{s}^{-1}$)	0.0137	0.3720	0.1230	0.0441	0.0126
ν ($\text{cm}^2 \text{s}^{-1}$)	0.0090	0.2373	0.0786	0.0283	0.0083
β (K^{-1})	0.226	0.215	0.223	0.227	0.242
Δ_{expt} (K)	0.628	0.732	0.413	0.353	0.172
λ_{BL} (cm)	0.095	0.611	0.376	0.218	0.130
l_{BL}	8.7	6.0	6.7	7.5	8.3
Decades	2.4	1.8	1.9	2.1	2.3
Dyads	8.0	5.9	6.4	7.1	7.5
$\tilde{\kappa}$	1.5×10^{-5}	30×10^{-5}	15×10^{-5}	6×10^{-5}	2.4×10^{-5}
$\tilde{\nu}$	0.984×10^{-5}	19.1×10^{-5}	9.59×10^{-5}	3.85×10^{-5}	1.58×10^{-5}
$N+1$	12	9	10	11	11
Δ (mK)	1.33	4.88	2.04	1.21	0.44
τ (s)	21.9	10.5	16.1	21.6	36.5
L (cm)	141	114	115	126	138
$\bar{\epsilon}_\Theta^{(0)}$ ($\text{K}^2 \text{s}^{-1}$)	8.1×10^{-8}	2.3×10^{-6}	2.6×10^{-7}	6.7×10^{-8}	5.3×10^{-9}
$\tilde{\Delta}_{\text{expt}}$	472	150	203	292	392
$\tilde{L}_{\text{expt}}$	0.283	0.350	0.349	0.317	0.289
$\tilde{\Theta}_c$	6.1	4.9	5.2	5.7	5.9
$\tilde{u}_c$	10.8	9.3	10.0	10.4	10.6
$\tilde{\epsilon}_u$	5.0	4.3	4.4	4.7	4.8
$\partial_t \tilde{\Theta}_c$	235	60	80	130	184
$\partial_t \tilde{u}_c$	238	83	107	150	197
$\tilde{\lambda}_T^u$	0.034	0.14	0.10	0.067	0.043
$\tilde{\lambda}_T^\Theta$	0.015	0.073	0.051	0.032	0.019
Re_λ^u	21 000	3 900	6 300	10 000	17 000
Re_λ^Θ	9 500	2 100	3 100	5 000	7 500
l_d	7.6	5.1	5.5	6.3	7.1
$l_{10\eta}$	7.9	5.0	5.7	6.6	7.4

TABLE II. Same as Table I but with kinetic plume forcing in addition to thermal stirring by plumes. Its strength $\bar{e}_u^{(l)}$ is determined by (3.7b) and (3.8) with $c = 1$. All other quantities of Table I are hardly influenced by the additional kinetic plume forcing and are therefore not repeated, but Table II is meant to describe the RB flow with combined thermal and kinetic plume stirring as complete as Table I does for solely thermal stirring.

Ra	7.30×10^{10}	1.10×10^8	6.01×10^8	4.04×10^9	2.50×10^{10}
$\bar{e}_u$	15.5	8.2	10.4	12.4	14.0
$\partial_t \bar{\Theta}_c$	282	65	93	149	213
$\partial_t \bar{u}_c$	365	101	141	206	283
$\tilde{\lambda}_T^u$	0.019	0.100	0.068	0.041	0.025
Re_λ^u	12 000	2 800	4 100	6 400	9 800

Sec. VII. Our units and some other (calculated) quantities for those five turbulent states, whose spectra can be found in Fig. 4 of Ref. [5], are presented in Tables I and II, together with the experimental control parameters taken from [22].

V. SCALE DEPENDENCES IN RB FLOW

In this section we shall report our results from the numerical solutions of the fluid dynamical equations (2.3) for the turbulent state with $Pr = 0.656$ and with $Ra = 7.3 \times 10^{10}$. As stated above, we choose $\tilde{\kappa} = 1.5 \times 10^{-5}$ and $\tilde{\nu} = 0.656\tilde{\kappa}$. From (3.5) the smallest scale that is forced is $l_{BL} = 9$.

A. Spectra

1. Purely thermal plume forcing

In this subsection we assume that the plumes cause no kinetic stirring, $\mathbf{f}_u = 0$. In Fig. 1(a) all calculated spectra are presented. The $-2/5$ scaling of the thermal spectrum $\langle |\Theta(p^{(l)})|^2 \rangle$ in the SSR holds by choice of our basic input parameter condition (3.6). We controlled the energy and thermal intensity budgets; Fig. 2(a) shows the scale dependence of the thermal input $\bar{e}_\Theta^{(l)}$ according to (3.6) and also displays how the different constituents (2.6) – (2.8) of the balance equations (2.5) depend on the scale $p^{(0)}/p^{(l)}$. Although the thermal input (3.6) is quite different from the stirring on only the largest scale (which we used in Ref. [8]), the individual contributions

to the kinetic-energy balance Eq. (2.5a) hardly change. The spectral power of the velocity is still governed by the buoyant energy input $e_g^{(0)}$ on the outer scale. $e_g^{(l)}$ as a function of l behaves like a geometrically decaying series and does not contribute significantly for $l > 2$. So the kinetic-energy transfer $T_u^{l \rightarrow l+1}$ essentially becomes constant with l [see Fig. 2(a)] resulting in the classical $-2/3$ -scaling of $\langle |\mathbf{u}(p^{(l)})|^2 \rangle$, as observed in Fig. 1(a).

Our arguments have the consequence that any reasonable forcing of the heat equation *alone*, i.e., without an explicit velocity stirring $\mathbf{f}_u$, will result in $-2/3$ scaling of $\langle |\mathbf{u}(p^{(l)})|^2 \rangle$. This classical scaling exponent seems to be embodied in the dynamics by the structure of the Navier-Stokes equation.

Of course, this will become different, if the effects of ascending and descending plumes from the boundary layers will be present directly by the stirring force $\mathbf{f}_u$ in Eq. (2.3a) in addition to the thermal force $\mathbf{f}_\Theta$ in (2.3b). The resulting changes will be discussed in the next subsection.

The cross spectrum of velocity and temperature, i.e., the stirring power $e_g^{(l)} = \beta g \text{Re} \langle u_3^*(p^{(l)}) \Theta(p^{(l)}) \rangle$ turns out to be definitely steeper than one would naively guess from the scaling of $\beta g \sqrt{\langle |u_3(p^{(l)})|^2 \rangle \langle |\Theta(p^{(l)})|^2 \rangle}$, which in the SSR is $(p^{(l)})^{-1/3-1/5} = (p^{(l)})^{-8/15}$, see Fig. 1(a). If we fit a power law $e_g^{(l)} \propto (p^{(l)})^{\zeta_{u_3\Theta}}$ to our data, we get $\zeta_{u_3\Theta} \approx -1.14 \pm 0.05$. This strong deviation from $-8/15$ is similar to what we found in [8] for the case of thermal input into

FIG. 1. Calculated wave-number spectra $\langle |\mathbf{u}(p^{(l)})|^2 \rangle$ ($\bullet$), $\langle |\Theta(p^{(l)})|^2 \rangle$ ($\diamond$), and $e_g^{(l)} = \beta g \text{Re} [\langle u_3^*(p^{(l)}) \Theta(p^{(l)}) \rangle]$ ($\circ$) vs inverse scale $\log_2(p^{(l)}/p^{(0)}) = \text{level } l$. For comparison, $\beta g \sqrt{\langle |u_3(p^{(l)})|^2 \rangle \langle |\Theta(p^{(l)})|^2 \rangle}$ is also shown ($\ast$). $\tilde{\kappa} = 1.5 \times 10^{-5}$, $\tilde{\nu} = \tilde{\kappa} Pr = 0.984 \times 10^{-5}$, $\bar{e}_\Theta^{(0)} = 1$, $\beta g = 1$. The data are obtained by numerical integration of Eqs. (2.3) with the Bulirsch-Stoer method [40]. The time average extends over about 500 time units τ . In (a) there is thermal forcing only and the $\bar{e}_\Theta^{(l)}$ are chosen according to (3.6) with $l_{BL} = 9$, whereas in (b) the plumes also force kinetically according to (3.7b) and (3.8) with $c = 1$.

FIG. 2. Constituents of the time-averaged kinetic energy [(a₁) and (b₁)] and thermal spectral intensity [(a₂) and (b₂)] balance Eqs. (2.5) vs inverse scale $\log_2(p^{(l)}/p^{(0)}) = \text{level } l$ according to the numerical solution of the Boussinesq equations. The shaded areas between the curves are equal, proving stationarity and perfect energy balances. (a) refers to purely thermal plume forcing, whereas in (b) we consider thermal *and* kinetic plume forcing.

the largest scale only, namely, -1.50 ± 0.05 instead of the corresponding naive guess $-2/3$. As expected the effect of gravity $\mathbf{g}$ is very small on small scales; the large values of $|\zeta_{u_3\theta}|$ mean a *very fast* approach to isotropy with decreasing eddy scale. For small scales the correlation between $u_3(p^{(l)})$ and $\Theta(p^{(l)})$ numerically is nearly zero, less than $1/200$ of $\beta g \sqrt{\langle |u_3(p^{(l)})|^2 \rangle \langle |\Theta(p^{(l)})|^2 \rangle}$, although numerically we find a tiny finite anisotropy remaining.

Note that a positive $u_3\Theta$ correlation implies negative $u_1\Theta$ and $u_2\Theta$ correlations because of the incompressibility condition (2.3c). We find from the numerical solution $\langle u_{1,2}^*(p^{(l)})\Theta(p^{(l)}) \rangle \approx -0.1 \langle u_3^*(p^{(l)})\Theta(p^{(l)}) \rangle$ on time average and wave-vector average on each level.

2. Thermal and kinetic plume forcing

Now we assume that in addition there is the kinetic stirring effect by plumes according to (3.7b) and (3.8) with $c = 1$. Again we have a well-defined set of ODES, now stirred thermally and kinetically, which we solved numerically. The resulting spectra are displayed in Fig. 1(b). According to (3.7b) the kinetic stirring effect is supposed to increase with decreasing scale, i.e., with increasing l . The buoyant forcing $e_g^{(l)}$, instead, steeply decreases with scale as we already saw in Sec. V A 1. The value of c determines, on which scale the kinetic plume input $e_u^{(l)}$ (2.7b) surmounts the buoyant coupling input $e_g^{(l)}$ (2.7a). With the above choice of $c = 1$ we find that $e_u^{(l)}$ becomes larger than $e_g^{(l)}$ for $l > 2$, i.e., the crossover between buoyant and plume driving happens an order of magnitude below the outer scale, see Fig. 2(b₁). The decay of the velocity spectrum is now slower than before. In principle, the SSR splits into two subranges: The very large scales are dominated by the buoyant input $e_g^{(l)}$ and

approximately show a $-2/3$ power law in the velocity spectrum as before (SSR1). But this stirring subrange is now followed by another stirring subrange dominated by the kinetic plume input $e_u^{(l)}$ which enforces a slower decay of the velocity spectrum (SSR2) due to an ever increasing forcing. Then comes the ISR (which again is not developed) and finally the VSR. The choice $c = 0$ in Sec. V A 1 means that the buoyant input (2.7a) *always* dominates the kinetic plume input (2.7b) and therefore the SSR2 does not develop. For increasing value of c , SSR2 becomes more and more visible. In Fig. 1(b), where $c = 1$, the SSR1 is practically suppressed already. It would be of great interest if the crossover from SSR1 to SSR2 could be observed experimentally in the velocity spectrum. From such measurements it should be possible to determine the model parameters $\bar{e}_u^{(l)}$ for the kinetic plume input strength.

To summarize, the essential difference between the two plume stirring mechanisms (with and without additional kinetic input ascribed to the plumes) lies in the scaling of the velocity spectrum. To decide experimentally which of the alternatives of Secs. V A 1 and V A 2 (or which value of c) better corresponds to reality, it would be very helpful to have measurements of the velocity fluctuations in addition to the bolometer measurements of the temperature fluctuations [1–5, 11, 22].

B. Exponential decay in the VSR

For smaller scales the spectra are increasingly influenced by viscous damping and thermal dissipation, see Fig. 1. To identify more clearly the pure scaling behavior in the whole range one may take explicit care of the contribution of viscosity. In Ref. [20] an exponential decay for large momenta was predicted by analysis of eigen-

function systems of the Navier-Stokes equation. So we fit our spectra with the three parameter function

$$E(p) = \text{const} \times \left(\frac{p}{p^{(0)}} \right)^{-\zeta} \exp \left(-\frac{p}{p_d} \right). \quad (5.1)$$

For wave numbers larger than p_d the dissipation becomes important. The quantity $l_d = \log_2(p_d/p^{(0)})$ represents the length of the SSR (in dyads). The l_d for various Ra are given in Table I. They nearly coincide with the number of dyads over which the plots of the experimental spectra show scaling. The power-law exponents ζ slightly differ from those which we obtained from a fit to the original data of Fig. 1. To demonstrate the quality of the scaling behavior after separating the influence of viscosity, we plot $E(p)\exp(p/p_d)$ against the wave number, see Fig. 3, which corresponds to Fig. 4 of Ref. [5]. It is the scaling range in these plots to which we have adopted the numerical value of $\tilde{\kappa}$, see Sec. IV. Equation (5.1) describes our numerical data remarkably well. So the exponential decay [20] also results from our Fourier-Weierstrass analysis of the dynamical equations.

Note that our numerical data points in the modified temperature spectra for the level $l = 9$ are definitely larger than expected from pure $-2/5$ scaling. This is due to the strong thermal plume intensity input on the level $l = 9$, Eq. (3.7), which represents the smallest scale input of the plumes, namely, of its stems, and of the swirls. This length scale is well inside the dissipation range, because $\lambda_{BL} < 10\eta$, the viscous crossover length, see below. Interestingly enough a similar increase of the (modified) temperature frequency power for the smallest scales is also observed in the experimental data [5]. The authors [5] ascribe this to electronic noise, but we will not exclude that it has the same origin as the increase in our numerical data, i.e., the strong thermal intensity input of the plumes at small scales—particularly, since this experimental phenomenon is Ra dependent (see Figs. 4 and 5 of [5]) like the width of the boundary layer and thus the smallest plume input scale.

FIG. 3. Calculated wave-number spectra multiplied by $\exp(p/p_d)$ vs inverse scale $\log_2(p^{(l)}/p^{(0)}) = \text{level } l$. The moments $p_{d,uu}$, $p_{d,\theta\theta}$, and $p_{d,u_3\theta}$ are gained by a fit of (5.1) to the first eight dyads of the spectra of Fig. 1 as in experiment [5]. In (a) $f_u = 0$, whereas in (b) $f_u \neq 0$, while $f_\theta \neq 0$ in both cases.

FIG. 4. The structure functions corresponding to the wave-number spectra of Fig. 1 vs eddy size r . (a) Purely thermal plume forcing. (b) Thermal and kinetic plume forcing.

C. Structure functions

The scaling in $\mathbf{p}$ space has its correspondence in the scaling of the $\mathbf{r}$ -space structure functions $D_{uu}(r)$, $D_{\theta\theta}(r)$, and $D_{u_3\theta}(r)$. They can be calculated by expressions like Eq. (2.11) and are displayed in Figs. 4(a) and 4(b). In the VSR all structure functions vary $\propto r^2$. The relations (2.12) are numerically well fulfilled. The change from SSR behavior to VSR behavior occurs at a length scale which well coincides with $(p_d)^{-1}$ gained by the fit (5.1) to the wave-number spectra.

In the case of purely thermal plume forcing the velocity structure function scales like $\propto r^{2/3}$ in the SSR, see Fig. 4(a). This is the same power law as for the velocity structure function in homogeneous, isotropic Navier-Stokes turbulence. In that case a characteristic quantity for the dynamics is the constant b in the ISR structure function [33, 38]

$$D_{uu}(r) = b(\epsilon_u r)^{2/3}. \quad (5.2)$$

Because of the same $2/3$ scaling of $D_{uu}(r)$ we will apply this formula also for RB turbulence if there is *no kinetic* plume forcing. Assuming this $2/3$ scaling throughout one may apply Poisson's sum formula for the resulting geometric series in (2.11) and obtains [7]

$$D_{uu}(r) = \frac{27}{10} \frac{\Gamma(4/3)}{\ln(2)} \left(\sum_{\mathbf{p} \in K_0} |\mathbf{p}|^{2/3} \langle |\mathbf{u}(\mathbf{p})|^2 \rangle \right) r^{2/3}. \quad (5.3)$$

The expression in the large parentheses and ϵ_u can be calculated within our numerical solution. Comparing (5.2) and (5.3) then gives $b = 130$.

Presumably in RB turbulence (with negligible kinetic plume forcing) b will be of about the same magnitude as for Navier-Stokes turbulence, where it is $b = 8.4$ [26]. Thus our numerical value is still far away. This indicates that one needs more wave numbers in the Fourier-Weierstrass mode sum. Taking more $\mathbf{p}$'s, b appreciably

decreases [7]. The value $b = 130$ that we find here in the simulation of RB turbulence is slightly better than $b = 170$ achieved in [7] with 38 wave vectors per level for pure Navier-Stokes turbulence. We attribute this to the additional modes contributed by the temperature field which increase the mixing and energy transporting power of the complete dynamical system similarly as additional wave vectors do.

D. Time scales

It might seem that the Boussinesq equations (2.3) together with the multilevel input (3.6) due to the plumes give rise to two different internal time scales. One of them is defined by the velocity spectrum, the other one by the temperature spectrum:

$$\tau_u^{(l)} \equiv \frac{\langle |\mathbf{u}^{(l)}|^2 \rangle}{T_u^{l-l+1}}, \quad (5.4a)$$

$$\tau_\Theta^{(l)} \equiv \frac{\langle |\Theta^{(l)}|^2 \rangle}{T_\Theta^{l-l+1}}. \quad (5.4b)$$

These times can be calculated from our numerical solution of (2.3). We find $\tau_u^{(0)} \approx 15\tau$ and $\tau_\Theta^{(0)} \approx 10\tau$. Also for all smaller scales within the SSR, these two time scales differ by the same factor of about 1.5 since they both scale with $p^{(l)}$ as $\tau_u^{(l)} \propto \tau_\Theta^{(l)} \propto (p^{(l)})^{-0.80}$, see Fig. 5. So, in fact, Eqs. (5.4a) and (5.4b) seem to define SSR time scales analogously as in the VSR L_{expt}^2/ν and L_{expt}^2/κ do. Their ratio $\tau_\Theta^{(l)}/\tau_u^{(l)}$ can be considered as turbulent Prandtl number. We have $Pr_{\text{turb}} \approx 0.67$. The experimental values for Pr_{turb} scatter appreciably around 0.36 [26], and in a mean field theory [39] we found 0.40. We do not know, whether the agreement of Pr_{turb} and Pr in our present calculation is incidental or deeper. In the case of additional kinetic plume forcing the agreement in the power laws of $\tau_u^{(l)}$ and $\tau_\Theta^{(l)}$ is not only due to the buoyant coupling of the dynamical *bulk* equations (2.3a) and (2.3b), but also to the same scaling behavior (3.7a) and (3.7b) of the thermal and the kinetic plume input expected to hold from the *boundary-layer* dynamics.

FIG. 5. Time scales $\tau_u^{(l)}$ (●) and $\tau_\Theta^{(l)}$ (◊) as defined by (5.4) without (a) and with (b) additional velocity stirring by plumes.

The exponent -0.80 in the $p^{(l)}$ power law for the eddy time scale is exactly what one would expect from the power-law exponent $-2/5$ of the temperature spectrum and, remembering $T_\Theta^{l-l+1} = e_\Theta^{(\text{sum}, l)}$, from the exponent -0.4 of the summed thermal input (3.6), namely $-2/5 - 0.4 = -0.80$.

The surprisingly large factor of ≈ 15 (10) between the time scale $\tau_u^{(0)}$ ($\tau_\Theta^{(0)}$) and the time unit τ (which is in the range of the large eddy turnover time, see Sec. VI) is attributed (as the too large b , see Sec. VC) to the wave-number truncation; the factor decreases for a larger wave-number set K .

VI. TIME DEPENDENCE OF RB FLOW AND FREQUENCY SPECTRA

Up to now only wave-number spectra (cf. Fig. 1) were discussed. When comparing with experiment, we tacitly assumed that these spectra show the same scaling behavior as functions of $p^{(l)}$ as the measured power spectrum $P(\omega)$ as functions of ω . The aim of this section is to validate this assumption by direct numerical calculation.

To evaluate the temperature frequency spectrum $P(\omega)$ numerically, we proceed as the experimentalists do. We produce a time series $\Theta(\mathbf{x}, t_\lambda)$ for fixed position $\mathbf{x}$ and equidistant t_λ , fast Fourier transform and square it. We in addition can resolve $\Theta(\mathbf{x}, t_\lambda)$ for different scales l by taking [in Eq. (2.2)] sums over $\mathbf{p}_n^{(l)}$ for fixed l only. As an additional kinetic plume forcing hardly influences the

FIG. 6. Typical time series. Temperatures in units of Δ , times in units of τ . (a) $\Theta(\mathbf{x} = \mathbf{0}, t)$. (b) $\Theta^{(l)}(\mathbf{x} = \mathbf{0}, t)$ for $l=0, 4$, and 8 .

temperature time series, the results in this section are valid for both forcing mechanisms that we introduced in Secs. IIIB and IIIC.

Typical time series $\Theta(\mathbf{x} = 0, t)$ and $\Theta^{(l)}(\mathbf{x} = 0, t)$ for $l = 0, 4, 8$ are shown in Fig. 6. We confirm from Fig. 6(b) that the typical large eddy turnover time is in the range of our time unit τ .

Unfortunately, the adaptive stepsize routine controlling the Bulirsch-Stoer integration algorithm [40] for ODES does not produce equidistant t_λ . We cure this by spline interpolation and obtain $4^7 = 16384$ equidistant data points from the same number of nonequidistant Bulirsch-Stoer points. Raw and splined signals cannot be optically distinguished. The corresponding total time interval is about 150 time units τ , i.e., about 150 large eddy turnover times and thus considerably less than in the experiments [4, 22].

The Nyquist frequency f_c is $f_c = 1/(2\Delta t) = 16384/(2 \times 150\tau) = 55\tau^{-1}$, which is definitely larger than all l -level frequencies $f^{(l)} = (\tau^{(l)})^{-1} = 2^{0.8l}(\tau^{(0)})^{-1}$, as confirmed by Fig. 5(a). We therefore need not bother aliasing [40]. For logarithmic plots of the frequency spectrum we average the discrete fast Fourier-transformed data points over an exponentially growing range of discrete frequencies. So the statistical errors go down exponentially.

The resulting frequency spectra are displayed in Fig. 7. The partial spectra $P^{(l)}(\omega) = \langle |\Theta^{(l)}(\omega)|^2 \rangle$ dominate the total spectrum $P(\omega)$ within a frequency range between $f^{(l)}/2$ and $2f^{(l)}$ with $f^{(l)} = 2^{0.8l}\tau^{-1}$. Although $P(\omega) = \langle |\sum_l \Theta^{(l)}(\omega)|^2 \rangle \neq \sum_l P^{(l)}(\omega)$, the superposition of the l -level spectra gives a sufficiently good estimate for the total spectrum, because the correlation $\text{Corr}(\Theta^{(l)}, \Theta^{(l')})(t) \equiv \int_{-\infty}^{\infty} \Theta^{(l)}(t+t')\Theta^{(l')}(t')dt'$ or its

Fourier transform $\Theta^{(l)}(\omega)\Theta^{(l')*}(\omega)$ are small for $l \neq l'$ in comparison to the $l = l'$ case.

The power law $\langle |\Theta(p)|^2 \rangle \propto p^{-2/5}$ in the SSR can be recognized in the total temperature frequency spectrum at fixed position in the bulk $P(\omega) \propto \omega^{-2/5}$, Fig. 7(b). This also means that the Taylor hypothesis [21], at least in our approximate solution of the fluid equations in Fourier-Weierstrass analysis, is well satisfied.

VII. Ra DEPENDENCES IN RB FLOW

Up to now we mainly discussed scale dependences. But as we solved (within the Fourier-Weierstrass approximation) the dynamical equations, namely, the Boussinesq equations with a properly restricted number of modes, we can even evaluate the magnitude of the quantities of interest in RB turbulence and compare them with experiment.

A. Taylor-Reynolds numbers

A quantity characterizing the degree of turbulence is the velocity Taylor-Reynolds number defined (see e.g. [33]) by $\text{Re}_\lambda^u = u_1 \lambda_T^u / \nu$ where $u_1 = \langle u_1^2(\mathbf{x}) \rangle^{1/2}$ is the rms velocity component and λ_T^u is the velocity Taylor microscale $\lambda_T^u \equiv \{ \langle u_1^2(\mathbf{x}) \rangle / \langle [\partial_1 u_1(\mathbf{x})]^2 \rangle \}^{1/2}$. The derivative in the denominator can be obtained from the well-known [33] relation $\langle [\partial_1 u_1(\mathbf{x})]^2 \rangle = \epsilon_u / (15\nu)$.

Analogously we define the temperature Taylor length by $\lambda_T^\Theta \equiv \{ \langle \Theta^2(\mathbf{x}) \rangle / \langle [\partial_1 \Theta(\mathbf{x})]^2 \rangle \}^{1/2}$. Here the denominator is $\langle [\partial_1 \Theta(\mathbf{x})]^2 \rangle = \epsilon_\Theta / (3\kappa)$. This gives rise to the corresponding temperature Taylor-Reynolds number $\text{Re}_\lambda^\Theta = u_1 \lambda_T^\Theta / \nu$. Of course, our choice of $\tilde{\kappa}$, i.e., the adaptance to the experimental scaling range, influences

FIG. 7. The temperature frequency spectra in the bulk (a) for single-level contributions $P^{(l)}(\omega)$ for $l = 0, 2, 4, 6, 8, 10$ and (b) for the sum over all scales, $P(\omega)$, including also $l = 1, 3, 5, 7, 9, 11$ [which were not drawn in (a) for the sake of optical clarity]. $-2/5$ scaling is seen as expected from the Taylor hypothesis.

the Taylor-Reynolds numbers. The $Re_\lambda^{u,\Theta}$ turn out (cf. Tables I and II) to be roughly of order 2000 - 20 000. This is quite large and again may have its origin partly in the truncation of the admitted wave numbers. Taking more wave vectors in K leads to smaller Re_λ ; for detailed discussions see Ref. [7].

B. Length scales

The dissipation length η of the turbulent state is defined as usual [33], $\eta \equiv (\nu^3/\epsilon_u)^{1/4}$. Figure 8 displays our findings of how η varies with Ra .

The SSR and ISR (if present) together range from approximately L_{expt} to 10η as one concludes from the analysis of the experimental structure functions and from mean-field theory, see [25, 38]. Therefore, the quantity $l_{10\eta} = \log_2(L_{\text{expt}}/10\eta)$ should give the number of levels which are necessary to represent the total length of SSR (and ISR). Indeed, the $l_{10\eta}$ we get very well agree with the number l_d of levels representing the SSR, see Sec. V B and Table I.

It is striking furthermore that the viscous crossover scale 10η obtained from our solution obeys nearly the same Ra power law as the thickness of the boundary layer λ_{BL} calculated according to (3.4); in fact, they are roughly even *absolutely* equal for all Ra . Therefore, all the three, l_d , $l_{10\eta}$, and l_{BL} approximately agree. Physically one can understand this by the following argument: From any point directly at the solid wall all other points within a radius of λ_{BL} appear viscously connected by the fluid as it also happens for any point within the bulk of turbulence within a radius 10η . Note an important implication which we already pointed out above: Since λ_{BL} nearly equals 10η , the stirring subrange SSR is imme-

FIG. 8. Ra dependences of various characteristic lengths. $\lambda_{\text{BL}}/L_{\text{expt}}$ from the Nu scaling law (3.4) (●) and $10\eta/L_{\text{expt}}$ from our solutions of (2.3) (without (○) and with (★) additional kinetic plume input) (units on left coordinate axis). Taylor lengths $\lambda_T^u/L_{\text{expt}}$ [without (◁) and with (□) additional kinetic plume input] and $\lambda_T^\Theta/L_{\text{expt}}$ (▷) defined in (7.1) (right coordinate axis).

diately followed by the viscous subrange and no proper ISR can develop—of course, provided that the plumes are really efficient down to the scale λ_{BL} .

For a more detailed discussion of the various length scales in RB turbulence see Ref. [6].

C. Scaling of velocities and mean time derivatives

We had adopted our control parameters and units to the $Ra^{-1/7}$ dependence of $\Theta_{c,\text{expt}}$ [Eq. (4.1d)], so the numerical values for the temperature fluctuations vary with Ra correctly, of course. But what about the Ra dependences of the mean temperature time derivative, $\langle [\partial_t \Theta(\mathbf{x}, t)]^2 \rangle^{1/2}$, henceforth denoted as $\partial_t \Theta_c$, the Ra scaling of the mean velocity time derivative $\langle [\partial_t \mathbf{u}(\mathbf{x}, t)]^2 \rangle^{1/2} \equiv \partial_t u_c$ and of the typical velocity fluctuations in the bulk $u_c = \langle \mathbf{u}^2(\mathbf{x}) \rangle^{1/2}$ itself?

The latter can be compared with the power-law fit to the data

$$u_{c,\text{expt}} = 1.05 Pr^{-2/3} \frac{\nu}{L_{\text{expt}}} Ra^{3/7} \quad (7.1)$$

given in [2] based on the measurements of Tanaka and Miyata [23], who used the velocity unit $Pr^{-2/3}\nu/L_{\text{expt}}$ introduced by Kraichnan [41]. From our solution the velocity fluctuation $u_c = \tilde{u}_c L/\tau$ in terms of $\nu Pr^{-2/3}/L_{\text{expt}}$ is shown versus Ra in Fig. 9(a). The $3/7$ scaling is quite well reproduced. The absolute value is too large by a factor of 4, but, like the Taylor-Reynolds numbers, this factor depends on the value of $\tilde{\kappa}$. Taking $\tilde{\kappa}$, say, three times larger which still gives a well-developed scaling regime in the spectra, this factor reduces to about 2.5. Furthermore, the insufficient energy transport property of our truncated wave-vector cascade enlarges the factor. If we take more small scale wave numbers into account, this factor is reduced even more, so that there is quite reasonable agreement.

Next, the Ra dependence of our numerical results for $\partial_t \Theta_c$ can be compared with experiment. A power law for the quantity Q_Θ defined as

$$Q_\Theta \equiv \frac{L_{\text{expt}}^2 \partial_t \Theta_c}{\kappa \Theta_c} \quad (7.2)$$

is given in [5], $Q_{\Theta,\text{expt}} = 0.1 Ra^{0.67 \pm 0.04}$. We plot both this and our $\log_{10} Q_\Theta$ versus $\log_{10} Ra$ in Fig. 9(b). Here the slope of the straight line fitted to our numerical data is about 0.58, which does not well agree with the experimental value 0.67 ± 0.04 , i.e., our dissipative power seems to be smaller than what was measured in experiment.

What might be the origin? We tried to estimate the error of our slope. Given the length of the scaling range, there is some freedom in choosing $\tilde{\kappa}$. To get a rough estimate about the error of the slope resulting from that uncertainty, we assign those $\tilde{\kappa}$'s to a particular Rayleigh number which in fact belong to the next larger or smaller Ra numbers (in Table I). This procedure seems justified because the length of the scaling range (measured by its numbers l_d of dyads) in Table I at most differs by 1 for neighboring Ra . The values for Q_Θ , which result from these bounds in $\tilde{\kappa}$, are shown as dashed lines in Fig. 9(b). From the two extremest straight lines compatible with

FIG. 9. (a) The rms velocity fluctuations u_c in terms of $\nu Pr^{-2/3}/L_{\text{expt}}$ vs Ra from the measured power law (7.1) (●) and from our numerical solution (○). Results for $f_u = 0$ and $f_u \neq 0$ do not differ visibly; this can be understood from the dominance of the large eddies. If $\tilde{\kappa}$ is enlarged by a factor of 3, the dashed line is obtained. (b) The normalized temperature time derivative Q_θ defined in (7.2) vs Ra from the power law given in [5] (●) and from our calculation (○). The error due to the uncertainty in $\tilde{\kappa}$ is indicated. For optical clarity, only the results for $f_u = 0$ are given, but those for $f_u \neq 0$ are very similar. (c) The normalized velocity time derivative Q_u defined in (7.3) vs Ra , both curves from our calculations. (○) and (★) refer to the calculations without and with additional kinetic plume forcing, respectively.

the numerical data we conclude that the slope 0.58 has an error of about 0.02 due to the uncertainty in $\tilde{\kappa}$.

So the uncertainty in $\tilde{\kappa}$ cannot completely account for the discrepancy in the slope. Neither can it explain the

absolute disagreement. We cannot exclude that the different slopes of theory and experiment are a shortcoming of our model. The dissipative power might get stronger within a space resolving Fourier-Weierstrass ansatz [24] or alternatively by considering more small scale wave numbers. But further explanations are possible: Q_θ might not just depend on Ra but also in addition on the absolute temperature or the pressure within the RB cell. So giving pure Ra power laws for Q_θ might be too simple. Interestingly enough we exactly get the 0.67 power law if we take the control parameters from [42], which are measurements in a quite different pressure range.

In the region of even higher Ra numbers, $10^{11} - 10^{13}$, the exponent of the Ra power law for Q_θ is measured [5] to be 0.48 ± 0.06 in the bulk of turbulence whereas it is 0.60 ± 0.06 near the bottom boundary for all $Ra < 10^{13}$. For a possible explanation see Ref. [6].

Analogously to the normalized rms temperature time derivative Q_θ we define the normalized rms velocity time derivative by

$$Q_u \equiv \frac{L_{\text{expt}}^2 \partial_t u_c}{\nu u_c} \quad (7.3)$$

and display this new quantity in a log-log plot versus Ra , cf. Fig. 9(c). The slope is determined to be 0.57 or 0.54, depending on whether one allows for a direct kinetic plume stirring or not. So besides the velocity spectrum also the Ra power law for $Q_u(Ra)$, in principle, is suited to clarify, whether the plumes stir the bulk of turbulence also by a substantial *kinetic* energy input or not. The prediction of this power-law exponent is open to experimental verification; we are not aware of published data.

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