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Semi-global regulation of output synchronization for heterogeneous networks of non-introspective, invertible agents subject to actuator saturation

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SUMMARY

In this paper, we consider the semi-global regulation of output synchronization problem for heterogeneous networks of invertible linear agents subject to actuator saturation. That is, we regulate the output of each agent according to an *a priori* specified reference model. The network communication infrastructure provides each agent with a linear combination of its own output relative to that of neighboring agents, and it allows the agents to exchange information about their own internal observer estimates while some agents have access to their own outputs relative to the reference trajectory. Copyright © 2012 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The synchronization problem in a network has received substantial attention in recent years (see [1–4] and references therein). Active research is being conducted in this context, and numerous results have been reported in the literature; to name a few, see [5–14].

Much of the attention has been devoted to achieving *state synchronization* in *homogeneous* networks (i.e., networks where the agent models are identical), where each agent has access to a linear combination of its own state relative to that of neighboring agents (e.g., [2, 6, 7, 10, 11, 13, 15–18]). A more realistic case—that is, each agent receives a linear combination of its own output relative to that of neighboring agents—has been considered in [5, 8, 14, 19, 20]. A key idea in the work of [5], which was expanded upon by Yang, Stoorvogel, and Saberi [21], is the development of a distributed observer. This observer makes additional use of the network by allowing the agents to exchange information with their neighbors about their own internal estimates. Many results on the synchronization problem are rooted in the seminal work [22, 23].

1.1. Heterogeneous networks and output synchronization

Recent activities in the synchronization literature have been focused on achieving synchronization for heterogeneous networks (i.e., networks where the agent models are non-identical). This problem is challenging, and only some results are available; see, for instance, [24–29].

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In heterogeneous networks, the agents' states may have different dimensions. In this case, the state synchronization is not even properly defined, and it is more natural to aim for *output synchronization*—that is, asymptotic agreement on some output from each agent. Chopra and Spong [24] studied output synchronization for weakly minimum-phase nonlinear systems of relative degree one, using a pre-feedback to create a single-integrator system with decoupled zero dynamics. Kim, Shim, and Seo [26] considered the output synchronization for uncertain single-input single-output, minimum-phase linear systems, by embedding an identical model within each agent, the output of which is tracked by the actual agent output. The authors have considered in [30] the output synchronization problem for right-invertible linear agents, using pre-compensators and an observer-based pre-feedback within each agent to yield a network of agents that are to a large extent identical.

1.2. Introspective versus non-introspective agents

The designs mentioned in Section 1.1 generally rely on some sort of self-knowledge that is separate from the information transmitted over the network. More specifically, the agents may be required to know their own states or their own outputs. In [31, 32], we refer to agents that possess this type of self-knowledge as *introspective* agents to distinguish them from *non-introspective* agents—that is, agents that have no knowledge about their own states or outputs separate from what is received via the network.

To our best knowledge, the only result besides [31, 32] that clearly applies to heterogeneous networks of non-introspective agents is by Zhao, Hill and Liu [33]. However, the agents are assumed to be passive—a strict requirement that, among other things, requires that the agents are weakly minimum-phase and of relative degree one.

1.3. Contributions of this paper

The *regulation of output synchronization* problem, where the objective is not only to achieve output synchronization but also to make the synchronization trajectory follow an *a priori* given reference trajectory generated by an arbitrary autonomous exosystem, has been considered in [32]. In [32], we assume that the agents in the network are non-introspective except for some of the agents who know their own outputs relative to the reference trajectory. However, we do not have any constraints on the magnitude of the agent's input. In the real world, every physically conceivable actuator has bounds on its input, and thus, actuator saturation is a common phenomenon. In this paper, we extend the results in [32] to the case where all the agents are subject to actuator saturation, which introduces significant complexities in terms of the analysis and design.

1.4. Notations

Given a matrix $A \in \mathbb{R}^{m\times n}$, A' denotes its transpose. $\mathrm{Im}A$ is the range space of a matrix $A \in \mathbb{R}^{m\times n}$ defined as

$$\mathrm{Im}\, A := \{Ax \mid x \in \mathbb{R}^n\}.$$

$A \in \mathbb{R}^{n\times n}$ is said to be Hurwitz stable if all its eigenvalues are in the open left-half complex plane. The Kronecker product between two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is defined as the $\mathbb{R}^{mp\times nq}$ matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix},$$

where a_{ij} denotes element (i, j) of A. I_n denotes the identity matrix of dimension n. Similarly, 0_n denotes the square matrix of dimension n with all zero elements. We sometimes drop the subscript if the dimension is clear in the context. When clear form the context, $\mathbf{1}$ denotes the column vector with all entries equal to one. For a given vector $v \in \mathbb{C}^n$, $\mathrm{re}\ v \in \mathbb{R}^n$ and $\mathrm{im}\ v \in \mathbb{R}^n$ denote respectively vectors whose entries are the real part and imaginary part of the vector v.

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2. PROBLEM FORMULATION AND MAIN RESULT

2.1. Problem formulation

Consider a network of N multiple-input multiple-output invertible agents of the form

$$\dot{x}_i = A_i x_i + B_i \sigma(u_i), \tag{1a}$$
$$y_i = C_i x_i + D_i \sigma(u_i), \tag{1b}$$

for $i \in \{1, \ldots, N\}$, where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^p$, $y_i \in \mathbb{R}^p$, and

$$\sigma(u_i) = [\sigma_1(u_{i,1}), \ldots, \sigma_1(u_{i,p})]',$$

where $\sigma_1(u)$ is the standard saturation function

$$\sigma_1(u) = \text{sgn}(u) \min\{1, |u|\},$$

and where the quadruple (A_i, B_i, C_i, D_i) is invertible.

The network provides each agent with a linear combination of its own output relative to that of other agents. In particular, each agent i has access to the quantity

$$\zeta_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j), \tag{2}$$

where $a_{ij} \geqslant 0$ and $a_{ii} = 0$ with $i, j \in \{1, \ldots, N\}$. This network can be described by a weighted directed graph (digraph) $\mathcal{G}$ with nodes corresponding to the agents in the network and edges with weight given by the coefficients a_{ij}. In particular, $a_{ij} > 0$ means that there exists an edge with weight a_{ij} from agent j to agent i, where agent j is called a parent of agent i, and agent i is called a child of agent j.

We also define a matrix $G = [g_{ij}]$, where $g_{ii} = \sum_{j=1}^{N} a_{ij}$ and $g_{ij} = -a_{ij}$ for $j \neq i$. The matrix G, known as the *weighted Laplacian* matrix of the digraph $\mathcal{G}$, has the property that the sum of the coefficients on each row is equal to zero. In terms of the coefficients g_{ij} of G, ζ_i given by (2) can be rewritten as

$$\zeta_i = \sum_{j=1}^{N} g_{ij} y_j. \tag{3}$$

In addition to ζ_i given by (3), we assume that the agents exchange information about their own internal estimates via the same network. That is, agent i has access to the quantity

$$\hat{\zeta}_i = \sum_{j=1}^{N} a_{ij}(\eta_i - \eta_j) = \sum_{j=1}^{N} g_{ij} \eta_j, \tag{4}$$

where $\eta_j \in \mathbb{R}^p$ is a variable produced internally by agent j. This value will be specified as we proceed with the design.

Our goal is to regulate the outputs of all agents towards an *a priori* specified reference trajectory $y_r(t)$, generated by an arbitrary autonomous exosystem

$$\dot{\omega} = S\omega, \quad \omega(0) = \omega_0 \in \Omega_0, \tag{5a}$$
$$y_r = C_r \omega, \tag{5b}$$

where $\omega \in \mathbb{R}^r$, $y_r \in \mathbb{R}^p$, and Ω_0 is a compact set of possible initial conditions for the exosystem. That is, for each agent $i \in \{1, \ldots, N\}$, we wish to achieve $\lim_{t\to\infty}(y_i - y_r) = 0$. Equivalently, we wish to regulate the synchronization error variable

$$e_i := y_i - y_r$$

to zero asymptotically, where the dynamics of e_i is governed by

$$\begin{bmatrix} \dot{x}_i \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} x_i \\ \omega \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} \sigma(u_i), \tag{6a}$$

$$e_i = \begin{bmatrix} C_i & -C_r \end{bmatrix} \begin{bmatrix} x_i \\ \omega \end{bmatrix} + D_i \sigma(u_i). \tag{6b}$$

In order to achieve our goal, in addition to ζ_i given by (3) and $\hat{\zeta}_i$ given by (4) provided by the network, it is clear that a non-empty subset of agents should observe its output relative to the reference trajectory y_r generated by (5) in order for the network of agents to follow the reference trajectory. Specifically, let $\mathcal{I} \subset \{1, \ldots, N\}$ denote such a subset. Then, each agent $i \in \{1, \ldots, N\}$ has access to the quantity

$$\psi_i = \iota_i (y_i - y_r), \quad \iota_i = \begin{cases} 1, & i \in \mathcal{I}, \\ 0, & i \notin \mathcal{I}. \end{cases} \tag{7}$$

Clearly, we need to restrict the initial conditions of the exosystem because, due to the input saturation, the agents will only be able to track a limited set of reference trajectories. This is formulated in the above by assuming that $\omega(0) \in \Omega_0$ with the set Ω_0 known a priori. Regarding the initial conditions of the agents, we would ideally like to design a controller that achieves $\lim_{t\to\infty} e_i(t) = 0$ for all initial conditions subject to $\omega(0) \in \Omega_0$, a problem that can be referred to as *global regulation of output synchronization*. However, from the literature on linear systems subject to actuator saturation, we know that global regulation of output synchronization in general requires nonlinear controllers. In this paper, we would like to use linear controllers of the form

$$\dot{x}_i^c = A_{i,c} x_i^c + B_{i,c} \begin{bmatrix} \zeta_i \\ \hat{\zeta}_i \\ \psi_i \end{bmatrix}, \tag{8a}$$

$$u_i = C_{i,c} x_i^c, \quad \forall i \in \{1, \ldots, N\}, \tag{8b}$$

where $x_i^c \in \mathbb{R}^{c_i}$ is the state of the controller for agent i. Thus, we restrict attention to the *semi-global regulation of output synchronization* problem, which is defined as follows.

Problem 1 (Semi-global regulation of output synchronization)
Consider a network of N agents as given by (1) and the reference model given by (5) with initial conditions in an *a priori* given compact set $\Omega_0 \subset \mathbb{R}^r$. The semi-global regulation of output synchronization problem is to find, if possible, for certain integers c_i, $i \in \{1, \ldots, N\}$ a family of controllers of the form (8) parameterized in a parameter ε such that for any arbitrarily large bounded sets $\mathcal{X}_i \subset \mathbb{R}^{n_i}$ and $\mathcal{P}_i \subset \mathbb{R}^{c_i}$, $i \in \{1, \ldots, N\}$, there exists ε small enough for which

$$\lim_{t\to\infty} e_i(t) = 0, \quad \forall i \in \{1, \ldots, N\}, \tag{9}$$

for all initial conditions $x_i(0) \in \mathcal{X}_i$, $x_i^c(0) \in \mathcal{P}_i$, and $\omega(0) \in \Omega_0$.

Remark 1
We would like to emphasize that our definition of the aforementioned semi-global regulation of output synchronization problem does not view the set of initial conditions of the agents' model (1) and their controllers (8) as given data. The set of given data consists of the models of the agent (1), the exosystem (5), and the set Ω_0 of possible initial conditions for the exosystem. Therefore, the solvability conditions must be independent of the set of initial conditions of the agents, $\mathcal{X}_i$, and the set of initial conditions for the controllers, $\mathcal{P}_i$.

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2.2. *Assumptions*

In this section, we present the assumptions about the network topology, the individual agents, and the reference model for solving the semi-global regulation of output synchronization problem as defined in Problem 1.

Assumption 1
Every node of the digraph $\mathcal{G}$ is a member of a directed tree whose root is contained in $\mathcal{I}$.

Remark 2
It is possible for $\mathcal{I}$ to consist of a single node, in which case Assumption 1 requires this node to be the root of a directed spanning tree of $\mathcal{G}$.

Assumption 2
For each agent $i \in \{1, \ldots, N\}$ as given in (1)

1. all the eigenvalues of A_i are in the closed left-half complex plane;
2. the pair (A_i, B_i) is stabilizable; and
3. the pair (C_i, A_i) is observable;

Remark 3
Conditions 2 and 3 are natural assumptions. Condition 1 is a necessary condition, since if A_i has one observable eigenvalue in the open right-half complex plane for some $i \in \{1, \ldots, N\}$, then for sufficiently large initial conditions $x_i(0)$, the output of that system y_i will be exponentially growing, and the bounded input $\sigma(u_i)$ can influence this exponentially growing signal only in a limited sense. Therefore, we cannot guarantee that this output will track y_r.

Assumption 3
For the reference model (5),

1. the pair (C_r, S) is observable;
2. all the eigenvalues of S are in the closed right-half complex plane; and
3. the matrix S is neutrally stable.

Remark 4
Condition 1 is a natural assumption. Condition 2 is made without loss of generality because asymptotically stable modes vanish asymptotically, and therefore they play no role asymptotically. Condition 3 is reasonable because the output of an agent cannot be expected to track exponentially growing signals with a bounded input. Polynomially growing reference signals can be easily included, but it requires very restrictive solvability conditions in case of input saturation and hence, for ease of presentation, we have excluded this case.

Assumption 4
The equations

$$\Pi_i S = A_i \Pi_i + B_i \Gamma_i, \tag{10a}$$

$$C_r = C_i \Pi_i + D_i \Gamma_i, \tag{10b}$$

commonly known as the *regulator equations* are solvable with respect to $\Pi_i \in \mathbb{R}^{n_i \times r}$ and $\Gamma_i \in \mathbb{R}^{p \times r}$, and there exists a $\delta > 0$ such that for each agent $i \in \{1, \ldots, N\}$,

$$\|\Gamma_i \omega(t)\|_\infty \leqslant 1 - \delta, \tag{11}$$

for all $t > 0$ and all $\omega(t)$ with $\omega(0) \in \Omega_0$.

Remark 5
Note that if the regulator equations (10) have a solution, then the solution is unique, as a consequence of the invertibility of the quadruple (A_i, B_i, C_i, D_i). Therefore, one can easily verify (11).

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2.3. Necessity of Assumption 4

Assumptions 1, 2, and 3 are natural as discussed in Remarks 3 and 4. On the other hand, Assumption 4 is critical. Essentially, this assumption is necessary for solving the semi-global regulation of output synchronization problem as defined in Problem 1. The following lemma, which is proven in Appendix A, shows this fact and gives the necessary condition for solving Problem 1.

Lemma 1
Suppose that each agent $i \in \{1, \ldots, N\}$ has access to full information. Assume that Ω_0 contains the origin in its interior. Then for any initial condition $\omega(0) \in \Omega_0$, there exist initial conditions $x_i(0)$ and an input $u_i(t)$ that leads to $e_i(t) \to 0$ as $t \to \infty$ only if the regulator equations (10) are solvable, and moreover the solution of the regulator equation must satisfy

$$\|\Gamma_i \omega(t)\|_\infty \leqslant 1 \tag{12}$$

for all $t > 0$.

2.4. Main result

Theorem 1
Consider a network of N agents as given by (1) and the reference model given by (5). Let Assumptions 1, 2, 3, and 4 hold. Then the semi-global regulation of output synchronization problem as defined in Problem 1 is solvable.

Proof
The proof of Theorem 1 is given in Section 3 by explicit construction of a controller for each agent. □

3. DESIGN OF CONTROL LAW FOR EACH AGENT

In this section, we describe the construction of a controller for each agent to solve the semi-global regulation of output synchronization problem as defined in Problem 1. The construction is carried out in three steps.

In Step 1, we construct a new state $\bar{x}_i$, via a transformation of x_i and ω, such that the dynamics of the synchronization error variable e_i can be described by equations

$$\dot{\bar{x}}_i = \bar{A}_i \bar{x}_i + \bar{B}_i \sigma(u_i) := \begin{bmatrix} A_i & 0 \\ 0 & \bar{A}_{i22} \end{bmatrix} \bar{x}_i + \begin{bmatrix} B_i \\ 0 \end{bmatrix} \sigma(u_i), \tag{13a}$$

$$e_i = \bar{C}_i \bar{x}_i + \bar{D}_i \sigma(u_i) := \begin{bmatrix} C_i & -\bar{C}_{i2} \end{bmatrix} \bar{x}_i + D_i \sigma(u_i). \tag{13b}$$

The purpose of this state transformation is to reduce the dimension of the model underlying e_i—the dimension of $\bar{x}_i$ is generally lower than that of $[x_i', \omega']'$—by removing redundant modes that have no effect on e_i. In particular, the model (6) may be unobservable, but the model (13) is always observable.

In Step 2, we construct a low-gain state feedback for u_i assuming $\bar{x}_i$ is known. This feedback is parameterized in ε and regulates e_i to zero for any arbitrarily large bounded set of initial conditions of the agent's models by choosing the low-gain parameter ε sufficiently small. Moreover, by making the low-gain parameter ε small enough, we can guarantee that the amplitude of the control law is less than any given α, where $1 - \delta < \alpha < 1$. Because the agent i has neither the internal state x_i nor the state ω of the exosystem available, this controller is not directly implementable. This brings us to Step 3 of the design.

In Step 3, we follow the procedure as given in our previous paper [32], that is, we construct a decentralized high-gain observer that makes an estimate of $\bar{x}_i$ available to agent i. However, as we shall see later, our state feedback design and high-gain observer are coupled. This will be illustrated in Section 3.1.

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3.1. Design procedure for agent i

Step 1*: State transformation*
Let O_i be the observability matrix corresponding to the system (6).

$$\mathcal{O}_i = \begin{bmatrix} C_i & -C_r \\ \vdots & \vdots \\ C_i A_i^{n_i+r-1} & -C_r S^{n_i+r-1} \end{bmatrix}.$$

Let q_i denote the dimension of the null space of matrix $\mathcal{O}_i$ and define $r_i = r - q_i$. Next, define $\Lambda_{iu} \in \mathbb{R}^{n_i \times q_i}$ and $\Phi_{iu} \in \mathbb{R}^{r \times q_i}$ such that

$$\mathcal{O}_i \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = 0, \quad \text{rank} \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = q_i.$$

Because the pair (C_i, A_i) and the pair (C_r, S) are observable, it is easy to see that Λ_{iu} and Φ_{iu} have full column rank (see [32, Appendix A]). Let therefore Λ_{io} and Φ_{io} be defined such that $\Lambda_i := [\Lambda_{iu}, \Lambda_{io}] \in \mathbb{R}^{n_i \times n_i}$ and $\Phi_i := [\Phi_{iu}, \Phi_{io}] \in \mathbb{R}^{r \times r}$ are nonsingular.

From the proof of [31, Lemma 2], we know that

$$S\Phi_i = \Phi_i R_i, \tag{14}$$

where

$$R_i = \begin{bmatrix} U_i & R_{i12} \\ 0 & R_{i22} \end{bmatrix}.$$

Because S is anti-Hurwitz stable and neutrally stable, we know that S is diagonalizable, and hence, R_i is diagonalizable. This implies that R_i has r independent right eigenvectors. Let $v_{i,1}, \cdots, v_{i,r}$ be r independent right eigenvectors of R_i, such that

$$v_{i,j} = \begin{bmatrix} \tilde{v}_{i,j} \\ 0 \end{bmatrix}$$

for $j = 1, \ldots, q_i$, where $\tilde{v}_{i,j}$ are right eigenvectors of U_i. In that case, we choose $V_{i11} \in \mathbb{R}^{q_i \times q_i}$ such that

$$\text{Im}\, V_{i11} = \text{span}\{\text{re } \tilde{v}_{i,j}, \text{ im } \tilde{v}_{i,j} \mid j = 1, \cdots, q_i\},$$

and we choose $V_{i12} \in \mathbb{R}^{q_i \times r_i}$ and $V_{i22} \in \mathbb{R}^{r_i \times r_i}$ such that

$$\text{Im} \begin{bmatrix} V_{i12} \\ V_{i22} \end{bmatrix} = \text{span}\{\text{re } v_{i,j}, \text{ im } v_{i,j} \mid j = q_i + 1, \cdots, r\}.$$

We then construct

$$V_i = \begin{bmatrix} V_{i11} & V_{i12} \\ 0 & V_{i22} \end{bmatrix}.$$

It can be easily verified that span$\{\text{re } v_{i,j}, \text{ im } v_{i,j}\}$ is an invariant subspace of R_i for any $j = 1, \ldots r$. This implies

$$R_i V_i = V_i \begin{bmatrix} \Lambda_{i1} & 0 \\ 0 & \Lambda_{i2} \end{bmatrix}. \tag{15}$$

One way of choosing the matrix V_i is choosing

$$\begin{bmatrix} \Lambda_{i1} & 0 \\ 0 & \Lambda_{i2} \end{bmatrix}$$

to be the real Jordan form of R_i ordered in such a way that Λ_{i1} is the real Jordan form of U_i.

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From (15), we obtain that

$$V_{i11}^{-1} U_i V_{i11} = \Lambda_{i1}, \quad V_{i22}^{-1} R_{i22} V_{i22} = \Lambda_{i2}, \tag{16}$$

and

$$U_i V_{i12} - V_{i12}\Lambda_{i2} = -R_{i12} V_{i22}. \tag{17}$$

We then define

$$\bar{\Phi}_i := [\bar{\Phi}_{iu}, \bar{\Phi}_{io}] = \Phi_i \begin{bmatrix} I_{q_i} & V_{i12}V_{i22}^{-1} \\ 0 & I_{r_i} \end{bmatrix}. \tag{18}$$

We then define a new state variable $\bar{x}_i \in \mathbb{R}^{n_i + r_i}$ as

$$\bar{x}_i = \begin{bmatrix} \bar{x}_{i1} \\ \bar{x}_{i2} \end{bmatrix} := \begin{bmatrix} x_i - \Lambda_i M_i \bar{\Phi}_i^{-1}\omega \\ N_i \bar{\Phi}_i^{-1}\omega \end{bmatrix},$$

where $M_i \in \mathbb{R}^{n_i \times r}$ and $N_i \in \mathbb{R}^{r_i \times r}$ are defined as

$$M_i = \begin{bmatrix} I_{q_i} & 0 \\ 0 & 0 \end{bmatrix}, \quad N_i = \begin{bmatrix} 0 & I_{r_i} \end{bmatrix}.$$

Note that the system (6) can be transformed into the system (13), with a block upper-triangular structure if we use the transformation Φ_i as shown in [32]. However, with the matrix $\bar{\Phi}_i$ given by (18), which is a special case of the transformation previously used in [32], everything from our previous results still holds. Moreover, the system (13) has a block-diagonal structure. The following lemma, which is proven in Appendix B, shows this.

Lemma 2
The synchronization error variable e_i is governed by dynamical equations of (13), where the pair $(\bar{C}_i, \bar{A}_i)$ is observable, and the eigenvalues of $\bar{A}_{i22}$ are a subset of the eigenvalues of S.

Remark 6
If the unforced system for an agent i is the same as the exosystem, that is, if $C_i = C_r$ and $A_i = S$, then it is easy to see that the dynamics of system (13) reduces to the dynamics of system (1).

Step 2: State feedback control design
For any arbitrarily large bounded set $\mathcal{X}_i$, we design a controller as a function of $\bar{x}_i$ such that $\lim_{t\to\infty} e_i(t) = 0$ for all $x_i(0) \in \mathcal{X}_i$ and $\omega(0) \in \Omega_0$. Consider the following regulator equations with unknowns $\Pi_i^r \in \mathbb{R}^{n_i \times r_i}$ and $\Gamma_i^r \in \mathbb{R}^{p \times r_i}$ for system (13)

$$\Pi_i^r \bar{A}_{i22} = A_i \Pi_i^r + B_i \Gamma_i^r, \tag{19a}$$

$$\bar{C}_{i2} = C_i \Pi_i^r + D_i \Gamma_i^r. \tag{19b}$$

The following lemma shows that the regulator equations (19) are solvable if and only if the regulator equations (10) are solvable and gives the mapping between the solutions of the two regulator equations. Note that if the regulator equations (19) (or the regulator equations (10)) have a solution, then it is unique due to the invertibility of the quadruple (A_i, B_i, C_i, D_i).

Lemma 3
If (Π_i^r, Γ_i^r) is the solution of the regulator equations (19), then (Π_i, Γ_i) given as

$$\Pi_i = \Pi_i^r N_i \bar{\Phi}_i^{-1} + \Lambda_i M_i \bar{\Phi}_i^{-1}, \quad \Gamma_i = \Gamma_i^r N_i \bar{\Phi}_i^{-1} \tag{20}$$

is the solution of the regulator equations (10). On the other hand, if (Π_i, Γ_i) is the solution of the regulator equations (10), then (Π_i^r, Γ_i^r) given as

$$\Pi_i^r = \Pi_i \bar{\Phi}_{io}, \quad \Gamma_i^r = \Gamma_i \bar{\Phi}_{io} \tag{21}$$

is the solution of the regulator equations (19).

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Proof
Let $\left(\Pi_i^r, \Gamma_i^r\right)$ be the solution of the regulator equations (19). And define (Π_i, Γ_i) by (20) and $W_i = \begin{bmatrix} I_{q_i} & 0 \end{bmatrix}$. From (20), it is easy to see that

$$\Pi_i = \begin{bmatrix} \Pi_i^r & 0 \end{bmatrix} \begin{bmatrix} N_i \\ W_i \end{bmatrix} \bar{\Phi}_i^{-1} + \Lambda_i M_i \bar{\Phi}_i^{-1}, \quad \Gamma_i = \begin{bmatrix} \Gamma_i^r & 0 \end{bmatrix} \begin{bmatrix} N_i \\ W_i \end{bmatrix} \bar{\Phi}_i^{-1}.$$

With some algebra, we obtain that

$$\begin{aligned} \Pi_i S \bar{\Phi}_i &= \begin{bmatrix} \Pi_i^r & 0 \end{bmatrix} \begin{bmatrix} N_i \\ W_i \end{bmatrix} \bar{\Phi}_i^{-1} S \bar{\Phi}_i + \Lambda_i M_i \bar{\Phi}_i^{-1} S \bar{\Phi}_i \\ &= \begin{bmatrix} 0 & \Pi_i^r \end{bmatrix} \begin{bmatrix} U_i & 0 \\ 0 & \bar{A}_{i22} \end{bmatrix} + \begin{bmatrix} \Lambda_{iu} & 0 \end{bmatrix} \begin{bmatrix} U_i & 0 \\ 0 & \bar{A}_{i22} \end{bmatrix} \\ &= \begin{bmatrix} \Lambda_{iu} U_i & \Pi_i^r \bar{A}_{i22} \end{bmatrix}, \end{aligned} \tag{22}$$

where we have used that $S \bar{\Phi}_i = \bar{\Phi}_i \bar{R}_i$ shown in Appendix B. Moreover,

$$\begin{aligned} (A_i \Pi_i + B_i \Gamma_i) \bar{\Phi}_i &= A_i \begin{bmatrix} \Pi_i^r & 0 \end{bmatrix} \begin{bmatrix} N_i \\ W_i \end{bmatrix} + A_i \Lambda_i M_i + B_i \begin{bmatrix} \Gamma_i^r & 0 \end{bmatrix} \begin{bmatrix} N_i \\ W_i \end{bmatrix} \\ &= \begin{bmatrix} 0 & A_i \Pi_i^r \end{bmatrix} + \begin{bmatrix} A_i \Lambda_{iu} & 0 \end{bmatrix} + \begin{bmatrix} 0 & B_i \Gamma_i^r \end{bmatrix} \\ &= \begin{bmatrix} \Lambda_{iu} U_i & A_i \Pi_i^r + B_i \Gamma_i^r \end{bmatrix}, \end{aligned} \tag{23}$$

where we have used that $A_i \Lambda_{iu} = \Lambda_{iu} U_i$, shown in our previous paper [32].

From (19a), (22), and (23), it is then easy to see that $\Pi_i S \bar{\Phi}_i = (A_i \Pi_i + B_i \Gamma_i) \bar{\Phi}_i$. Because $\bar{\Phi}_i$ is non-singular, this implies that (10a) is satisfied.

Similarly, we obtain that

$$C_r \bar{\Phi}_i = \begin{bmatrix} C_r \bar{\Phi}_{iu} & C_r \bar{\Phi}_{io} \end{bmatrix} = \begin{bmatrix} C_i \Lambda_{iu} & \bar{C}_{i2} \end{bmatrix}, \tag{24}$$

where we have used that $C_r \bar{\Phi}_{iu} = C_i \Lambda_{iu}$ and $\bar{C}_{i2} = C_r \bar{\Phi}_i N_i' = C_r \bar{\Phi}_{io}$, shown in our previous paper [32]. Moreover,

$$\begin{aligned} (C_i \Pi_i + D_i \Gamma_i) \bar{\Phi}_i &= C_i \begin{bmatrix} \Pi_i^r & 0 \end{bmatrix} \begin{bmatrix} N_i \\ W_i \end{bmatrix} + C_i \Lambda_i M_i + D_i \begin{bmatrix} \Gamma_i^r & 0 \end{bmatrix} \begin{bmatrix} N_i \\ W_i \end{bmatrix} \\ &= \begin{bmatrix} C_i \Lambda_{iu} & C_i \Pi_i^r + D_i \Gamma_i^r \end{bmatrix}. \end{aligned} \tag{25}$$

From (19b), (24), and (25), it is then easy to see that $\bar{C}_r \bar{\Phi}_i = (C_i \Pi_i + D_i \Gamma_i) \bar{\Phi}_i$. Because $\bar{\Phi}_i$ is non-singular, this implies that (10b) is satisfied. Hence, (Π_i, Γ_i) given by (20) is the solution of the regulator equations (10).

Now let (Π_i, Γ_i) be the solution of the regulator equations (10). And define $\left(\Pi_i^r, \Gamma_i^r\right)$ by (21). With just a little bit algebra, we obtain that

$$A_i \Pi_i^r + B_i \Gamma_i^r = A_i \Pi_i \bar{\Phi}_{io} + B_i \Gamma_i \bar{\Phi}_{io} \tag{26}$$

and

$$\Pi_i^r \bar{A}_{i22} = \Pi_i \bar{\Phi}_{io} \bar{A}_{i22} = \Pi_i S \bar{\Phi}_{io}, \tag{27}$$

where we have used that $S \bar{\Phi}_{io} = \bar{\Phi}_{io} \bar{A}_{i22}$, which follows from the fact that $S \bar{\Phi}_i = \bar{\Phi}_i \bar{R}_i$.

From (10a), (26), and (27), it is easy to see that $\Pi_i^r \bar{A}_{i22} = A_i \Pi_i^r + B_i \Gamma_i^r$, that is, (19a) is satisfied.

Finally, we obtain that

$$C_i \Pi_i^r + D_i \Gamma_i^r = C_i \Pi_i \bar{\Phi}_{io} + D_i \Gamma_i \bar{\Phi}_{io}. \tag{28}$$

This together with the fact that $\bar{C}_{i2} = C_r \bar{\Phi}_i N_i' = C_r \bar{\Phi}_{io}$ and (10b) yields $\bar{C}_{i2} = C_i \Pi_i^r + D_i \Gamma_i^r$, that is, (19b) is satisfied. Hence, $\left(\Pi_i^r, \Gamma_i^r\right)$ given by (21) is the solution of the regulator equations (19). □

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Remark 7
In view of Lemma 3 and (11) of Assumption 4, we see that $\|\Gamma_i^r \bar{x}_{i2}\|_\infty = \|\Gamma_i \omega\|_\infty \leqslant 1 - \delta$.

Because agent i is subject to actuator saturation, we design the state feedback controller by using a *low-gain* technique, which is widely used for the semi-global stabilization problem for linear systems subject to actuator saturation, see for instance, [34, 35]. There exist in the literature several low-gain design algorithms. For conceptual clarity, we use here the one based on the solution of a continuous-time algebraic Riccati equation, parameterized in a low-gain parameter $\varepsilon \in (0, 1]$. More specifically, we form a family of parameterized state feedback gain matrices $F_{i,\varepsilon}$ for $\bar{x}_{i1}$ as

$$F_{i,\varepsilon} = -B_i' P_{i,\varepsilon},$$

where $P_{i,\varepsilon} = P_{i,\varepsilon}' > 0$ is the unique solution of the continuous-time algebraic Riccati equation defined as

$$P_{i,\varepsilon} A_i + A_i' P_{i,\varepsilon} - P_{i,\varepsilon} B_i B_i' P_{i,\varepsilon} + \varepsilon I_{n_i} = 0. \tag{29}$$

It follows from Lemma 3 and Condition 1 of Assumption 4 that the regulator equations (19) have a unique solution $\left(\Pi_i^r, \Gamma_i^r\right)$. We use the unique $\left(\Pi_i^r, \Gamma_i^r\right)$ and the feedback gain matrix $F_{i,\varepsilon}$ to define a family of parameterized state feedback controllers in terms of $\bar{x}_i$ as

$$u_i = \begin{bmatrix} F_{i,\varepsilon} & \Gamma_i^r - F_{i,\varepsilon}\Pi_i^r \end{bmatrix} \bar{x}_i. \tag{30}$$

Then for any given arbitrarily large bounded set of initial conditions, there exists an $\varepsilon^* \in (0, 1]$, such that for all $\varepsilon \in (0, \varepsilon^*]$, the family of linear state feedback controllers of the form (30) ensures that $\lim_{t\to\infty} e_i(t) = 0$ for all initial conditions belong to the given arbitrarily large bounded set and $\omega(0) \in \Omega_0$. This is a well-known result, see [35, Theorem 3.3.2].

Remark 8
If the unforced system for an agent i is the same as the exosystem, that is, if $C_i = C_r$ and $A_i = S$, then it is easy to see that $\Pi_i = I$ and $\Gamma_i = 0$ is the solution of regulator equations (10). Thus, Assumption 4 is always satisfied for that agent.

Step 3*: Observer-based implementation*
Following the design procedure given in the proof of [35, Theorem 3.3.4], one can obtain, for a given set of initial conditions, suitable state feedback controllers for which input saturation is not active. This is performed by properly choosing the low-gain parameter ε. Then such a state feedback law must be implemented by a suitable designed distributed observer. This will be performed next.

We will design a high-gain decentralized observer to produce an estimate of $\bar{x}_i$, denoted by $\hat{\bar{x}}_i$. We follow the procedure as given in our previous paper [32], to be self-contained, we reproduce the design here.

Let $\bar{n}$ denotes the maximum order among the all the systems (13) for $i \in \{1, \ldots, N\}$, that is, $\bar{n} = \max_{i=1,\ldots,N}(n_i + r_i)$. Define $\chi_i = T_i \bar{x}_i$, where

$$T_i = \begin{bmatrix} \bar{C}_i \\ \vdots \\ \bar{C}_i \bar{A}_i^{\bar{n}-1} \end{bmatrix}.$$

Note that T_i is injective because the pair $(\bar{C}_i, \bar{A}_i)$ is observable, which implies that $T_i' T_i$ is nonsingular.

In term of χ_i, we can write the system equations

$$\dot{\chi}_i = (\mathcal{A} + \mathcal{L}_i)\chi_i + \mathcal{B}_i \sigma(u_i), \quad \chi_i(0) = T_i \bar{x}_i(0), \tag{31a}$$
$$e_i = \mathcal{C}\chi_i + \mathcal{D}_i \sigma(u_i), \tag{31b}$$

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where

$$\mathcal{A} = \begin{bmatrix} 0 & I_{p(\bar{n}-1)} \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} I_p & 0 \end{bmatrix}, \quad \mathcal{L}_i = \begin{bmatrix} 0 \\ L_i \end{bmatrix}, \quad \mathcal{B}_i = T_i \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad \mathcal{D}_i = D_i,$$

for some matrix $L_i \in \mathbb{R}^{p \times \bar{n}p}$. Note that the matrices $\mathcal{A}$ and $\mathcal{C}$ are the same for all the agents $i \in \{1, \dots, N\}$, and the special form of these matrices implies that $(\mathcal{C}, \mathcal{A})$ is observable.

Next, define the matrix $\bar{G} = G + \text{Diag}(\iota_1, \dots, \iota_N)$ and $\tau = \min_{i=1,\dots,N} \text{Re}(\lambda_i(\bar{G})) > 0$. Let $\mathcal{P} = \mathcal{P}' > 0$ be the unique solution of the algebraic Riccati equation

$$\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}' - \tau\mathcal{P}\mathcal{C}'\mathcal{C}\mathcal{P} + I_{\bar{n}p} = 0. \tag{32}$$

We then we design the observer

$$\dot{\hat{\chi}}_i = (\mathcal{A} + \mathcal{L}_i)\hat{\chi}_i + \mathcal{B}_i\sigma(u_i) + S(\ell)\mathcal{P}\mathcal{C}'(\zeta_i - \hat{\zeta}_i) + S(\ell)\mathcal{P}\mathcal{C}'(\psi_i - \iota_i(\mathcal{C}\hat{\chi}_i + D_i\sigma(u_i))), \tag{33a}$$

$$\hat{\bar{x}}_i = (T_i'T_i)^{-1}T_i'\hat{\chi}_i, \tag{33b}$$

where $S(\ell) = \text{blk diag}(I_p\ell, I_p\ell^2, \dots, I_p\ell^{\bar{n}})$ and $\ell > 1$ is a high-gain parameter.

On the basis of the observer estimate, we define the variable $\eta_i = \mathcal{C}\hat{\chi}_i + \mathcal{D}_i\sigma(u_i)$ to be shared with the other agents via the network communication infrastructure as described in Section 2.1 and the observer-based control law

$$u_i = \begin{bmatrix} F_{i,\varepsilon} & \Gamma_i^r - F_{i,\varepsilon}\Pi_i^r \end{bmatrix} \hat{\bar{x}}_i. \tag{34}$$

Together, the observers for agents $i \in \{1, \dots, N\}$ form a distributed observer parameterized by a high-gain parameter ℓ. It has been shown in [32, Lemma 4] that the estimation errors dynamics are globally exponentially stable, that is, $\lim_{t\to\infty}(\bar{x}_i - \hat{\bar{x}}_i) = 0$, by choosing the high-gain parameter ℓ sufficiently large.

Remark 9

If all the agents have the same dynamics, it is not necessary to design an observer based on the high-order system (31), and one can design an observer based on the original system (13).

In summary, for any given arbitrarily large bounded sets $\mathcal{X}_i \subset \mathbb{R}^{n_i}$ and $\mathcal{P}_i \subset \mathbb{R}^{p\bar{n}}$, there exist ε^* with the property that for any $\varepsilon \in (0, \varepsilon^*]$ there exists ℓ^* such that for $\ell \geqslant \ell^*$, the observer-based implementation (33) and (34), ensure that

$$\lim_{t\to\infty} e_i(t) = 0, \quad \forall i \in \{1, \dots, N\}, \tag{35}$$

for all initial conditions $x_i(0) \in \mathcal{X}_i$, $\hat{\chi}_i(0) \in \mathcal{P}_i$, and $\omega(0) \in \Omega_0$.

3.2. Comparison with the case where the agents have no actuator magnitude constraints

Let us make a few comments to compare our result with the case where the agents do not have actuator saturation.

- The regulator equations (10) have to be solvable for the case with actuator magnitude constraints. In our previous work, for the case without saturation, we assumed existence of a solution of the regulator equations, but in that case, this existence is not necessary.
- For the case with actuator magnitude constraints, we only achieve semi-global regulation of output synchronization.
- For the case with actuator magnitude constraints, it is required that all the eigenvalues of agents' system matrices are in the closed left-half complex plane.
- For the case with actuator magnitude constraints, we have constraints on the size of the synchronized output trajectory as given by (11).

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4. EXAMPLE

In this section, we illustrate our design procedure by considering a network of 10 agents. Agents 1 and 2 are composed as the cascade of a second-order oscillator and a single integrator:

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D_i = 0.$$

Agents 3, 4, and 5 have the following dynamics:

$$A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_i = 2.$$

Agents 6, 7, and 8 have the following dynamics:

$$A_i = 0, \quad B_i = 1, \quad C_i = 1, \quad D_i = 1.$$

Finally, Agents 9 and 10 are second-order mass-spring-damper systems:

$$A_i = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_i = 0.$$

The reference trajectory y_r is generated by an exosystem with

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad C_r = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

and initial conditions $\Omega_0 = \{\omega \in \mathbb{R}^3 : \|\omega\| \leqslant 0.1\}$.

The communication topology of the network is given by the digraph depicted in Figure 1, and agent 2 has access to the information $y_2 - y_r$.

Step 1

For illustrative purpose, we give the details for agent 3. In Step 1,

$$O_3 = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Longrightarrow q_3 = 1, \ r_3 = 2,$$

We may choose

$$\Lambda_{3u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Phi_{3u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

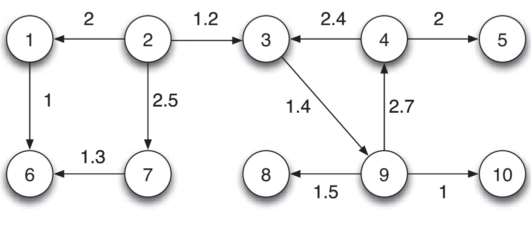

Figure 1. Network topology.

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and hence, we can set $\Lambda_3 = I_2$ and $\Phi_3 = I_3$. Following the design procedure, we have

$$V_{311} = 1,\ V_{312} = \begin{bmatrix} 0 & 1 \end{bmatrix},\ \text{and}\ V_{322} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

for (16) and (17). Therefore, from (18), we obtain that

$$\bar{\Phi}_3 = \Phi_i \begin{bmatrix} I_{q_3} & V_{312}V_{322}^{-1} \\ 0 & I_{r_i} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

thus, it follows that

$$\bar{x}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} x_3 - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \omega,$$

then the dynamics of $\bar{x}_i$ with output e_i takes the form of (13) with

$$\bar{A}_{322} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\quad \bar{C}_{32} = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

Step 2

We now need to solve the regulator equations (19), which are easily found to have the unique solution

$$\Pi_3^r = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\quad \Gamma_3^r = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

We then select the matrix $F_{3,\varepsilon} = -B_3' P_{3,\varepsilon}$, where $P_{3,\varepsilon} = P_{3,\varepsilon}'$ is the unique solution of (29), and the value of ε will be determined later.

We perform the same procedure for the other agents, to identify appropriate state feedbacks. For agents 1 and 2, there is no need for solving the regulator equations (19); for agents 6, 7, and 8, we obtain

$$\Pi_6^r = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \end{bmatrix},\quad \Gamma_6^r = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \end{bmatrix},$$

and for agents 9 and 10, the system (6) is observable, moreover $\bar{x}_{i2} = \omega$. We then find the unique solution of the regulator equations (10) as

$$\Pi_9 = \Pi_9^r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\quad \Gamma_9 = \Gamma_9^r = \begin{bmatrix} 2 & 2 & 1 \end{bmatrix}.$$

Note that

$$\Gamma_9 \omega \leqslant 0.5,$$

therefore, we choose $\delta = 0.5$, such that

$$\Gamma_9 \omega \leqslant 1 - \delta$$

for all $\omega(0) \in \Omega_0$. It is also easy to check that $\delta = 0.5$ works for all other agents.

Step 3

In Step 3, we design the decentralized observer that allows the feedbacks to be implemented based on observer estimates. It is easy to check that $\bar{n} = 5$, then we have

$$\mathcal{A} = \begin{bmatrix} 0 & I_4 \\ 0 & 0 \end{bmatrix},\quad \mathcal{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.$$

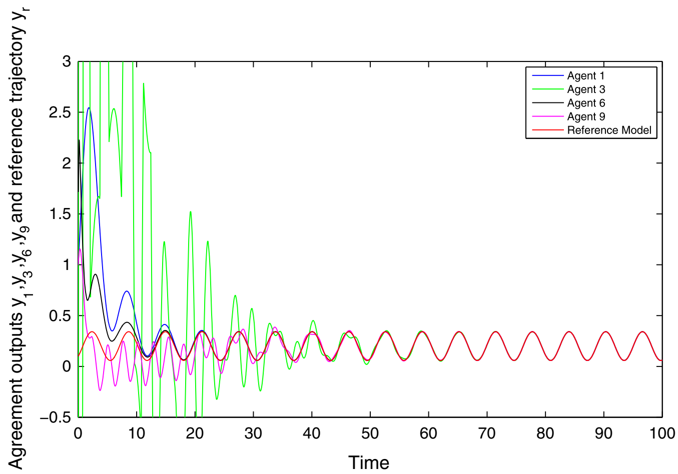

Figure 2. Output trajectories for agents 1, 3, 6, 9, and reference model.

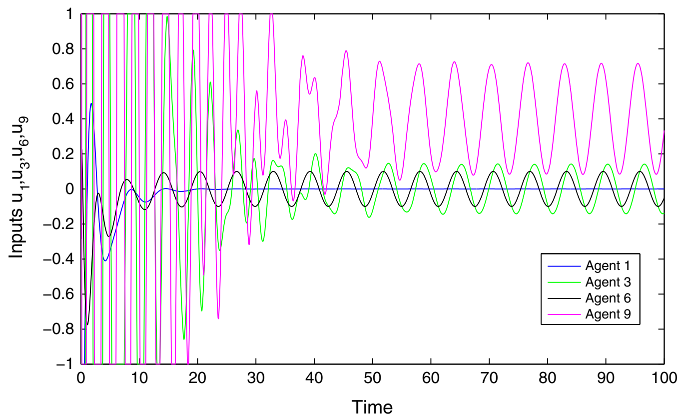

Figure 3. Input trajectories for agents 1, 3, 6, and 9.

Note that in order to implement the observer-based feedback (33) and (34), we need to determine the value of the low-gain parameter $\varepsilon \in (0, \varepsilon^*]$. For the set given by $\mathcal{X}_i = \{x_i \in \mathbb{R}^{n_i} : \|x_i\| \leqslant 1\}$ and $\mathcal{P}_i = \left\{x_i \in \mathbb{R}^{c_i} : \|x_i^c\| \leqslant 1\right\}$, we can confirm that $\varepsilon^* = 0.1$, thus we choose $\varepsilon = \varepsilon^* = 0.1$. Now, we construct the weighted Laplacian G from the digraph in Figure 1, note that the digraph contains a directed spanning tree with agent 2 being the root. Given fact that $\iota_2 = 1$ while $\iota_i = 0$ for all other i, we find that $\tau = \min_{i=1,\dots,10} \operatorname{re}(\lambda_i(G + \operatorname{Diag}(\iota_1, \dots, \iota_{10}))) \approx 0.2749$. Solving the algebraic Riccati equation (32) and implementing observer-based feedback (33) and (34), we find that we achieve stability with $\ell = 2$. Figure 2 shows the resulting simulated output of four agents and the synchronization trajectory, whereas Figure 3 shows the resulting simulated input of four agents.

APPENDIX A: PROOF OF LEMMA 1

Proof

If the quadruple (A_i, B_i, C_i, D_i) has no invariant zeros that are eigenvalues of the matrix S, then the existence of solutions to the regulator equations follows from the fact that the system is right-invertible (Corollary 2.5.1 of [35]).

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On the other hand, assume that the quadruple (A_i, B_i, C_i, D_i) has an invariant zero λ that is an eigenvalue of the matrix S. In that case, let (v, w) be such that

$$\begin{pmatrix} v' & w' \end{pmatrix} \begin{pmatrix} A_i - \lambda I & B_i \\ C_i & D_i \end{pmatrix} = 0 \tag{A.1}$$

and ω_0 such that

$$S\omega_0 = \lambda\omega_0.$$

Because Ω_0 contains 0 in its interior, we can, without loss of generality, assume that $\omega_0 \in \Omega_0$.

We first assume that $w'C_r\omega_0 \neq 0$ and we will establish a contradiction with the fact that there exists for $\omega(0) = \omega_0$, an input u_i and an appropriate initial condition $x_i(0)$ such that $e_i(t) \to 0$ as $t \to \infty$.

Because (A_i, B_i, C_i, D_i) is right-invertible, we note that the subsystem from u to $z = w'y$ (which has a scalar output) can be described by a polynomial description:

$$d\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) z(t) = N\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) u(t),$$

where $N(s)$ is a non-zero polynomial row vector, whereas $d(s)$ is a scalar polynomial. Because, the subsystem from u to z is right-invertible and has a zero in λ, we find that N has a zero in λ. Moreover, if d also has a zero in λ, then N has a zero in λ whose order is at least one higher than the zero in λ of d. We define

$$\bar{z}(t) = e^{-\lambda t} z(t), \qquad \bar{u}(t) = e^{-\lambda t} u(t),$$

and

$$\bar{d}(s) = d(s + \lambda), \qquad \bar{N}(s) = N(s + \lambda).$$

We note that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \lambda\right) \bar{z}(t) = e^{-\lambda t} \frac{\mathrm{d}}{\mathrm{d}t} z(t),$$

and similarly for $u, \bar{u}$. Hence,

$$\bar{d}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \bar{z}(t) = e^{-\lambda t} d\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) z(t),$$

and

$$\bar{N}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \bar{u}(t) = e^{-\lambda t} N\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) u(t).$$

Assume that the input u is such that tracking is achieved, then we have

$$z(t) \to w'C_r\omega(t) = e^{\lambda t} w'C_r\omega_0$$

as $t \to \infty$ and hence

$$\bar{z}(t) \to w'C_r\omega_0$$

as $t \to \infty$. Without loss of generality, we assume that $w'C_r\omega_0 = \delta > 0$. In that case, there exists $t_0 > 0$ such that we have

$$\frac{1}{2}\delta \leqslant \bar{z}(t) \leqslant \frac{3}{2}\delta$$

for all $t > t_0$. On the other hand, given that λ is on the imaginary axis and that $u(t)$ is bounded, we have that there exists an $M > 0$ such that

$$\|\bar{u}(t)\| \leqslant M$$

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for all $t > 0$. We have

$$\bar{d}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bar{z}(t) = \bar{N}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bar{u}(t)$$

Define

$$\bar{d}(s) = d_i s^i + d_{i+1}s^{i+1} + \cdots + d_n s^n,$$

and

$$\bar{N}(s) = N_{i+1}s^{i+1} + \cdots + N_n s^n,$$

such that $d_i \neq 0$. Here, we used that N had a zero in λ and if d has a zero as well in λ, then it is of strictly lower order. We find that

$$\left|\underbrace{\int_{t_1}^{t_2}\int_{t_1}^{t_2}\cdots\int_{t_1}^{t_2}}_{n}\bar{d}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bar{z}(t)\right| \geqslant \left(|d_i|(t_2-t_1)^{n-i} - 3\sum_{j=i+1}^{n}|d_j|(t_2-t_1)^{n-j}\right)\frac{1}{2}\delta$$

for all $t_2, t_1 > t_0$. On the other hand,

$$\left|\underbrace{\int_{t_1}^{t_2}\int_{t_1}^{t_2}\cdots\int_{t_1}^{t_2}}_{n}\bar{N}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bar{u}(t)\right| \leqslant M\sum_{j=i+1}^{n}\|N_i\|(t_2-t_1)^{n-j}$$

for all $t_2, t_1 > t_0$. This yields a contradiction as $t_2 \to \infty$ because we have

$$\underbrace{\int_{t_1}^{t_2}\int_{t_1}^{t_2}\cdots\int_{t_1}^{t_2}}_{n}\bar{d}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bar{z}(t) = \underbrace{\int_{t_1}^{t_2}\int_{t_1}^{t_2}\cdots\int_{t_1}^{t_2}}_{n}\bar{N}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bar{u}(t),$$

and our inequalities imply that the left-hand side grows like $(t_2-t_1)^{n-i}$, whereas the right-hand side can at most grow like $(t_2-t_1)^{n-i-1}$.

Because, assuming that $w'C_r\omega_0 \neq 0$, we obtain a contradiction, we must have that $w'C_r\omega_0 = 0$.

Using this property, we will establish that (10) has a solution. Without loss of generality and using Assumption 3, we can assume that

$$S = \begin{pmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \omega_r \end{pmatrix}, \quad C_r = \left(C_{r,1}\cdots C_{r,r}\right),$$

and we also decompose the potential solutions of the regulator equations as

$$\Pi_i = \left(\Pi_{i,1}\cdots\Pi_{i,r}\right), \quad \Gamma_i = \left(\Gamma_{i,1}\cdots\Gamma_{i,r}\right).$$

We obtain that (10) is equivalent to

$$\begin{aligned} \Pi_{i,j}\omega_j &= A_i\Pi_{i,j} + B_i\Gamma_{i,j}, \\ C_{r,j} &= C_i\Pi_{i,j} + D_i\Gamma_{i,j} \end{aligned}$$

for $j = 1, \ldots, r$. This can be rewritten as

$$\begin{pmatrix} A_i - \omega_j I & B_i \\ C_i & D_i \end{pmatrix}\begin{pmatrix} \Pi_{i,j} \\ \Gamma_{i,j} \end{pmatrix} = \begin{pmatrix} 0 \\ C_{r,j} \end{pmatrix},$$

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which is solvable if

$$\operatorname{Im}\begin{pmatrix} 0 \\ C_{r,j} \end{pmatrix} \subset \operatorname{Im}\begin{pmatrix} A_i - \omega_j I & B_i \\ C_i & D_i \end{pmatrix},$$

and the latter condition is equivalent to

$$\begin{pmatrix} v' & w' \end{pmatrix}\begin{pmatrix} A_i - \omega_j I & B_i \\ C_i & D_i \end{pmatrix} = 0 \Longrightarrow \begin{pmatrix} v' & w' \end{pmatrix}\begin{pmatrix} 0 \\ C_{r,j} \end{pmatrix} = 0.$$

Because the latter is equivalent to $w'C_r e_j = 0$ where $Se_j = \omega_j e_j$, we note that this implication is exactly the condition that we have proven earlier.

The fact that we need (12) is a consequence of Corollary 3.3.1 in [35]. □

APPENDIX B: PROOF OF LEMMA 2

Proof

If we use the transformation Φ_i, from the proof of [31, Lemma 1], we know that e_i is governed by the following dynamical equations

$$\dot{\bar{x}}_i = \bar{A}_i \bar{x}_i + \bar{B}_i \sigma(u_i) := \begin{bmatrix} A_i & \bar{A}_{i12} \\ 0 & \bar{A}_{i22} \end{bmatrix} \bar{x}_i + \begin{bmatrix} B_i \\ 0 \end{bmatrix} \sigma(u_i), \tag{B.1a}$$

$$e_i = \bar{C}_i \bar{x}_i + \bar{D}_i \sigma(u_i) := \begin{bmatrix} C_i & -\bar{C}_{i2} \end{bmatrix} \bar{x}_i + D_i \sigma(u_i), \tag{B.1b}$$

where

$$\bar{A}_{i12} = \Lambda_i \begin{bmatrix} R_{i12} \\ 0 \end{bmatrix}, \quad \bar{A}_{i22} = R_{i22}, \quad \bar{C}_{i2} = C_r \Phi_i N_i'.^{\S}$$

Note that $\bar{A}_i$ of the system (B.1) is block-upper triangular. Therefore, we need to show that with the transformation $\bar{\Phi}_i$ given by (18), the system (B.1) is block-diagonal.

From (16) and (17), it is easy to show that

$$\begin{bmatrix} U_i & R_{i12} \\ 0 & R_{i22} \end{bmatrix}\begin{bmatrix} I_{q_i} & V_{i12}V_{i22}^{-1} \\ 0 & I_{r_i} \end{bmatrix} = \begin{bmatrix} I_{q_i} & V_{i12}V_{i22}^{-1} \\ 0 & I_{r_i} \end{bmatrix}\begin{bmatrix} U_i & 0 \\ 0 & R_{i22} \end{bmatrix}. \tag{B.2}$$

Now post multiplying both sides of (14) by

$$\begin{bmatrix} I_{q_i} & V_{i12}V_{i22}^{-1} \\ 0 & I_{r_i} \end{bmatrix},$$

we obtain that $S\bar{\Phi}_i = \bar{\Phi}_i \bar{R}_i$, where

$$\bar{R}_i = \begin{bmatrix} U_i & 0 \\ 0 & R_{i22} \end{bmatrix}. \tag{B.3}$$

Thus $\bar{A}_{i12} = 0$. □

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§Note that the variable $\bar{x}_{i2}$ has a sign difference from that of [32].

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