Convexity in Stochastic Cooperative Situations*

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Abstract

This paper introduces a new model concerning cooperative situations in which the payoffs are modeled by random variables. First, we study adequate preference relations of the agents. Next, we define corresponding cooperative games and we introduce and study various basic notions like an allocation, the core and marginal vectors. Furthermore, we introduce three types of convexity, namely coalitional-merge, individual-merge and marginal convexity. The relations between these definitions are studied and in particular, as opposed to the deterministic counterparts for TU games, we show that these three types of convexity are not equivalent. However, all types imply that the core of the game is nonempty and the first two types even imply that each subgame has a nonempty core. In particular, we show that the Shapley value, the average of the marginal vectors, belongs to the core of the convex game for certain types of preferences and for any type of convexity.

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1 Introduction

In many real-life situations payoffs to agents are uncertain. For example, consider two musicians, a pianist and a violinist. Each of them has a contract with a hotel to give small performances. Their payoffs consist of a small wage and the tips they receive during their performances. At the end of the month their contracts will end and both their employers offer them a new contract with the same conditions. Until now these musicians always performed separately, although recently they started studying some pieces for violin and piano together. This is because they found a (third) hotel that is willing to contract both of them. This contract says that both musicians only perform for this hotel and their individual payoffs consist of a small wage. Ten percent of all the tips they receive during their

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performances will be for the hotel and the remaining 90 percent will be divided among the pianist and the violinist. Before the end of the month the musicians have to decide whether to cooperate or not. In both cases their payoffs will be uncertain because they depend on the uncertain amount of tips to be received during performances.

Another situation with uncertain payoffs is the following. Consider a firm that goes bankrupt. An intermediary is appointed who will settle the remaining financial matters of the firm. All creditors claim their money while the intermediary finds out that the only money left in the firm is a portfolio consisting of shares and options. When the creditors agree upon a distribution of this portfolio among themselves, then this distribution will be executed. Each of them receives a small portfolio with an uncertain value because the prices at the shares and options markets change over time. This kind of situations are called bankruptcy situations and we will return to it later in this paper.

In 'classical' cooperative game theory, payoffs to coalitions of agents are known with certainty. Therefore, situations with uncertain payoffs in which the agents cannot await the realizations of these payoffs, cannot be modeled according to this theory. CHARNES and GRANOT (1973) and SUIJS, BORM, DE WAEGENAERE and TIJS (1999) introduced new models that can handle uncertain payoffs.

CHARNES and GRANOT (1973) introduced games in stochastic characteristic function form. These are games where the payoff to coalition S, $V(S)$, is allowed to be a random variable. To allocate the payoff of the grand coalition to the players, the authors suggest a two-stage procedure. In the first stage, so called prior-payoffs are promised to the agents. These prior-payoffs are determined such that there is a relatively high chance that the promises can be realized. In the second stage the realizations of the payoffs are awaited and, if necessary, the prior-payoff vector has to be adjusted to this realization. Research on this subject was continued in CHARNES and GRANOT (1976, 1977) and GRANOT (1977).

In SUIJS and BORM (1999) a different and more extensive model is studied. They consider a set A_S of actions that coalition S can take. The stochastic value of this coalition then depends on which action $a \in A_S$ is chosen and is denoted by $X_S(a)$. This, however, is not the main feature we like to stress. It is the fact that an allocation of $X_S(a)$ to the members of coalition S is described as the sum of two parts. The first part is a monetary transfer between the agents and the second part is an allocation of fractions of $X_S(a)$. Work on this model was started in SUIJS, BORM, DE WAEGENAERE and TIJS (1999) and an application in insurance can be found in SUIJS, DE WAEGENAERE and BORM (1998).

In this paper we introduce a model that, when compared to the previous two models, looks the most like the model of SUIJS et al. but there are some major differences. First, in the model of SUIJS et al., a collection $\mathcal{V}(S)$ of stochastic payoffs is assigned to each coalition S of agents. Each $X \in \mathcal{V}(S)$ is a possible stochastic payoff to coalition S. In our model, we assign a *single* random value $R(S)$ to each coalition of agents. This value contains all the information that the coalition knows about its payoff.

Second, allocations are defined differently. In the model of SUIJS et al. allocations are defined as follows. Let S be a coalition of agents and let $X \in \mathcal{V}(S)$ be a stochastic payoff for this coalition.

An allocation of X to the agents in S is represented by a pair $(d, r) \in \mathbb{R}^S \times \mathbb{R}^S$ with $\sum_{i \in S} d_i \leq 0$, $\sum_{i \in S} r_i = 1$ and $r_i \geq 0$ for all agents $i \in S$. Given such a pair (d, r), agent $i \in S$ receives the stochastic payoff $d_i + r_i X$. The second part, $r_i X$, describes the *fraction* of X that is allocated to agent i. The first part, d_i, describes the deterministic transfer payments between the agents. When $d_i \geq 0$ then agent i receives money while $d_i < 0$ means that this agent pays money. The purpose of these transfer payments is that the agents compensate among themselves for transfers of random payoffs. For example, a risk-averse agent (that is an agent who 'hates' uncertainty) who receives a large fraction of X can be compensated by the other agents if they give him an adequate positive amount d_i. The set of allocations that coalition S can obtain, contains all such allocations for all $X \in \mathcal{V}(S)$.

In our model, allocations are defined as follows. Let S be a coalition of agents. An allocation of the single random value $R(S)$ to the agents in S is a division of this stochastic payoff where each agent receives a *multiple* of $R(S)$. Given a vector $p \in \mathbb{R}^S$, $pR(S)$ is an allocation (in terms) of $R(S)$ where agent $i \in S$ receives the (possibly negative) multiple $p_i R(S)$. Such an allocation is efficient if $\sum_{i \in S} p_i = 1$. Thus, we see that the model of SUIJS et al. only allocates fractions of stochastic payoffs while our model allocates multiples of such payoffs. Furthermore, the model of SUIJS et al. allows for deterministic transfer payments while our model does not allow for this. In specific applications these payments do not always seem very realistic. If you recall the second example at the beginning of this section about creditors claiming their money from a firm that went bankrupt, then it seems very unlikely that the creditors will decide upon deterministic transfer payments among themselves once the portfolio will be distributed.

First, we set up a framework that defines how each agent compares any two stochastic payoffs that may be allocated to him. In this framework we introduce a so-called embedding map α_i for agent i that 'embeds' one stochastic payoff in the other as follows. For specific pairs of stochastic payoffs X and Y where $Y \neq 0$, $\alpha_i(X, Y)$ is the real number α such that agent i is indifferent between receiving X or αY. Our assumptions on the preferences imply that this number α is uniquely determined and so, we have for each agent a unique embedding of X into Y.

Special attention will be paid to convexity in cooperative situations with uncertain payoffs. We define three types of convexity for games corresponding to these situations. The three types are coalitional-merge convex, individual-merge convex and marginal convex. The first two are based on the marginal contributions of a coalition of agents and a single agent, respectively, while the third type, marginal convexity, is based on whether or not all the marginal vectors belong to the core of the game. We show that coalitional-merge convexity implies individual-merge convexity, which in turn implies marginal convexity. Examples show that reverse relations need not hold. In particular, each marginal convex game has a nonempty core as well as each subgame of an individual-merge convex game. Besides, we extend the definition of the Shapley value for TU games as the average of the marginal vectors to our class of stochastic cooperative games. We show that the Shapley value is an element of the core of a marginal convex game for certain types of preferences.

The remainder of this paper is organized as follows. Allocations of random variables and the preference relations of the agents over these allocations are defined in section 2. After this, we give a formal description of our model in section 3 and we extend several basic notions from deterministic TU games, like allocations, imputations, superadditivity, the core and marginal vectors, to our model. We end this section with an explicit example of a bankruptcy game to illustrate the new notions. In section 4 we introduce and study the three types of convexity as discussed above. As opposed to deterministic TU games, we show that these types are not equivalent. However, they all imply that the core of the game is nonempty and for certain types of preferences this core contains the Shapley value. Furthermore, while deterministic bankruptcy games are convex, an example indicates that bankruptcy games with an uncertain estate may or may not satisfy any of the introduced convexity types.

2 Preference relations

A complete preference relation of an agent describes which one of two alternatives this agent weakly prefers to the other, for any two alternatives. Here, the alternatives are random variables allocated to this agent. For this, we have to define allocations of random variables before we can turn our attention to the preference relations. But we will start with the probability space.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is the outcome space, $\mathcal{F}$ is a σ-algebra in Ω and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. A stochastic variable $X \in \mathbb{R}\mathcal{F}$ is a measurable function that assigns to each outcome $\omega \in \Omega$ a real number $X(\omega)$. The set of all stochastic variables X with a finite expectation is denoted by $\mathcal{L}$ and $\mathcal{L}_+$ is the set of all nonnegative stochastic variables in $\mathcal{L}$. By 0 we denote the stochastic variable that takes the value zero for sure. Note that $0 \in \mathcal{L}_+$.

A deterministic cooperative game with transferable utility, or TU game, is described by a pair (N, v) where N is the set of agents and $v : 2^N \to \mathbb{R}$ is the characteristic function assigning to each coalition $S \subset N$ a value $v(S)$ and $v(\emptyset) = 0$. If we introduce uncertainty into this model, such that coalitions of agents may not know for sure what payoff they will receive, then the payoffs will be random variables. Denote by $R(S)$ the stochastic payoff (reward) in $\mathcal{L}_+$ to coalition S. For a nonempty coalition S of agents, an *allocation* (in terms) of $R(S)$ is a distribution of multiples of $R(S)$. If $p \in \mathbb{R}^S$ then $pR(S)$ is an allocation in terms of $R(S)$ where agent $i \in S$ receives $p_i R(S)$. Such an allocation is efficient if $\sum_{i \in S} p_i = 1$. For ease of notation define $\Delta^*(S) = \{p \in \mathbb{R}^S | \sum_{i \in S} p_i = 1\}$.

Now that we know how the payoffs of the coalitions can be distributed, it is time to see how agents compare two stochastic payoffs. First we restrict ourselves to nonzero random payoffs and after that we include the zero payoffs. Let $A = \{R(S) | S \subset N,\ S \neq \emptyset,\ R(S) \neq 0\}$ be the set of all nonzero payoffs to coalitions of agents. Rename them such that $A = \{R(S_1), R(S_2), \ldots, R(S_m)\}$ for some integer number m. Because allocations are multiples of payoffs, the set of all possible stochastic payoffs restricted to the random values in A equals $B = \{pR(S_k) | p \in \mathbb{R},\ R(S_k) \in A\}$. By $\succsim_i$ we denote the preference relation of agent $i \in N$ over B. If for some stochastic payoffs X, Y it holds that $X \succsim_i Y$ then the agent weakly prefers receiving the stochastic payoff X to receiving Y while $X \succ_i Y$ means that the agent strictly prefers X to Y. If $X \succsim_i Y$ and $Y \succsim_i X$ then we write $X \sim_i Y$, the agent is indifferent between receiving X or Y. We make the following assumption about how an agent compares two payoffs in B.

Assumption 2.1 *For each agent $i \in N$ there exists a function $f^i : \mathbb{R} \to \mathbb{R}^m$, which is surjective, continuous and monotone increasing, such that*

1. $f^i_k(t)R(S_k) \succsim_i f^i_l(t')R(S_l)$ *if and only if* $t \geq t'$,
2. $f^i_k(0) = 0$

for any $k, l \in \{1, 2, \ldots, m\}$.

This kind of preferences is particularly suitable for our model, which will be presented in the next section. So, when agent i compares the payoffs $pR(S_k)$ and $qR(S_l)$ then $pR(S_k) \succsim_i qR(S_l)$ if and only if $t = (f^i_k)^{-1}(p) \geq t' = (f^i_l)^{-1}(q)$. The assumptions on f^i imply that these inverse functions exist and the condition $f^i_k(0) = 0$ is a normalization condition. One may interpret the function $(f^i_k)^{-1}$ as some kind of utility function with respect to multiples of $R(S_k)$ only.

We say that a preference relation $\succsim_i$ is *reflexive* if $X \succsim_i X$ is true for all $X \in B$. Secondly, it is *transitive* if $X \succsim_i Y$ and $Y \succsim_i Z$ implies that $X \succsim_i Z$ for any $X, Y, Z \in B$ and thirdly, it is *monotone increasing* when $pR(S_k) \succsim_i qR(S_k)$ if and only if $p \geq q$. The following theorem shows that these properties hold.

Theorem 2.2 *If a preference relation $\succsim_i$ satisfies assumption 2.1 then it is reflexive, transitive and monotone increasing.*

The proof is left to the reader. Another implication of assumption 2.1 is that $R(S_k) \succ_i 0$ for all $R(S_k) \in A$ because $1R(S_k) = R(S_k) \succ_i 0R(S_k) = 0$ if and only if $t > t'$ where $1 = f^i_k(t)$ and $0 = f^i_k(t') \Leftrightarrow t' = 0$. This is true because f^i is monotone increasing. Similarly it follows that

$$\begin{cases} pR(S_k) \succ_i 0 & \Leftrightarrow \quad p > 0 \\ pR(S_k) \sim_i 0 & \Leftrightarrow \quad p = 0 \\ pR(S_k) \prec_i 0 & \Leftrightarrow \quad p < 0. \end{cases}$$

The following example presents two preference relations that satisfy assumption 2.1.

Example 2.3 The first type of preferences we discuss here concerns expected values of random variables. Suppose that the preferences of agent i are such that $X \succsim_i Y$ if and only if $\mathrm{E}(X) \geq \mathrm{E}(Y)$ for any $X, Y \in B$, where $\mathrm{E}(X)$ is the expectation of X. We call this type of preferences *expectation-preferences*. Then $f^i_k(t) = t/\mathrm{E}(R(S_k))$ for all $k \in \{1, 2, \ldots, m\}$ makes sure that $\succsim_i$ satisfies assumption 2.1.

A second type of preferences involves quantiles of random variables. Let $u^X_{\beta_i} = \sup\{t \in \mathbb{R} | \Pr\{X \leq t\} \leq \beta_i\}$ be the β_i-quantile of X, where $0 < \beta_i < 1$ is such that $u^{R(S_k)}_{\beta_i} > 0$ for all $R(S_k) \in A$. Define the utility function U_i by $U_i(X) = u^X_{\beta_i}$ if $\mathrm{E}(X) \geq 0$ and $U_i(X) = u^X_{1-\beta_i}$ otherwise. Suppose that $X \succsim_i Y$ if and only if $U_i(X) \geq U_i(Y)$ for any $X, Y \in B$. We call this type of preferences *quantile-preferences*. The functions $f^i_k(t) = t/u^{R(S_k)}_{\beta_i}$ for all $k \in \{1, 2, \ldots, m\}$ will do the job. □

Note that for both expectation- and quantile-preferences all the functions f_k^i are *linear*, that is, $f_k^i(t) = f_k^i(1)t$ for all $i \in N, k \in \{1, \ldots, m\}$. We will return to this type of preference relations later in this section.

An important consequence of assumption 2.1 is given in the next theorem, where $B_{-0} = \{pR(S_k) \in B |\, p \neq 0\}$.

Theorem 2.4 *For all $X \in B$, $Y \in B_{-0}$ and $i \in N$ there exists a unique number $\alpha \in \mathbb{R}$ such that $X \sim_i \alpha Y$.*

Proof. Let $X \in B, Y \in B_{-0}$ and $i \in N$, then $X = pR(S_k)$ for some $p \in \mathbb{R}$, $k \in \{1, 2, \ldots, m\}$ and $Y = qR(S_l)$ for some $q \neq 0, l \in \{1, 2, \ldots, m\}$. By assumption 2.1 there exists a number $t \in \mathbb{R}$ such that $f_k^i(t) = p$. By definition of f^i it holds that $f_l^i(t)R(S_l) \sim_i f_k^i(t)R(S_k) = X$. We know that $t = (f_k^i)^{-1}(p)$ and this gives

$$\begin{aligned} X \sim_i f_l^i((f_k^i)^{-1}(p))R(S_l) &= f_l^i((f_k^i)^{-1}(p))/q \cdot qR(S_l) \\ &= f_l^i((f_k^i)^{-1}(p))/q \cdot Y. \end{aligned}$$

We conclude that $\alpha = f_l^i((f_k^i)^{-1}(p))/q$. The function f^i is monotone increasing and this implies that this number α is unique. □

To be able to keep track of which α is connected to which variables X, Y, and i we define for all agents $i \in N$ the embedding function $\alpha_i : B \times B_{-0} \to \mathbb{R}$ by $\alpha_i(X, Y) = f_l^i((f_k^i)^{-1}(p))/q$, and so, $X \sim_i \alpha_i(X, Y)Y$, where $X = pR(S_k)$ and $Y = qR(S_l), q \neq 0$. Thus, the embedding function α_i gives a complete description of the preference relation of agent i.

What happens if $R(S) = 0$ for some coalition S of agents? For all $X \in B_{-0}$ it holds that either $X \succ_i 0$ or $X \prec_i 0$. Because $\alpha R(S) = 0$ for any $\alpha \in \mathbb{R}$ it follows that for all $i \in N$ there exists no $\alpha \in \mathbb{R}$ such that $X \sim_i \alpha R(S)$. The only thing we will define in this kind of situation is $\alpha_i(0, 0) = 1$. The following example shows what the embedding functions look like for the preferences in the previous example.

Example 2.5 Consider the preference relations in the previous example and take $X \in B$, $X = pR(S_k)$, and $Y \in B_{-0}, Y = qR(S_l)$. For the expectation-preferences it holds that

$$\alpha_i(X, Y) = f_l^i((f_k^i)^{-1}(p))/q = p\mathrm{E}(R(S_k))/(q\mathrm{E}(R(S_l))) = \mathrm{E}(X)/\mathrm{E}(Y)$$

and for the quantile-preferences it follows that

$$\alpha_i(X, Y) = pu_{\beta_i}^{R(S_k)}/(qu_{\beta_i}^{R(S_l)})$$

where $0 < \beta_i < 1$ is such that $u_{\beta_i}^{R(S_l)} > 0$ for all $l \in \{1, 2, \ldots, m\}$. □

The next theorem states some nice properties of the function α_i.

Theorem 2.6 *For all $i \in N$*

1. $\alpha_i(hZ, Z) = h$ *for any* $h \in \mathbb{R}$, $Z \in B_{-0}$,

2. $\alpha_i(\alpha_i(X, Y)Y, Z) = \alpha_i(X, Z)$ *for any* $X \in B$ *and* $Y, Z \in B_{-0}$,

3. $\alpha_i(pR(S_k), qR(S_l)) = p\alpha_i(R(S_k), R(S_l))/q$ *for any* $pR(S_k) \in B$ *and* $qR(S_l) \in B_{-0}$ *if the functions* f_k^i, $k \in \{1, \dots, m\}$, *are linear.*

Proof. For the first item, let $h \in \mathbb{R}$ and $Z \in B_{-0}$, then $hZ \sim_i \alpha_i(hZ, Z)Z$ by definition of α_i. From theorem 2.2 we know that $\succsim_i$ is monotone increasing and this implies that $h = \alpha_i(hZ, Z)$.

To prove the second item, let $X \in B$ and $Y, Z \in B_{-0}$. Then $X \sim_i \alpha_i(X, Y)Y$ and $\alpha_i(X, Y)Y \sim_i \alpha_i(\alpha_i(X, Y)Y, Z)Z$ by definition of α_i. According to theorem 2.2, $\succsim_i$ is transitive, and so, $X \sim_i \alpha_i(\alpha_i(X, Y)Y, Z)Z$. Hence, $\alpha_i(\alpha_i(X, Y)Y, Z) = \alpha_i(X, Z)$ because $\succsim_i$ is also monotone increasing.

Finally, let $pR(S_k) \in B$ and $qR(S_l) \in B_{-0}$. If the functions f_k^i are linear then

$$\begin{aligned}\alpha_i(pR(S_k), qR(S_l)) &= f_l^i((f_k^i)^{-1}(p))/q = pf_l^i((f_k^i)^{-1}(1))/q \\ &= p\alpha_i(R(S_k), R(S_l))/q,\end{aligned}$$

which concludes the proof. □

3 The model

In this section we will describe our model in more detail. We define the corresponding games where coalitions of players receive random values. After this we extend some basic definitions in cooperative game theory to our model and illustrate these concepts with an example of a bankruptcy game.

Given a set of agents $N = \{1, \dots, n\}$, variables $R(S) \in \mathcal{L}_+$ and preference relations $\succsim_i$ for all $i \in N$, a game (N, R, α) is a cooperative game where N denotes the set of players, the map R assigns to each nonempty coalition in N a random value in $\mathcal{L}_+$, and $\alpha = (\alpha_i)_{i \in N}$ with α_i the previously defined function that describes what multiple of one stochastic variable player i finds equivalent to another stochastic variable.

We will now extend various notions from deterministic TU games to cooperative games with stochastic payoffs. Recall from the previous section that if $p \in \mathbb{R}^S$ then $pR(S)$ is an allocation (in terms) of $R(S)$ and such an allocation is efficient if $p \in \Delta^*(S) = \{p \in \mathbb{R}^S | \sum_{i \in S} p_i = 1\}$. An allocation $pR(S)$ for coalition S is *individual rational* if $p_i R(S) \succsim_i R(\{i\})$ for all $i \in S$. We will denote the set of all efficient individual rational allocations of $R(S)$ for coalition S by $IR(S)$.

An allocation of $R(N)$ is called an *imputation* if it is individual rational and efficient. The *imputation set* $I(N, R, \alpha)$ is the set of all imputations.

$$I(N, R, \alpha) = \{\{p_i R(N)\}_{i \in N} | p \in \Delta^*(N);\ p_i R(N) \succsim_i R(\{i\}) \text{ for all } i \in N\}$$

Note that $I(N, R, \alpha) = IR(N)$. Depending upon the random values of the various coalitions we can say something more about the structure of the imputation set.

Lemma 3.1 *$I(N,R,\alpha) \subset \{pR(N) | \, p \in \Delta^*(N), \; p_i \geq 0$ for all $i \in N\}$ if $R(N) \neq 0$. If $R(N) = 0$ and $R(\{i\}) = 0$ for all $i \in N$ then $I(N,R,\alpha) = \{pR(N) | \, p \in \Delta^*(N)\}$. If $R(N) = 0$ and $R(\{i\}) \neq 0$ for some $i \in N$ then $I(N,R,\alpha) = \emptyset$.*

Proof. Let $R(N) \neq 0$. If $I(N,R,\alpha) = \emptyset$ then we are done. Otherwise take an imputation $pR(N)$. Then $p_i R(N) \succsim_i R(\{i\})$, which is equivalent to $p_i \geq \alpha_i(R(\{i\}), R(N)) \geq 0$ where the first inequality follows from monotonicity of the preferences and the second one from $R(N) \neq 0$ and $R(\{i\}) \in \mathcal{L}_+$. The two remaining statements are trivial. □

The game (N,R,α) is *superadditive* if for all $S, T \subset N$ with $S \cap T = \emptyset$, $S \neq \emptyset$ and $T \neq \emptyset$, for all $pR(S) \in IR(S)$ and for all $qR(T) \in IR(T)$ there exists an allocation $rR(S \cup T)$, $r \in \Delta^*(S \cup T)$, such that all players are weakly better off:

$$\begin{cases} r_i R(S \cup T) \succsim_i p_i R(S) & \text{for all } i \in S, \\ r_i R(S \cup T) \succsim_i q_i R(T) & \text{for all } i \in T. \end{cases}$$

Notice that $rR(S \cup T) \in IR(S \cup T)$. We also could have formulated superadditivity in the following way: for all $T_1, T_2, \ldots, T_k \subset N$, $k \geq 2$, such that $T_i \neq \emptyset$ and $T_i \cap T_j = \emptyset$ for all $i \neq j$, and for all $p^i R(T_i) \in IR(T_i)$, $i = 1, 2, \ldots, k$ there exists an allocation $rR(\cup_{i=1}^k T_i)$, $r \in \Delta^*(\cup_{i=1}^k T_i)$, such that

$$r_j R(\cup_{i=1}^k T_i) \succsim_i p_j^i R(T_i) \text{ for all } j \in T_i, \; i = 1, 2, \ldots, k, \tag{1}$$

all players are weakly better off. Obviously, this alternative definition implies superadditivity. The other way around is also true, as is shown hereafter.

Lemma 3.2 *If a game (N,R,α) is superadditive then it satisfies the alternative definition (1).*

Proof. Assume that the game (N,R,α) is superadditive. Then condition (1) is satisfied for $k = 2$. We will use induction on the number k of coalitions to show that this game satisfies condition (1). So, suppose that (1) is satisfied for k coalitions, $2 \leq k < n$ with n the total number of players. Take coalitions $T_1, T_2, \ldots, T_{k+1} \subset N$ such that $T_i \neq \emptyset$ and $T_i \cap T_j = \emptyset$ for $i \neq j$ and let $p^i R(T_i) \in IR(T_i)$ for $i = 1, 2, \ldots, k+1$. By induction there exists an allocation $rR(\cup_{i=1}^k T_i)$, $r \in \Delta^*(\cup_{i=1}^k T_i)$, such that

$$r_j R(\cup_{i=1}^k T_i) \succsim_j p_j^i R(T_i) \text{ for all } j \in T_i, \; i = 1, 2, \ldots, k. \tag{2}$$

Note that $p^i R(T_i) \in IR(T_i)$ implies that $rR(\cup_{i=1}^k T_i) \in IR(\cup_{i=1}^k T_i)$. It follows from superadditivity and $p^{k+1} R(T_{k+1}) \in IR(T_{k+1})$ that there exists an allocation $sR(\cup_{i=1}^{k+1} T_i)$, $s \in \Delta^*(\cup_{i=1}^{k+1} T_i)$ such that

$$\begin{cases} s_j R(\cup_{i=1}^{k+1} T_i) \succsim_j r_j R(\cup_{i=1}^k T_i) & \text{for all } j \in \cup_{i=1}^k T_i \\ s_j R(\cup_{i=1}^{k+1} T_i) \succsim_j p_j^{k+1} R(T_{k+1}) & \text{for all } j \in T_{k+1}. \end{cases}$$

Transitivity of the preference relations and (2) imply that

$$s_j R(\cup_{i=1}^{k+1} T_i) \succsim_j p_j^i R(T_i) \text{ for all } j \in T_i, \; i = 1, 2, \ldots, k+1.$$

Hence, (1) is satisfied for $k+1$ coalitions. $\square$

This result implies the following relation between superadditive games and the sets $IR(S)$ of individual rational allocations.

Lemma 3.3 *If a game (N, R, α) is superadditive then $IR(S) \neq \emptyset$ for all nonempty coalitions S.*

Proof. Let the game (N, R, α) be superadditive and take a coalition $S \subset N$, $S \neq \emptyset$. According to lemma 3.2 the alternative definition (1) is satisfied. Let s be the number of players in S and define $T_i = \{i\}$ for $i = 1, \ldots, s$. Then $IR(T_i) = IR(\{i\}) = \{R(\{i\})\}$ and $\cup_{i=1}^{s} T_i = S$. By (1) there exists an allocation $rR(S)$, $r \in \Delta^*(S)$, such that $r_i R(S) \succsim_i R(\{i\})$ for all $i \in S$. Thus $rR(S) \in IR(S)$. $\square$

For all $S \subset N$, $S \neq \emptyset$, the set $\text{dom}(S)$ contains the allocations of $R(N)$ restricted to coalition S that are dominated by this coalition, i.e., there exists an allocation $qR(S)$, $q \in \Delta^*(S)$, that is strictly preferred by all members of S.

$$\text{dom}(S) = \Big\{ pR(N) \,\Big|\, p \in \mathbb{R}^S,\ \exists q \in \Delta^*(S) :\ q_i R(S) \succ_i p_i R(N) \text{ for all } i \in S \Big\}$$

The set of allocations that are not dominated by some coalition can take many forms, depending upon the random values. Let D be a set of allocations of $R(N)$ that satisfy some restrictions. We say that D is a *convex set of allocations* if and only if the set $\{p |\, pR(N) \in D\}$ is a convex set in $\mathbb{R}^N$. Furthermore let $p_S = \{p_i\}_{i \in S}$ be the restriction of $p \in \mathbb{R}^N$ to coalition S.

Lemma 3.4 *Let $S \subset N$ be a nonempty set of players. Then*

$$p_S R(N) \notin \mathit{dom}(S) \Leftrightarrow \begin{cases} p \in \mathbb{R}^N & \text{if } R(S) = 0 \text{ and } R(N) = 0, \\ p_i \geq 0 \text{ for some } i \in S & \text{if } R(S) = 0 \text{ and } R(N) \neq 0. \end{cases}$$

If $R(S) \neq 0$ and $R(N) = 0$ then $p_S R(N) \in \mathit{dom}(S)$ for all $p \in \mathbb{R}^N$. Furthermore, if all the functions f_k^i are linear, $R(S) \neq 0$ and $R(N) \neq 0$ then the set $\{pR(N) |\, p_S R(N) \notin \mathit{dom}(S)\}$ is convex.

Proof. We only prove the last statement. The remaining parts of the lemma are trivial.

Let (N, R, α) be a cooperative game with stochastic payoffs and let $S \subset N$ be a nonempty set of players. Assume that $R(S) \neq 0$ and $R(N) \neq 0$. Then $p_S R(N) \notin \text{dom}(S)$ if and only if there exists no vector $q \in \Delta^*(S)$ such that $q_i R(S) \succ_i p_i R(N)$ for all $i \in S$. By monotonicity of the preferences we have

$$\nexists q \in \Delta^*(S) \ :\ \alpha_i(q_i R(S), R(N)) > p_i \text{ for all } i \in S.$$

By property 3 in theorem 2.6 this is equivalent to

$$\nexists q \in \Delta^*(S) \ :\ q_i \alpha_i(R(S), R(N)) > p_i \text{ for all } i \in S,$$

so,

$$\nexists q \in \Delta^*(S) \; : \; q_i > p_i/\alpha_i(R(S), R(N)) \text{ for all } i \in S.$$

Hence,

$$\sum_{i \in S} p_i/\alpha_i(R(S), R(N)) \geq 1.$$

Define $h \in \mathbb{R}^S$ by $h_i = 1/\alpha_i(R(S), R(N))$. Then $h_i > 0$ for all $i \in S$ and $p_S R(N) \notin \text{dom}(S)$ if and only if $\sum_{i \in S} h_i p_i \geq 1$. We conclude that the set $\{pR(N) | \, p_S R(N) \notin \text{dom}(S)\}$ is convex. □

The *core* of (N, R, α), denoted by $C(N, R, \alpha)$, consists of all payoff vectors attainable for the grand coalition that are not dominated by any coalition S, that is

$$C(N, R, \alpha) = \{pR(N) \, | p \in \Delta^*(N), \; p_S R(N) \notin \text{dom}(S) \text{ for all } S \subset N, \; S \neq \emptyset\} \, .$$

Because

$$p_i R(N) \notin \text{dom}(\{i\}) \Leftrightarrow p_i R(N) \succsim_i R(\{i\})$$

holds for all $i \in N$, the core is a subset of the imputation set, $C(N, R, \alpha) \subset I(N, R, \alpha)$, for all games (N, R, α). In particular, $C(N, R, \alpha) = I(N, R, \alpha)$ for 2-person games. Using the results in the theorems 3.1 and 3.4 we can show that the core is convex if the functions f_k^i of every agent are linear.

Theorem 3.5 *Let (N, R, α) be a cooperative game with stochastic payoffs where all the functions f_k^i are linear. Then the core $C(N, R, \alpha)$ is a convex set.*

Proof. Let $\text{undom}(S) = \{pR(N) | \, p \in \Delta^*(N), \; p_S R(N) \notin \text{dom}(S)\}$ be the set of efficient allocations of $R(N)$ that are not dominated by coalition S. Then

$$I(N, R, \alpha) = \cap_{i \in N} \text{ undom}(\{i\})$$

and this implies that

$$C(N, R, \alpha) = \cap_{S \subset N, \, S \neq \emptyset} \text{ undom}(S) = I(N, R, \alpha) \cap \left(\cap_{S \subset N, \, |S| \geq 2} \text{ undom}(S) \right).$$

Firstly, suppose that $R(N) = 0$. If $R(S) \neq 0$ for some $S \subset N$ then $\text{undom}(S) = \emptyset$ according to theorem 3.4 and by this $C(N, R, \alpha) = \emptyset$. If $R(S) = 0$ for all $S \subset N$ then according to the same theorem $\text{undom}(S) = \Delta^*(N)$ for all $S \subset N$ and so $C(N, R, \alpha) = \Delta^*(N)$, which is a convex set.

Secondly, if $R(N) \neq 0$ then $\text{undom}(\{i\}) = \{pR(N) | \, p \in \Delta^*(N), \; p_i R(N) \succsim_i R(\{i\})\} = \{pR(N) | \, p \in \Delta^*(N), \; p_i \geq \alpha_i(R(\{i\}), R(N))\}$ is a convex set for all $i \in N$ and so is $I(N, R, \alpha)$. If $R(S) = 0$ for some $S \subset N$ then $\text{undom}(S) = \{pR(N) | \, p \in \Delta^*(N), \; p_i \geq 0 \text{ for some } i \in S\}$ according to theorem 3.4 and by theorem 3.1 it follows that $\text{undom}(S) \supset I(N, R, \alpha)$. This implies that $\text{undom}(S) \cap I(N, R, \alpha) = I(N, R, \alpha)$, which is a convex set. If $R(S) \neq 0$ then it follows from theorem 3.4 that undom(S) is a convex set. We conclude that also in case $R(N) \neq 0$ it holds that $C(N, R, \alpha)$ is a convex set. $\square$

A permutation σ of the players in N is a function from $\{1, 2, \ldots, n\}$ to N and $\sigma(i)$ denotes which player in N is at position i. Let $\Pi(N)$ be the set of all permutations of N. Denote by $S_i^\sigma = \{\sigma(k) | \, k \leq i\}$ the set of the first i players according to permutation σ, $i \in \{1, 2, \ldots, n\}$, and let $S_0^\sigma = \emptyset$. In a deterministic TU game (N, v) the marginal vector $m^\sigma(v)$ is defined by

$$m_{\sigma(k)}^\sigma(v) = v(S_k^\sigma) - v(S_{k-1}^\sigma)(= v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\}))$$

for each $k \in \{1, 2, \ldots, n\}$.

In cooperative games with stochastic payoffs *marginal vectors* can be defined in a similar way. For this we need the following assumption on cooperative games with stochastic payoffs.

Assumption 3.6 *If $R(T) = 0$ for some coalition $T \subset N$ then $R(S) = 0$ for all $S \subset T$, $S \neq \emptyset$.*

The first player according to σ, i.e., $\sigma(1)$, receives $Y_{\sigma(1)}^\sigma = R(\{\sigma(1)\})$. If the second player, $\sigma(2)$, joins then the two players together can get $R(S_2^\sigma)$. By assumption 3.6 $\alpha_{\sigma(1)}(Y_{\sigma(1)}^\sigma, R(S_2^\sigma))$ exists and because $Y_{\sigma(1)}^\sigma \sim_{\sigma(1)} \alpha_{\sigma(1)}(Y_{\sigma(1)}^\sigma, R(S_2^\sigma))R(S_2^\sigma)$ the marginal contribution $R_{\sigma(2)}^\sigma$ of player $\sigma(2)$ to coalition S_1^σ is

$$\begin{aligned} Y_{\sigma(2)}^\sigma &= R(S_2^\sigma) - \alpha_{\sigma(1)}(Y_{\sigma(1)}^\sigma, R(S_2^\sigma))R(S_2^\sigma) \\ &= \left[1 - \alpha_{\sigma(1)}(Y_{\sigma(1)}^\sigma, R(S_2^\sigma))\right] R(S_2^\sigma). \end{aligned}$$

Similarly, the marginal contribution of the third player is

$$\begin{aligned} Y_{\sigma(3)}^\sigma &= R(S_3^\sigma) - \alpha_{\sigma(1)}(Y_{\sigma(1)}^\sigma, R(S_3^\sigma))R(S_3^\sigma) - \alpha_{\sigma(2)}(Y_{\sigma(2)}^\sigma, R(S_3^\sigma))R(S_3^\sigma) \\ &= \left[1 - \sum_{k=1}^{2} \alpha_{\sigma(k)}(Y_{\sigma(k)}^\sigma, R(S_3^\sigma))\right] R(S_3^\sigma) \end{aligned}$$

and the marginal contribution of the i^{th} player, $\sigma(i)$, to coalition S_{i-1}^σ is

$$Y_{\sigma(i)}^\sigma = \left[1 - \sum_{k=1}^{i-1} \alpha_{\sigma(k)}(Y_{\sigma(k)}^\sigma, R(S_i^\sigma))\right] R(S_i^\sigma)$$

for all $i \in \{1, 2, \ldots, n\}$. Then the marginal vector M^σ is defined by

$$M_{\sigma(i)}^\sigma = \alpha_{\sigma(i)}(Y_{\sigma(i)}^\sigma, R(N))R(N)$$

for $i = 1, 2, \ldots, n$, and so, this marginal vector is an efficient allocation of $R(N)$. Based on these marginal vectors we define the *Shapley value* ϕ as the average of the $n!$ marginal vectors,

$$\phi = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} M^\sigma$$

just like its counterpart for deterministic TU-games (cf. SHAPLEY (1953)).

To conclude this section, we give an example of a bankruptcy game that illustrates the concepts introduced in this section. In deterministic bankruptcy situations, an estate $e \geq 0$ has to be divided among the agents in N. Agent $i \in N$ claims the amount $d_i \geq 0$ and the total amount claimed exceeds the estate, $\sum_{i\in N} d_i \geq e$. The value of a coalition $S \subset N$ in the corresponding game is given by (cf. O'NEILL (1982))

$$v(S) = \max\left\{e - \sum_{i\in N\setminus S} d_i,\, 0\right\}.$$

Situations where the estate e is not known with certainty may also occur. One can think for example of the following. A widower just passed away and since none of his children wants to have any of his properties they will all be sold at some future point in time. However, at present each child claims a deterministic part of the total stochastic revenue. Our model offers a way to 'translate' these deterministic claims in a justifiable way into multiples of the total amount of properties (and hence into multiples of the eventual realized revenue) without having to await the specific outcome of the property sale.

To model these situations as a cooperative game with stochastic payoffs, denote the uncertain estate by $E \in \mathcal{L}_+$ and the deterministic claim of agent i by d_i. These claims are such that they always exceed the estate, that is, the event $E \leq \sum_{i\in N} d_i$ takes place with probability 1. Let N be the set of claimants. Then

$$R(S) = \max\left\{E - \sum_{i\in N\setminus S} d_i,\, 0\right\}$$

is the payoff to the coalition $S \subset N$ of claimants.

Example 3.7 Consider the following bankruptcy situation. There are three agents, $N = \{1, 2, 3\}$, and their claims on the estate are $d_1 = 200$, $d_2 = 180$, $d_3 = 100$. The estate E equals

$$E = \begin{cases} 200 & \text{w.p. } 1/4, \\ 300 & \text{w.p. } 1/2, \\ 400 & \text{w.p. } 1/4, \end{cases}$$

where w.p. means 'with probability'. From $R(S) = \max\{E - \sum_{i\in N\setminus S} d_i, 0\}$ it follows that the values of the various coalitions are

$$R(\{1\}) = \max\{E - 280, 0\} = \begin{cases} 0 & \text{w.p. } 1/4 \\ 20 & \text{w.p. } 1/2 \\ 120 & \text{w.p. } 1/4 \end{cases}$$

$$R(\{2\}) = \max\{E - 300, 0\} = \begin{cases} 0 & \text{w.p. } 3/4 \\ 100 & \text{w.p. } 1/4 \end{cases}$$

and so on. We notice that $R(N) = \max\{E, 0\} = E$. The preference relations of the players are as follows. Player 1 has expectation-preferences and the players 2 and 3 have quantile-preferences with $\beta_2 = 0.75$ and $\beta_3 = 0.9$. Thus for player 1 it holds that for all $X, Y \in B$ $X \succsim_1 Y$ if and only if $\mathrm{E}(X) \geq \mathrm{E}(Y)$. In section 2 we showed that $\alpha_1(X,Y) = \mathrm{E}(X)/\mathrm{E}(Y)$ for all $X \in B$, $Y \in B_{-0}$. For players 2 and 3 and for all $X, Y \in B$ it holds that $X \succsim_i Y$ if and only if $U_i(X) \geq U_i(Y)$ with $U_i(X) = u^X_{\beta_i}$ if $\mathrm{E}(X) \geq 0$ and $U_i(X) = u^X_{1-\beta_i}$ otherwise, for $i = 2, 3$. Then $\alpha_i(X,Y) = pu^{R(S_k)}_{\beta_i}/(qu^{R(S_l)}_{\beta_i})$ for all $X \in B$, $Y \in B_{-0}$ such that $X = pR(S_k)$ and $Y = qR(S_l)$. Because $\beta_i \geq 0.75$ for $i = 2, 3$ it holds that $u^{R(S)}_{\beta_i} > 0$ for all $S \subset N$, $S \neq \emptyset$.

All individual rational allocations are in the set

$$I(N,R,\alpha) = \{pR(N)\,|\,p \in \Delta^*(N),\ p_1 \geq 2/15,\ p_2 \geq 1/4,\ p_3 \geq 1/20\}$$

and the core equals

$$C(N,R,\alpha) = \left\{ pR(N) \in I(N,R,\alpha) \,\middle|\, \begin{array}{l} 9p_1 + 8p_2 \geq 6,\ 11p_1 + 8p_3 \geq 22/5, \\ p_2 + p_3 \geq 1/2 \end{array} \right\}.$$

Next, we will calculate the six marginal vectors. Let $\sigma_1 = (1,2,3)$. Then player $\sigma(1)$=1 receives $R(\{1\})$. Player 2 is the second player according to σ and because $R(\{1\}) \sim_1 1/5R(\{1,2\})$, his marginal contribution to coalition $\{1\}$ is $(1-1/5)R(\{1,2\}) = 4/5R(\{1,2\})$. From $R(\{1\}) \sim_1 2/15R(N)$ and $4/5R(\{1,2\}) \sim_2 3/5R(N)$ it follows that the marginal contribution of player 3 to coalition $\{1,2\}$ equals $(1-2/15-3/5)R(N) = 4/15R(N)$. Thus $M^{\sigma_1} = (2/15, 3/5, 4/15)R(N)$. The other five marginal vectors are as follows.

$$\begin{array}{ll}
\sigma_2 = (1,3,2) & M^{\sigma_2} = (2/15, 1/2, 11/30)R(N) \\
\sigma_3 = (2,1,3) & M^{\sigma_3} = (4/9, 1/4, 11/36)R(N) \\
\sigma_4 = (2,3,1) & M^{\sigma_4} = (1/2, 1/4, 1/4)R(N) \\
\sigma_5 = (3,1,2) & M^{\sigma_5} = (4/11, 129/220, 1/20)R(N) \\
\sigma_6 = (3,2,1) & M^{\sigma_6} = (1/2, 9/20, 1/20)R(N)
\end{array}$$

It is easy to check that M^{σ_i} belongs to the core for $i = 3,4,5,6$. The other marginal vectors, M^{σ_1} and M^{σ_2}, only belong to the imputation set. The Shapley value ϕ, which is the average of the six marginal vectors, equals $\phi = (1027/2970, 29/66, 29/135)R(N)$. □

4 Three types of convexity

The following three statements about a deterministic TU game (N, v) are equivalent (cf. SHAPLEY (1971) and ICHIISHI (1981)).

i. For all $U \subset N$ and for all $S \subset T \subset N \setminus U$ it holds that $v(S \cup U) - v(S) \leq v(T \cup U) - v(T)$.

ii. For all $i \in N$ and for all $S \subset T \subset N \setminus \{i\}$ it holds that $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$.

iii. All $n!$ marginal vectors m^σ of (N, v) belong to the core $C(v)$.

A game (N, v) that satisfies these statements is called a convex game. Based on these statements we define three types of convexity for cooperative games with stochastic payoffs.

Similar to SUIJS and BORM (1999), statement i can be interpreted as follows. The marginal contribution of coalition U to coalition S, $v(S \cup U) - v(S)$, is smaller than the contribution of U to T, $v(T \cup U) - v(T)$. Thus, when allocations of $v(S)$, $v(T)$ and $v(S \cup U)$ are proposed and if coalition S is willing to let U join, that is, the members of S get from $v(S \cup U)$ at least as much as what they get from $v(S)$, then there exists an allocation of $v(T \cup U)$ that makes all players in $T \cup U$ better off. The players in T get at least as much from $v(T \cup U)$ as from $v(T)$ and the players in U get at least as much from $v(T \cup U)$ as from $v(S \cup U)$. If we take into account that players will only consider individual rational allocations, then we can define a first kind of convexity as follows.

A cooperative game with stochastic payoffs is called *coalitional-merge convex* if and only if it is superadditive and if for all $U \subset N$, $U \neq \emptyset$, for all $S \subset T \subset N \setminus U$ such that $S \neq \emptyset$ and $S \neq T$, for all $pR(S) \in IR(S)$, for all $qR(T) \in IR(T)$ and for all $rR(S \cup U) \in IR(S \cup U)$ such that $r_i R(S \cup U) \succsim_i p_i R(S)$ for all $i \in S$, there exists an allocation $sR(T \cup U)$, $s \in \Delta^*(T \cup U)$, such that

$$\begin{cases} s_i R(T \cup U) \succsim_i q_i R(T) & \text{for all } i \in T, \\ s_i R(T \cup U) \succsim_i r_i R(S \cup U) & \text{for all } i \in U. \end{cases}$$

Notice that if we allow for $S = \emptyset$ in the second part of this definition and will define $R(\emptyset) = 0$ and $IR(\emptyset) = \emptyset$, then that part implies superadditivity. Thus we can drop the first part of the definition. In our opinion, the present definition allows for a better interpretation without ad hoc definitions for the empty set. This is why we prefer this bipartite definition.

If we restrict ourselves to $U = \{i\}$ for all $i \in N$ then we arrive at a second type of convexity, which is related to statement *ii*. For the same reason as above this definition is split into two parts. A cooperative game with stochastic payoffs is called *individual-merge convex* if and only if the following two conditions hold. In the first place, for all $i \in N$, for all $T \subset N \setminus \{i\}$ such that $T \neq \emptyset$ and for all $qR(T) \in IR(T)$ there exists an allocation $sR(T \cup \{i\})$, $s \in \Delta^*(T \cup \{i\})$, such that

$$\begin{cases} s_j R(T \cup \{i\}) \succsim_j q_j R(T) & \text{for all } j \in T, \\ s_i R(T \cup \{i\}) \succsim_i R(\{i\}). \end{cases}$$

Secondly, for all $i \in N$, for all $S \subset T \subset N \setminus \{i\}$ such that $S \neq \emptyset$ and $S \neq T$, for all $pR(S) \in IR(S)$, for all $qR(T) \in IR(T)$ and for all $rR(S \cup \{i\}) \in IR(S \cup \{i\})$ such that $r_j R(S \cup \{i\}) \succsim_j p_j R(S)$ for all $j \in S$, there exists an allocation $sR(T \cup \{i\})$, $s \in \Delta^*(T \cup \{i\})$, such that

$$\begin{cases} s_j R(T \cup \{i\}) \succsim_j q_j R(T) & \text{for all } j \in T, \\ s_i R(T \cup \{i\}) \succsim_i r_i R(S \cup \{i\}). \end{cases}$$

Finally, we call a cooperative game with stochastic payoffs *marginal convex* if and only if all its marginal vectors M^σ belong to its core. This provides a sufficient condition for the Shapley value to belong to the core when each player either has expectation- or quantile-preferences.

Theorem 4.1 *Let (N, R, α) be a marginal convex game where all the functions f_k^i are linear. Then the Shapley value belongs to the core $C(N, R, \alpha)$.*

Proof. According to theorem 3.5 the core $C(N, R, \alpha)$ is a convex set. Because all the marginal vectors belong to the core, so does their average, the Shapley value. $\square$

When we consider other types of preferences then this result need not hold, as is shown in the next example.

Example 4.2 Consider the game (N, R, α) with $N = \{1, 2, 3\}$, $R(\{1\}) = R(\{2\}) = R(\{3\}) = 0$, $R(\{1, 2\}) = R(\{1, 3\}) = R(\{2, 3\}) = 1$ and $R(N) \sim U([2, 3])$, that is, $R(N)$ is uniformly distributed over the interval [2,3]. The players 1 and 3 have expectation preferences and the preference relation of player 2 is represented by the function

$$f^2(t) = \begin{cases} (t, t, t, t^{1/6}/2) & , t \geq 0, \\ (t, t, t, 2t/5) & , t < 0. \end{cases}$$

The coordinates of f^2 correspond to the coalitions $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and N, respectively. The core of this game,

$$C(N, R, \alpha) = \left\{ pR(N) \,\middle|\, \begin{array}{l} p \in \Delta^*(N),\ p_1 \geq 0,\ p_2 \geq 0,\ p_3 \geq 0,\ 5p_1 + 128(p_2)^6 \geq 2, \\ 5p_1 + 5p_3 \geq 2,\ 128(p_2)^6 + 5p_3 \geq 2 \end{array} \right\},$$

consists of two disjoint sets in $\Delta^*(N)$. It contains all the marginal vectors and therefore this game is marginal convex. Nevertheless, the Shapley value $\phi = (19/60, 11/30, 19/60)R(N)$ is not an element of the core since it belongs to both dom$(\{1, 2\})$ and dom$(\{2, 3\})$. $\square$

From the definitions it follows immediately that a coalitional-merge convex game is superadditive. If there exists a coalition $S \neq \emptyset$ such that $IR(S) = \emptyset$ then the game (N, R, α) is not superadditive by lemma 3.3 and hence it is not coalitional-merge convex. The following theorem states a similar result with respect to marginal convexity.

Theorem 4.3 *If there exists a coalition $S \neq \emptyset$ with $IR(S) = \emptyset$ then the game (N, R, α) is not marginal convex.*

Proof. Recall that

$$IR(S) = \{pR(S) \,|\, p \in \Delta^*(S),\ p_i R(S) \succsim_i R(\{i\}) \text{ for all } i \in S\}.$$

Let $S \neq \emptyset$ be a coalition with $IR(S) = \emptyset$ such that $IR(T) \neq \emptyset$ for all subsets T of S. Note that S should contain at least two players since $IR(\{i\}) = \{R(\{i\})\} \neq \emptyset$ for all $i \in N$. Without loss of generality assume that $S = \{1, 2, \ldots, s\}$. Let σ be a permutation of N such that $\sigma(i) = i$. We have seen before that the marginal vector M^σ is defined by

$$M^\sigma_{\sigma(j)} = \alpha_{\sigma(j)}(Y^\sigma_{\sigma(j)}, R(N))R(N)$$

for all $j \in N$ where

$$Y^\sigma_{\sigma(j)} = \left[1 - \sum_{k=1}^{j-1} \alpha_{\sigma(k)}(Y^\sigma_{\sigma(k)}, R(S^\sigma_{\sigma(j)}))\right] R(S^\sigma_{\sigma(j)}).$$

coalitional-merge convex

$\Downarrow$

individual-merge convex

$\Downarrow$

marginal convex

Figure 4.1: Relations between the three types of convexity.

If $Y_j^\sigma \prec_j R(\{j\})$ for some $j < s$ then $M_j^\sigma \prec_j R(\{j\})$ because $M_j^\sigma \sim_j Y_j^\sigma$. Thus $M^\sigma \notin I(N, R, \alpha)$. Otherwise, if $R_j^\sigma \succsim_j R(\{j\})$ for all $j < s$ then $IR(S) = \emptyset$ implies that $Y_s^\sigma \prec_s R(\{s\})$. From $M_s^\sigma \sim_s Y_s^\sigma$ we obtain, once again, that $M^\sigma \notin I(N, R, \alpha)$. Consequently, $M^\sigma \notin C(N, R, \alpha)$ since $I(N, R, \alpha) \supset C(N, R, \alpha)$. We conclude that this game is not marginal convex. $\square$

Our definitions of convexity are not equivalent for cooperative games with stochastic payoffs, while the corresponding notions are equivalent for deterministic TU games. Figure 4.1 shows the relations between the three definitions. The latter relation, individual-merge convex games are marginal convex, is shown in the next theorem.

Theorem 4.4 *Let (N, R, α) be a cooperative game with stochastic payoffs. If it is individual-merge convex then it is marginal convex.*

Proof. Let (N, R, α) be an individual-merge convex game and take a permutation $\sigma \in \Pi(N)$. Without loss of generality assume that $\sigma(i) = i$ for all $i \in N$. Furthermore, let $Z^{\sigma,k}$ be an efficient allocation of $R(\{1, \ldots, k\})$ defined by $Z_i^{\sigma,k} = \alpha_i(Y_i^\sigma, R(\{1, \ldots, k\}))R(\{1, \ldots, k\})$ for $i = 1, 2, \ldots, k$ and $k = 1, 2, \ldots, n$. Notice that $Z^{\sigma,n} = M^\sigma$. We show that $Z^{\sigma,k}$ is a core-element of the subgame Γ^k with player set $\{1, 2, \ldots, k\}$ by induction on k.

If $k = 1$ then it is clear that $Z^{\sigma,1} \in C(\Gamma^1)$. Next, assume that $Z^{\sigma,k} \in C(\Gamma^k)$ for $k = 1, 2, \ldots, m-1$ where $m \leq n$. We have to prove that $Z^{\sigma,m} \in C(\Gamma^m)$. Consider a coalition $S \subset \{1, 2, \ldots, m-1\}$. Then it follows from $Z^{\sigma,m-1} \in C(\Gamma^{m-1})$ and $Z_j^{\sigma,m} \sim_j Z_j^{\sigma,m-1}$ for all $j \in S_{m-1}^\sigma$ that $\{Z_j^{\sigma,m}\}_{j \in S} \notin \mathrm{dom}(S)$. Therefore, coalition S has no incentives to leave the coalition $\{1, 2, \ldots, m\}$.

Next, we show that also the coalition $S \cup \{m\}$ has no incentive to leave the coalition $\{1, \ldots, m\}$ if $Z^{\sigma,m}$ is allocated. Let $pR(S) \in IR(S)$ be such that $\sum_{j \in S} \alpha_j(p_j R(S), R(S \cup \{m\}))$ is minimized. Define $r_j = \alpha_j(p_j R(S), R(S \cup \{m\}))$ then $r_j R(S \cup \{m\}) \sim_j p_j R(S)$ for all $j \in S$ and $r_m := 1 - \sum_{j \in S} r_j$ is as large as possible. So, due to monotonicity of the preferences, $r_m R(S \cup \{m\})$ is the best payoff player m can expect when cooperating with coalition S. Let $T = \{1, \ldots, m-1\}$ and $i = m$, then $T \cup \{i\} = \{1, \ldots, m\}$. Because $Z^{\sigma,m-1} \in C(\Gamma^{m-1}) \subset I(\Gamma^{m-1})$ and $I(\Gamma^{m-1}) = IR(\{1, \ldots, m-1\})$ it holds that $Z^{\sigma,m-1} \in IR(T)$. Since the game (N, R, α) is individual-merge convex there exists an allocation $sR(\{1, \ldots, m\})$, $s \in \Delta^*(\{1, \ldots, m\})$, such that

$$\begin{cases} s_j R(\{1, \ldots, m\}) \succsim_j Z_j^{\sigma,m-1} & \text{for } j \in \{1, \ldots, m-1\}, \\ s_m R(\{1, \ldots, m\}) \succsim_m r_m R(S \cup \{m\}) & \end{cases}$$

From $s_j R(\{1,\ldots,m\}) \succsim_j Z_j^{\sigma,m-1} \sim_j Z_j^{\sigma,m} = \alpha_j(Y_j^\sigma, R(\{1,\ldots,m\}))R(\{1,\ldots,m\})$ and theorem 2.2 we derive $s_j \geq \alpha_j(Y_j^\sigma, R(\{1,\ldots,m\}))$ for $j = 1,\ldots,m-1$. Thus

$$\begin{aligned} s_m &= 1 - \sum_{j\in S} s_j \\ &\leq 1 - \sum_{j\in S} \alpha_j(Y_j^\sigma, R(\{1,\ldots,m\})) = \alpha_m(Y_m^\sigma, R(\{1,\ldots,m\})), \end{aligned}$$

where the last equality holds because $Z^{\sigma,m}$ is an efficient allocation of $R(\{1,\ldots,m\})$. So, $Z_m^{\sigma,m} \succsim_m s_m R(\{1,\ldots,m\})$. Together with $s_m R(\{1,\ldots,m\}) \succsim_m r_m R(S \cup \{m\})$ and transitivity we obtain $Z_m^{\sigma,m} \succsim_m r_m R(S\cup\{m\})$. But we stated before that $r_m R(S\cup\{m\})$ is the best payoff player m can obtain when cooperating with coalition S. Therefore there exists no individual rational allocation for coalition $S\cup\{m\}$ that yields player m a strictly better payoff then $Z_m^{\sigma,m}$. Hence, coalition $S\cup\{m\}$ has no incentive to part company with coalition S_m^σ if $Z^{\sigma,m}$ is allocated. Consequently, we have that $Z^{\sigma,m} \in C(\Gamma^m)$. Taking $m = n$ then gives that $M^\sigma = Z^{\sigma,n} \in C(\Gamma^n) = C(N,R,\alpha)$. □

For deterministic convex games it is well known that each of its subgames has a nonempty core. We can derive a similar result for games with stochastic payoffs.

Theorem 4.5 *Let (N,R,α) be a cooperative game with stochastic payoffs. If it is individual-merge convex then all of its subgames have a nonempty core.*

Proof. When the game (N,R,α) is individual-merge convex then each subgame (S,R,α), $S \neq \emptyset$, is also individual-merge convex. According to theorem 4.4 each subgame is marginal convex. We conclude that all subgames have a nonempty core. □

For two-person games it holds that all three types of convexity are equivalent. In particular it holds that marginal convex games are individual-merge convex. The following example shows that this need not hold for games with three or more players. Because coalitional-merge convex games are by definition individual-merge convex, it follows immediately from the next example that a marginal convex game also need not be coalitional-merge convex.

Example 4.6 Consider the following game (N,R,α) where $N = \{1,2,3\}$, $R(\{i\}) = 0$ for all $i \in N$, $R(\{1,2\}) = 3$, $R(\{1,3\}) = 2$, $R(\{2,3\}) = 6$ and $R(N) \sim U([5,15])$. All players have quantile-preferences with $\beta_1 = 0.1$, $\beta_2 = 0.5$ and $\beta_3 = 0.9$. In particular it holds for $q_i \neq 0$ and $p_i \in \mathbb{R}$ for all $i \in N$ that

$$\begin{aligned} &\alpha_i(p_i R(S), q_i R(T)) = p_i u_{\beta_i}^{R(S)} / (q_i u_{\beta_i}^{R(T)}), \\ &p_1 u_{\beta_1}^{R(N)} = 6p_1,\ p_2 u_{\beta_2}^{R(N)} = 10p_2 \text{ and } p_3 u_{\beta_3}^{R(N)} = 14p_3. \end{aligned}$$

The set of imputations is

$$I(N,R,\alpha) = \{pR(N) \,|\, p_1 + p_2 + p_3 = 1,\ p_i \geq 0 \text{ for all } i \in N\}.$$

and the core equals

$$C(N,R,\alpha) = \{pR(N) \in I(N,R,\alpha) \,|\, 6p_1 + 10p_2 \geq 3,\ 6p_1 + 14p_3 \geq 2,\ 10p_2 + 14p_3 \geq 6\}.$$

Take permutation $\sigma_1 = (1,2,3)$. Then player 1 receives $R(\{1\}) = 0$. Next, player 2 gets his marginal contribution to coalition $\{1\}$, which is

$$(1-\alpha_1(R(\{1\}), R(\{1,2\})))\, R(\{1,2\}) = (1-0)R(\{1,2\}) = R(\{1,2\}) = 3.$$

Player 3 receives all that is left of $R(N)$:

$$\begin{aligned}&(1-\alpha_1(R(\{1\}), R(N)) - \alpha_2(R(\{1,2\}), R(N)))\, R(N)\\ &= (1-0-3/10)R(N) = 7/10R(N).\end{aligned}$$

So, $M^{\sigma_1} = (0, 3/10, 7/10)R(N)$. In the same way all the other marginal vectors can be calculated and it is easy to check that they all belong to the core.

However, this game is neither individual-merge nor coalitional-merge convex. Let $U = \{1\}$, $S = \{2\}$ and $T = \{2,3\}$. Then $S \cup U = \{1,2\}$ and $T \cup U = N$. Furthermore, let $pR(S) = R(\{2\})$, $qR(T) = (1,0)R(\{2,3\})$ and $rR(S \cup U) = (1,0)R(\{1,2\})$. Then $pR(S) \in IR(S)$, because $R(\{2\}) \succsim_2 R(\{2\})$, $qR(T) \in IR(T)$, because $R(\{2,3\}) \succsim_2 R(\{2\})$ and $0 \succsim_3 R(\{3\})$, and $rR(S \cup U) \in IR(S \cup U)$ satisfies $0 \succsim_2 pR(S)$. If there exists an allocation $sR(T \cup U) = sR(N)$, $s \in \Delta^*(N)$, such that

$$\begin{cases} s_1 R(N) \succsim_1 r_1 R(S \cup U) = R(\{1,2\}) \\ s_2 R(N) \succsim_2 q_2 R(T) = R(\{2,3\}) \\ s_3 R(N) \succsim_3 q_3 R(T) = 0 \end{cases}$$

then this is equivalent to

$$\begin{cases} 6s_1 & \geq 3 \\ 10s_2 & \geq 6 \\ 14s_3 & \geq 0 \end{cases} \Leftrightarrow \begin{cases} s_1 \geq 1/2 \\ s_2 \geq 3/5 \\ s_3 \geq 0 \end{cases}$$

But this implies that $s_1 + s_2 + s_3 \geq 1/2 + 3/5 + 0 > 1$, which is in contradiction to $s \in \Delta^*(N)$. $\square$

By definition it holds that coalitional-merge convex games are individual-merge convex. One can easily see that the reverse relation will hold if the game has two players. The following theorem shows the same result for games with three players.

Theorem 4.7 *Let (N, R, α) be a cooperative game with stochastic payoffs and with three players. If the game is individual-merge convex, then it is coalitional-merge convex.*

Proof. Let (N, R, α) be a three-person game that is individual-merge convex. Firstly, we have to show that the game is superadditive. For this, let S and T be two nonempty coalitions in N such that $S \cap T = \emptyset$. Because there are only three players, we know that either S or T consists of one player. Assume without loss of generality that $S = \{i\}$ for some $i \in N$. Let $pR(S) \in IR(S)$, so, $pR(S) = R(\{i\})$. Because the game is individual-merge convex, it follows from theorems 4.3 and 4.4

that $IR(T) \neq \emptyset$. Let $qR(T) \in IR(T)$. Then there exists an allocation $sR(T \cup \{i\})$, $s \in \Delta^*(T \cup \{i\})$, such that

$$\begin{cases} s_i R(T \cup \{i\}) \succsim_i R(\{i\}) \\ s_j R(T \cup \{i\}) \succsim_j q_j R(\{T\}) \quad \text{for all } j \in T. \end{cases}$$

Hence, the game is superadditive.

Secondly, we have to show that the remaining condition of coalitional-merge convexity is satisfied. Let $U \subset N$. If $|U| = 1$ then this condition is equivalent to the second condition of individual-merge convexity with $U = \{i\}$ and thus it is satisfied. Next, if $U = \{i, j\} \subset N = \{i, j, k\}$ then there exist no coalitions S and T such that $S \subset T \subset N \setminus U$, $S \neq \emptyset$ and $S \neq T$ hold and consequently, there is nothing to check. If $U = N$ then there is also nothing to check. We conclude that the second condition of coalitional-merge convexity is satisfied. □

In case of four or more players, we were neither able to prove that individual-merge convex games are coalitional-merge convex nor could we find a counterexample. Hence, at this moment this remains open.

Finally, we return to the example of a bankruptcy game in the previous section and we check if it satisfies any of the convexity concepts introduced in this section. It is well-known that deterministic bankruptcy games are convex.

Example 4.8 Consider the same bankruptcy situation as in example 3.7. There we noticed that M^{σ_1} and M^{σ_2} do not belong to the core. Hence this game is not marginal convex and consequently it is neither individual- nor coalitional-merge convex. However, when we change the preferences of the agents such that all players have expectation-preferences then the corresponding bankruptcy game satisfies all the convexity concepts. □

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