

On multi-stage production/inventory systems under stochastic demand

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Abstract

This paper was presented at the 1992 Conference of the International Society of Inventory Research in Budapest, as a tribute to professor Andrew C. Clark for his inspiring work on multi-echelon inventory models both in theory and practice. It reviews and extends the work of the authors on periodic review serial and convergent multi-echelon systems under stochastic stationary demand. In particular, we highlight the structure of echelon cost functions which play a central role in the derivation of the decomposition results and the optimality of base stock policies. The resulting optimal base stock policy is then compared with an MRP system in terms of cost effectiveness, given a predefined target customer service level. Another extension concerns an at first glance rather different problem; it is shown that the problem of setting safety leadtimes in a multi-stage production-to-order system with stochastic lead times leads to similar decomposition structures as those derived for multi-stage inventory systems. Finally, a discussion on possible extensions to capacitated models, models with uncertainty in both demand and production lead time as well as models with an aborescent structure concludes the paper.

1. Introduction

This paper reviews and extends the work of the authors on the analysis of periodic review multi-stage production/inventory system under stationary stochastic demand. It was presented in a special session of the ISIR Conference in Budapest, August 1992, to honor the path-breaking work of Professor A.C. Clark on multi-echelon inventory models, starting at the Rand Corporation in the late 1950s and later applied when, e.g., developing multi-

indenture, multi-echelon inventory systems for the American Navy.

The publication of Forrester's study on the cyclical variation of stocks in large production–distribution chains [1] marked the start of a period of increasing interest in multi-stage inventory systems. The first rigorous analysis of serial multi-stage inventory systems under random demand was given by Clark and Scarf [2], in a discounted cost dynamic programming framework and based on Clark's observations of the fundamental role played by echelon stocks [3, 4]. Since then, the literature has expanded rapidly, considering not only serial systems but in particular also systems

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with an aborescence (divergent or distribution) structure and, to a minor extent, systems with an inverse aborescence (convergent or assembly) structure. Clark and Scarf [5] extended their previous work on serial systems with approximate models to handle fixed costs at the upstream stages. Langenhoff and Zijm [6] present an average cost unifying framework which can handle all three structures mentioned above. Cohen and Lee [7] present a completely different approach, based on queueing approximations, to deal with capacitated multi-echelon serial systems. Attention to convergent assembly structures was further given by Schmidt and Nahmias [8], Rosling [9] and Van Houtum and Zijm [10]; in particular the latter two papers establish equivalence properties with serial systems (also discussed in [6]). Most attention has been paid to systems with an aborescence (divergent) structure, more specific to the two-stage single-depot multi-warehouse system under stationary stochastic demand (a few references are [11–22]). Finally, many authors consider lot-sizing problems in deterministic multi-stage production systems. For further details, the reader is referred to the books of Schwarz [23] and Axsäter et al. [24].

This paper discusses periodic review stochastic multi-stage systems with both a serial and an assembly structure. Parts of the unifying framework to be discussed are published elsewhere [6, 10]. In Section 2, we provide some intuition for the structure of echelon cost functions, after which a brief summary of results obtained so far is given. New results on comparisons with MRP-controlled systems are presented in Section 3 and equivalence relations with safety lead time multi-stage models are discussed in Section 4. Finally, we explore the possibilities to extend the framework to capacitated systems (Section 5) as well as to a series of related but more general interesting models (Section 6).

2. Multi-stage serial systems and assembly systems: A review

Consider first a two-stage inventory system where between the two stages and after the most downstream stage material can be stocked in installations. Demand originates at the lowest (down-

stream) installation only. The stages (followed by their installations) are numbered such that material flows through stage 2 and stage 1 subsequently. A fixed lead time l_2 is required to transfer material to installation 2 while the transfer from installation 2 to installation 1 takes a fixed lead time l_1 . These transfers may represent a material supply and production stage, respectively. F_l denotes the distribution function of the l -period cumulative demand u_l . If $l = 1$, we suppress the index. At the end of each period (of unit size), material may be ordered at each installation (from the next higher installation or from the outside supplier). We assume that there is no material shortage at the outside supplier, hence every order of installation 2 will indeed be delivered l_2 periods later. Orders from installation 1, however, may be subject to an additional delay (apart from the lead time l_1) due to a material shortage at installation 2. We assume that any excess demand is backlogged. Furthermore, a holding cost h_2 is incurred at installation 2 and in transfer to installation 1 whereas in the latter installation a holding cost $h_2 + h_1$ is assumed (higher because of value added). Finally, a penalty cost p holds in case of shortage at installation 1 (unfilled demand). All these costs are linear, both in time and quantity. No fixed ordering costs are assumed. Furthermore, because we present an average cost analysis and all excess demand is backlogged, it is easy to verify that linear variable ordering costs do not influence any control policy and therefore can be omitted. All costs are incurred at the end of a period.

The concept of *echelon stock* for a given installation was introduced by Clark [3] to denote all stock at that installation plus in transit to or on hand at any installation downstream minus eventual backlogs at the most downstream installation. The chain under consideration is called the echelon. An echelon stock may be negative, indicating that the backlogs are larger than the total inventory in that echelon. Echelons are numbered according to the highest installation in that echelon. Finally, we define the *echelon inventory position* of an installation as its echelon stock plus all material in transfer to that installation.

Clark and Scarf [2] assumed the existence of convex cost functions which depend on echelon

stocks only. For the linear (inventory and penalty) cost structure defined earlier, we first have to develop a natural definition for the corresponding echelon cost functions. A careful analysis of possible situations easily leads to these natural expressions. Let v_n be the stock of echelon n ($n = 1, 2$) at the end of a period ($v_2 \geq v_1$). Then we have costs

$$h_2(v_2 - v_1) + (h_1 + h_2)v_1 = (h_1 + h_2)v_1 - h_2v_1 + h_2v_2$$

if $v_1 > 0, v_2 > 0,$

$$h_2(v_2 - v_1) + p(-v_1) = p(-v_1) - h_2v_1 + h_2v_2$$

if $v_1 \leq 0, v_2 > 0,$

$$h_2(v_2 - v_1) + p(-v_1) = p(-v_1) - h_2v_1 + h_2v_2$$

if $v_1 \leq 0, v_2 \leq 0.$

It therefore seems natural to attribute costs h_2v_2 to the echelon stock v_2 , independent of its sign, whereas to v_1 we attribute either $(h_1 + h_2)v_1 - h_2v_1$ (if $v_1 > 0$) or $p(-v_1) - h_2v_1$ (if $v_1 \leq 0$). Note that the second term in the costs of v_1 is again independent of its sign. As a result we find that, if at the beginning of a period the echelon stock of installation n is increased to x_n for $n = 1, 2$, then the expected costs incurred at the end of a period are equal to $L_1(x_1) + L_2(x_2)$, where

$$L_1(x_1) = (h_1 + h_2) \int_0^{x_1} (x_1 - u) dF(u) + p \int_{x_1}^{\infty} (u - x_1) \times dF(u) - h_2 \int_0^{\infty} (x_1 - u) dF(u)$$

$$= h_1 \int_0^{\infty} (x_1 - u) dF(u)$$

$$+ (p + h_1 + h_2) \int_{x_1}^{\infty} (u - x_1) dF(u)$$

for all $x_1,$

$$L_2(x_2) = h_2 \int_0^{\infty} (x_2 - u) dF(u) \quad \text{for all } x_2.$$

This defines the echelon cost functions. If now, at the beginning of period t , the echelon inventory position of installation 2 is increased to y_2 and, at the beginning of period $t + 1$, the echelon inventory position of installation 1 to y_1 or as far as possible if y_1 cannot be reached, then it is not hard to verify that the expected costs at the end of period $t + 1$ are equal to

$$D_2(y_1, y_2) = D_1(y_1) + \int_0^{\infty} L_2(y_2 - u_{t2}) dF_{t2}(u_{t2}) + \int_{y_2 - y_1}^{\infty} [D_1(y_2 - u_{t2}) - D_1(y_1)] dF_{t2}(u_{t2}),$$

where

$$D_1(y_1) = \int_0^{\infty} L_1(y_1 - u_{t1}) dF_{t1}(u_{t1}).$$

Langenhoff and Zijm [6] show that the optimal average cost policy is characterized by two parameters (S_1, S_2), where S_1 minimizes $D_1(y_1)$ and S_2 subsequently minimizes $D_2(S_1, y_2)$. In other words, the optimal base stock policy (S_1, S_2) prescribes that at the beginning of each period the echelon inventory position of echelon n is increased to S_n , for $n = 1, 2$. Analogous results are derived for N -stage serial systems, meaning that the N -dimensional optimization problem can be solved completely by successively solving a series of recursively defined one-dimensional problems. An extensive numerical analysis of the difficulties stemming from these recursive definitions is presented by Van Houtum and Zijm [10]. The equivalence of serial and assembly systems has been proved by Rosling [9] and Langenhoff and Zijm [6]. Finally, Van Houtum and Zijm [10] show how to extend the analysis to systems in which average holding costs have to be minimized, subject to a target customer service level.

Table 1
Basic parameter values for the two-stage assembly model

$h_0 = 5$	$h_1 = 1.5$	$h_2 = 1.5$	$h_3 = 2$	$\mu = 100$	$\sigma = 70$
$l_0 = 2$	$l_1 = 1$	$l_2 = 2$	$l_3 = 4$	$\tau = 95\%$	

3. Comparisons with MRP-controlled systems

The optimality of a base stock control policy for serial and assembly systems implies that safety stocks, meant to anticipate uncertainty in future demand, are spread over the entire chain. An alternative is to keep all safety stocks at one installation, close to the market. This is the policy advocated in the initial MRP literature (see e.g. [25]) where all uncertainty in a make and assemble to stock environment is covered at the end-item level. Often, the master production schedule (MPS) specifies quantities of end items to be assembled each period while the preceding phases (component manufacturing and subassembly) are driven deterministically by this MPS. More precisely, under conditions similar to those studied in this paper, the MPS of an MRP system should specify for each period how much end items have to be produced to return to an optimal reorder level which covers all demand uncertainty during the *full system lead time* (plus one period). In fact, this is similar to a single-stage inventory system; the only difference with the classical Newsboy problem is that holding costs are calculated relative to the manufacturing or assembly stage the material is in. Optimal reorder levels

for such a, basically single-stage, system are easily determined but it seems worthwhile to investigate the cost differences between such an optimal MRP policy and an overall optimal base stock policy. Note that the two policies coincide if no added value is counted in the system (i.e. $h_n = 0$ for $n \geq 2$).

We start with investigating these cost differences for a simple two-stage assembly system with three components and one end item, under a service level constraint. Table 1 lists the basic parameter values (where μ and σ denote the mean and standard deviation of the periodic demand, while τ is the target customer service level).

In the following tables, a number of cost comparisons between an optimal MRP policy and an overall optimal policy are made as a function of one varying parameter (e.g. σ in Table 2, τ in Table 3). The order-up-to levels for the optimal base stock policy as well as the (single) order-up-to level for the optimal MRP policy (S_{MRP}) are given, together with the inventory holding costs for both policies (C_{BS} and C_{MRP}). CD finally denotes the cost difference between the two policies, expressed both absolutely and as a percentage.

Table 2 illustrates the intuitively obvious result that the differences in cost between an optimal MRP policy and an overall optimal policy increase as a function of demand uncertainty (the standard variation). Table 3 shows that the cost difference is also (weakly) sensitive to the target service level; this sensitivity seems to increase for very high service levels.

Table 2
Comparisons between MRP and base stock policies for varying σ

σ	S_0	S_1	S_2	S_3	C_{BS}	S_{MRP}	C_{MRP}	CD	CD (%)
10	316.8	419.5	519.2	718.9	1174	714.2	1192	18	1.5
20	346.0	455.3	557.8	763.1	1480	751.2	1562	82	5.5
30	381.3	498.4	604.4	816.4	1869	796.3	2013	144	7.7
40	421.7	547.4	657.2	876.6	2317	847.7	2527	210	9.1
50	467.6	602.7	716.5	944.0	2827	905.1	3101	274	9.7
60	516.9	661.9	779.8	1016	3376	966.6	3716	340	10.1
70	573.0	728.6	850.5	1096	3995	1035	4397	402	10.1
80	628.0	794.1	920.4	1175	4607	1103	5078	471	10.2
90	691.3	868.1	993.3	1263	5304	1179	5838	534	10.1
100	772.4	961.7	1096	1371	6183	1271	6755	572	9.3

Table 3
Comparisons between MRP and base stock policies for varying τ

τ (%)	S_0	S_1	S_2	S_3	C_{BS}	S_{MRP}	C_{MRP}	CD	CD (%)
90	522.3	667.3	781.6	1015	3384	959.8	3698	314	9.3
91	530.1	676.8	792.4	1027	3478	971.5	3805	327	9.4
92	538.7	687.4	804.3	1041	3583	984.5	3925	342	9.5
93	548.5	699.2	817.6	1057	3701	999.0	4060	359	9.7
94	559.8	712.7	832.8	1075	3836	1015	4215	379	9.9
95	573.0	728.6	850.5	1096	3995	1035	4397	402	10.1
96	589.1	747.7	871.8	1120	4189	1058	4619	430	10.3
97	609.6	771.9	898.7	1151	4435	1087	4900	465	10.5
98	638.2	805.4	935.7	1194	4776	1127	5291	515	10.8
99	686.3	861.0	996.7	1263	5345	1193	5941	596	11.2

Table 4
Basic parameter values for the two-stage serial model

$$l_0 = 2 \quad l_1 = 4 \quad h_i = 2l_i \text{ for } i = 0, 1 \quad \mu = 100 \quad \sigma = 70 \quad \tau = 95\%$$

In the next tables we investigate the influence of the amount of value added in the manufacturing process, as well as the influence of the lead times. A simple two-stage serial system is used as an example (see Table 4). Since inventory holding costs are usually taken as a percentage of the value of an item at each moment, a large h_0 value reflects a high added value in the manufacturing process. Intuitively, one would expect a sharp increase in cost difference as h_0 grows larger and a relatively

minor influence of the ratios of the lead times. Tables 5 and 6 confirm these conjectures.

Finally, we investigate the influence of the number of installations in a serial system. Table 7 lists the basic parameters; the results are shown in Table 8.

In summary, we may conclude that the increase in costs as a result of the absence of buffer stocks in the component and subassembly stores (hence only stocks of end items) strongly depends on the amount of value added during the subsequent stages in the manufacturing process. The influence of the height of the target customer service level is less significant but a high demand variability may have a very serious impact on these cost differences.

Table 5
The influence of the value added in a two-stage serial model

h_0	h_1	S_0	S_1	C_{BS}	S_{MRP}	C_{MRP}	CD	CD (%)
0	10	1035	1035	5397	1035	5397	0	0.0
1	9	703.7	1036	5186	1035	5197	11	0.2
2	8	651.7	1040	4949	1035	4997	48	1.0
3	7	619.4	1045	4689	1035	4797	108	2.3
4	6	595.3	1052	4409	1035	4597	188	4.3
5	5	575.4	1062	4107	1035	4397	290	7.1
6	4	558.4	1075	3784	1035	4197	413	10.9
7	3	543.1	1093	3435	1035	3997	562	16.4
8	2	529.0	1121	3054	1035	3797	743	24.3
9	1	516.0	1171	2627	1035	3597	970	36.9
10	0	503.8	∞	2088	1035	3397	1309	62.7

Table 6
The influence of the lead time ratios in a two-stage serial model

l_0	l_1	S_0	S_1	C_{BS}	S_{MRP}	C_{MRP}	CD	CD (%)
1	5	510.4	1040	4981	1035	5077	96	1.9
2	4	610.7	1047	5517	1035	5677	160	2.9
3	3	715.8	1056	5695	1035	5877	182	3.2
4	2	821.7	1065	5509	1035	5677	168	3.0
5	1	928.0	1076	4965	1035	5097	112	2.3

Table 7
Basic parameters for investigating the influence of N

N	h_0	h_1	h_2	h_3	h_4	h_5	l_0	l_1	l_2	l_3	l_4	l_5
1	5	5					2	5				
2	5	4	1				2	4	5			
3	5	3	1	1			2	3	4	5		
4	5	2	1	1	1		2	2	3	4	5	
5	5	1	1	1	1	1	2	1	2	3	4	5
$\mu = 100$		$\sigma = 70$			$\tau = 95\%$							

4. A related model: Setting safety lead times in a multi-stage system

The problem of setting planning lead times in a multi-stage production system in which production is order driven and tardiness results in penalty costs has been investigated by several authors (see e.g. [26, 27]). Here the lead times are assumed to be random variables with a known distribution function. For the single-stage system, the problem of setting a planning or safety lead time leads to a Newsboy formulation. De Kok [28] recognized the striking similarity between the general problem

of setting safety lead times in an N -stage serial system and the results of the analysis presented in Section 2 for the N -stage inventory problem. Below, we show the equivalence of the two problems for the two-stage serial system.

Consider a two-stage serial, order driven, production system with stochastic lead times t_2 (for the upstream or manufacturing stage) and t_1 (for the final assembly stage). Let F_n denote the distribution functions of t_n for $n = 1, 2$. Suppose, furthermore, that a holding cost h_2 is incurred for a batch of products (components or subassemblies) in installation 2 or between installations 2 and 1, while a batch of final products is stored at a rate $h_2 + h_1$ if the batch is finished before its due date. A batch is temporarily stored in installation 2 if it arrives there before the planned starting time of the final stage. If the final stage of the batch is completed after its due date, a linear penalty cost p per unit per period is incurred. A batch is delivered to the customer at its due date or, if not possible, immediately after completion. The problem is to determine the best starting time for each phase, i.e. how to set planning or safety lead times to minimize average holding and penalty costs?

Table 8
The influence of the number of installations N

N	S_0	S_1	S_2	S_3	S_4	S_5	C_{BS}	S_{MRP}	C_{MRP}	CD	CD (%)
1	577.3	1192					4275	1162	4666	391	9.1
2	576.3	1084	1232				4225	1162	4666	441	10.4
3	575.5	975.6	1119	1237			4189	1162	4666	477	11.4
4	574.9	866.0	1004	1123	1239		4167	1162	4666	499	12.0
5	574.7	754.3	883.9	1006	1124	1239	4156	1162	4666	510	12.3

Define T_2 as the total planning lead time for the two stages together and T_1 as the planning lead time for the final (downstream) stage alone. Also, let

$$v_2 = T_2 - t_2, \quad v_1 = \min(v_2, T_1) - t_1.$$

Hence, v_n denotes the time difference between the completion of stage n and the final due date. Then the following cost structure appears:

$$\begin{aligned} & h_2(v_2 - v_1) + (h_1 + h_2)v_1 \\ &= (h_1 + h_2)v_1 - h_2v_1 + h_2v_2 \quad \text{if } v_1 > 0, \quad v_2 > 0, \\ & h_2(v_2 - v_1) + p(-v_1) \\ &= p(-v_1) - h_2v_1 + h_2v_2 \quad \text{if } v_1 \leq 0, \quad v_2 > 0, \\ & h_2(v_2 - v_1) + p(-v_1) \\ &= p(-v_1) - h_2v_1 + h_2v_2 \quad \text{if } v_1 \leq 0, \quad v_2 \leq 0. \end{aligned}$$

The reader may note that this is completely identical to the inventory cost structure in Section 2 which formed the basis of the echelon cost functions. For the safety lead time problem discussed here we find that the optimal planning lead times can be found by minimizing the following function:

$$\begin{aligned} C_2(T_1, T_2) &= C_1(T_1) + h_2 \int_0^\infty (T_2 - t_2) dF_2(t_2) \\ &+ \int_{T_2 - T_1}^\infty [C_1(T_2 - t_2) \\ &- C_1(T_1)] dF_2(t_2), \end{aligned}$$

where

$$\begin{aligned} C_1(T_1) &= (h_1 + h_2) \int_0^{T_1} (T_1 - t_1) dF_1(t_1) \\ &+ p \int_{T_1}^\infty (t_1 - T_1) dF_1(t_1) \\ &- h_2 \int_0^\infty (T_1 - t_1) dF_1(t_1) \end{aligned}$$

$$\begin{aligned} &= h_1 \int_0^\infty (T_1 - t_1) dF_1(t_1) \\ &+ (p + h_1 + h_2) \int_{T_1}^\infty (t_1 - T_1) dF_1(t_1), \end{aligned}$$

where $C_1(T_1)$ is a convex function, which is minimized in $T_1 = L_1$, say, and $C_2(L_1, T_2)$ is a convex function in T_2 . Exactly the same recursive procedure as in Section 2 can therefore be applied to obtain the optimal (T_1, T_2) -tuple by successively minimizing two one-dimensional functions. Let the optimal values be L_1 and L_2 ; then the optimal planned lead times for the upstream and downstream stages equal $L_2 - L_1$ and L_1 , respectively. The generalization of the above arguments to general N -stage serial systems and to systems with an assembly structure are completely straightforward and are therefore left to the reader.

5. Extensions to capacitated multi-stage systems

In this section we seek to extend the theoretical framework to systems where transfer at each stage, i.e. between each two installations, is limited by a capacity constraint. Although, in general, we cannot expect that the results derived so far continue to hold in capacitated systems, an important question is whether base-stock policies, based on echelon stocks, are still close to optimal. If the answer to this question is affirmative, numerical schemes taking into account the capacity limitations have to be developed again.

Speck and Van der Wal [29, 30] have investigated the first question. We only briefly summarize their results here. Define a serial N -stage system as before and assume that transfer between stage $i + 1$ and stage i is limited by an amount c_i (the production or transfer capacity). Some reflection shows that, without any loss of generality, one may restrict oneself to systems with $c_{i+1} \leq c_i$. Speck and Van der Wal numerically show that for a two-stage serial system a so-called push-ahead effect may exist. That is, the replenishment decision of the downstream installation may depend on the

amount of material in transfer to installation 2 (upstream). On the other hand, Speck and Van der Wal provide evidence that in all cases there exist base-stock policies, the calculation of which is based on capacity limitations as well, which are very close to optimal.

In this section, we therefore concentrate on base-stock policies again. For the development of numerical procedures we first briefly study the single-stage system. Note that from the results of Federgruen and Zipkin [31] the optimality of an order-up-to level policy easily follows for a single-stage capacitated system, as long as no fixed ordering costs are assumed. Now, consider a two-stage linear system as described in Section 2 with the extra assumption of a limited transfer capacity c_2 per period to installation 2 while transfers between installation 2 and installation 1 are unconstrained. Let z_2 denote an order-up-to level for the echelon inventory position of installation 2 (i.e. if, at the beginning of a period, this echelon inventory position is equal to x_2 then we order $\min(z_2 - x_2, c_2)$ if $x_2 < z_2$ and nothing otherwise). Furthermore, let $y_{2,n}$ be the echelon inventory position of installation 2 just *after* ordering in period n . Then we can write the following traffic equation for $y_{2,n+1}$ and $y_{2,n}$:

$$z_2 - y_{2,n+1} = \max(z_2 - y_{2,n} + u - c, 0),$$

where u denotes the one-period (stochastic) demand. Hence, $z_2 - y_{2,n}$ satisfies the law of motion of the waiting time (excluding service) of an arriving customer in a $D/G/1$ queue with interarrival time c and service time u . If the mean demand $\mu < c$ (meaning that on average the system can fulfill all demand) then clearly $z_2 - y_{2,n} \rightarrow z_2 - y_2$ for $n \rightarrow \infty$ with probability one for some *random* variable y_2 . If we define $w := z_2 - y_2$ then the probability distribution function $G(w)$ of w satisfies the following Wiener-Hopf equation:

$$G(w) = \int_0^{w+c} G(w+c-u) dF(u).$$

Note that $G(w)$ has a discontinuity in 0 (since $G(0) > 0$) while $G(w)$ is differentiable for $w > 0$, with derivative $g(w)$, say. De Kok [32] developed

a method to quickly determine an approximation of $G(w)$ in terms of a mixture of Erlang distributions which has proven to be extremely accurate. Since the random variable $y_2 = z_2 - w$ equals the inventory position just after ordering we now easily find the optimal order-up-to level S_2 by minimizing the following expression:

$$E(S_1, z_2) = \int_0^{\infty} D_2(S_1, z_2 - w)g(w)dw + D_2(S_1, z_2)G(0),$$

where S_1 is determined in the first step of the recursive procedure presented in Section 2 for unconstrained systems while also $D_2(y_1, y_2)$ denotes the two-dimensional cost function given in Section 2. It can be shown, using the results of Federgruen and Zipkin [31], that this procedure still yields an overall optimal policy.

If there exists a capacity constraint c_1 on the periodic transfer of materials between installation 2 and installation 1 as well, then the term $D_1(y_1)$ in the recursive definition of $D_2(y_1, y_2)$ has to be replaced by a function $E_1(z_1)$ which is determined along lines similarly to the analysis given above. The term $D_1(y_2 - u_{12})$ is replaced by a more complicated expression to indicate that there may be two causes for not attaining the order-up-to level z_1 , i.e. a material shortage or a lack of capacity between installation 2 and installation 1. The analysis of these complications is beyond the scope of this paper; a detailed treatment will be given in a forthcoming publication.

9. Conclusions and suggestions for further research

In this paper we have reviewed and extended a theoretical framework for the planning of stochastic multi-stage serial and assembly systems. An equivalence property between these two systems is established. Numerical procedures to actually calculate order-up-to levels have been developed and perform satisfactorily. A translation to a more realistic situation where optimal policies have to be determined subject to a target customer service level constraint follows easily from the cost

expressions underlying the numerical evaluation. The numerical calculation scheme also allows us to compare the performance of the optimal base-stock policies with an MRP control structure, showing remarkable differences. Furthermore, the problem to determine safety lead times in a stochastic lead time multi-stage system with known demand appears to have a highly similar structure. Finally, we elaborate on possibilities to extend the theory to capacitated multi-stage systems.

Further extensions are possible and are highly recommended for investigation. The extension of the numerical calculation scheme to systems with a divergent (aborescence) structure is currently studied and will be treated in a forthcoming publication (see also [33]). Referring to Section 7, the combination of stochastic demand and stochastic lead times may lead to more realistic models covering unreliability and scheduling effects. Highly important seems to be the extension to systems with a non-stationary demand structure as well as to systems where demand originates not only at the end-item level. In particular, the combination of non-stationary demand (e.g. seasonal patterns) with capacity management issues may lead to a framework which perhaps cannot be treated by exact methods any longer but on the other hand provides us with a tool to understand and manage rather complex practical production/inventory systems.

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