Modeling and Feedback Control of a Flexible Arm of a Robot for Prescribed Frequency-domain Tolerances*

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Quantitative feedback theory can be used to achieve frequency-domain specifications for the tip of a flexible rotating beam by controlling the motor torque in the presence of plant parameter uncertainty or unknown disturbance.

Key Words—Flexible beam; feedback control; frequency-domain design.

Abstract—In this article we solve the problem of achieving quantitative specifications for the tip of a rotating flexible beam with uncertainty in some of its physical and geometric parameters. The frequency-domain design method used here is also helpful in clarifying some limitations on the feedback loop capabilities due to the distributed nature of the problem. Finally, time domain simulations, while proving the validity of the design method suggested in this work, pointed to some interesting mathematical problems concerning the influence of some of the parameters on the plant dynamics with possible physical consequences on the design of flexible beams.

1. INTRODUCTION

IN RECENT YEARS a significant research effort has been devoted to the study of flexible beams. Apart from the theoretical interest, this attention is motivated by important applications in robotics and flexible structures.

There are two (not decoupled) directions of research, with results both in state space and frequency-domain modeling and control of flexible beams, see Sakawa and Luo (1989), Balakrishnan (1988), Bontsema *et al.* (1988), Cannon and Schmitz (1984), Horowitz *et al.* (1989), Kruse *et al.* (1989) and Wells *et al.* (1990), to name a few. Most of the control problems for flexible beams solved so far dealt

with optimization, (robust) stabilization, vibration suppression, and related problems.

Our goal in this article is to go a step further, namely to achieve (whenever possible) for the closed loop desired (frequency domain) performance despite large uncertainty in the plant parameters. The importance of this problem cannot be overemphasized in view of the increasing need for accurate, but at the same time for fast and less energy consuming industrial robots.

The desired performance is given by quantitative specifications, and the actual design should take into consideration both the extent of uncertainty and the narrowness of performance tolerances to reduce the cost of feedback (i.e. the overdesign). An effective design method for approaching this type of problems is the quantitative feedback theory (QFT), see Horowitz and Sidi (1972). QFT was proposed in the early 1970s for lumped systems but its structure allows for various extensions, including that to distributed systems.

Specifically, we consider a slender flexible arm, i.e. a beam, which is rotated by a motor in a horizontal plane, about an axis through the fixed end of the arm. The arm may carry a payload on its tip, and there are frequency-domain specifications on the movement of the tip of the arm. Our problem will be to design a feedback loop for controlling the motor torque in such a way that the specifications should be met (whenever possible) in spite of the uncertainty about some of the parameters of the plant, or in the presence of unknown disturbances.

To solve this problem we first develop a mathematical model of the cantilevered beam

* Received 4 April 1991; revised 9 June 1992; revised 5 November 1992; received in final form 7 December 1992. The original version of this paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor M. Rodd under the direction of Editor H. Austin Spang III. Corresponding author Bagchi's Telephone 31-53-893452; Fax 31-53-340733; e-mail bagchi@math.utwente.nl.

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and the motor. We then express our task as a standard control problem. Finally, the feedback loop is found by extending QFT to our distributed case. In the process we point out some limitations in the feedback loop capabilities appearing in the control of flexible beams. We work with the non-reduced model of the plant, which proves to be useful in the robust stabilization of the system.

At the same time we want to obtain maximum benefit from a feedback loop based on a single and relatively simple measurement, i.e. that of the angle of the motor shaft. While it is known that more measurements can improve the performance of the controller, our approach is justified by the fact that in many practical situations there are quite severe constraints imposed on measurements. An advantage of our frequency-domain design method is that it allows for more freedom in the location of sensors than with a state space technique.

Finally, let us compare our design methodology with another QFT approach to the control of flexible beams, given in Yaniv and Horowitz (1990). We work with the original distributed plant, not with a lumped approximation of it (including the first mode of the beam) used in that paper. The gain is twofold: we can reduce some of the closed loop bandwidth because we may design closer to the instability region, see Remark 5.2; we can give a clear idea on the limitation in the feedback loop capability to control the beam, see Remark 3.3.

2. MODEL OF OPEN LOOP ELEMENTS

2.1. The beam

Most authors obtain the model of the flexible beam in rotating Cartesian coordinates. However, for our control purpose it is more appropriate to have it in fixed, polar coordinates. Thus, despite the compatibility of the two forms, we briefly deduce our form of the model for completeness.

As pointed out in the Introduction, we have a flexible beam of length R which is rotated by a motor, horizontally, about an axis through its fixed end. Therefore, we would like a model of the beam in which we could visualize its tip rotating. This is why we shall consider the plane in which the beam moves relative to fixed axes (ox, oy) , but in polar coordinates (r, θ) , see Fig. 1. Thus

$$x = r \cos \theta, \quad y = r \sin \theta,$$

and

$$\begin{aligned} \ddot{x} &= -r\ddot{\theta} \sin \theta - r\dot{\theta}^2 \cos \theta, \\ \ddot{y} &= r\ddot{\theta} \cos \theta - r\dot{\theta}^2 \sin \theta, \end{aligned} \quad (2.1)$$

FIG. 1. Bending of a flexible beam.

where the dot means differentiation with respect to time.

The equation of motion is obtained by considering at each moment the equilibrium of forces acting on a differential segment of the rotating beam. More precisely, we have to find the transverse (normal) displacement from the current equilibrium position of the beam, caused by the sum of projections on this normal direction of the forces acting on the beam, i.e. the shear force and the inertial force.

At each moment the transverse direction to the beam is given approximately, for small displacements of the beam from its steady state, by $(-\sin \theta, \cos \theta)$.

We shall consider the shear force as being normal to the beam.

The longitudinal force in the beam has no influence on the transverse dynamics, hence it can be discarded. Therefore, one can see from (2.1), if we neglect the almost longitudinal force, that the inertial force of the accelerating beam is approximately transverse to the beam. This approximation can be done if θ is close to θ_0 at every point of the beam. Here θ_0 is the angle of the motor shaft to the axis ox in Fig. 1.

Now at each moment the transverse displacement is given by $r \tan(\theta - \theta_0)$. Again when θ is close enough to θ_0 , this displacement is approximately $r(\theta - \theta_0)$.

By a standard procedure, see for instance Clough and Penzien (1975), we obtain the equation of motion

$$\rho r \ddot{\theta} + (EI) \frac{\partial^4 (\theta - \theta_0) r}{\partial r^4} = 0.$$

Here EI is the flexural rigidity and ρ is the mass per unit length of the beam. Both are considered as being uniform over the length R of the beam. If we recall that θ_0 cannot depend on r , if we add also a Voigt-type term accounting for the internal viscous damping with uniform coefficient ε , and if we consider the arc $\phi = r\theta$ as the new

dependent variable, we obtain a new (and more complete) version of the dynamics of the beam, i.e.

$$\begin{aligned} \sum(\phi(t, r)) := & \rho \ddot{\phi} + 2\varepsilon(EI) \frac{\partial^5 \phi}{\partial t \partial r^4} \\ & + (EI) \frac{\partial^4 \phi}{\partial r^4} = 0. \end{aligned} \quad (2.2)$$

The boundary conditions for this linear partial differential equation with constant coefficients are

$$\begin{aligned} \phi(t, 0) = 0, \quad \frac{\partial \phi}{\partial r}(t, 0) = \theta_0(t), \\ \frac{\partial^2 \phi}{\partial r^2}(t, R) = 0, \quad \frac{\partial^3 \phi}{\partial r^3}(t, R) = 0. \end{aligned} \quad (2.3)$$

These conditions are a direct consequence of the fact that the beam is a cantilever one, see for instance (Sokolnikoff, 1946), i.e. at one end it is clamped on the motor shaft, and the other end is free. We shall assume that the initial conditions are zero.

Remark 2.1. From (2.3) one may view the angle of the motor shaft, θ_0 , as a boundary input to the beam, which is for our purpose a plant to be controlled, while arc ϕ is the output.

Remark 2.2. The condition for a linear dynamical behavior of the beam calls for the angle $(\theta - \theta_0)$ to be sufficiently small over all the length of the beam.

Had we considered the dependent variable θ , the equation (2.3) would have remained linear, but with variable coefficients (depending on r).

Remark 2.3. Equation (2.2) provides a model for the bending vibrations of the flexible beam. A more general situation, see Sakawa and Luo (1989), is when there are also torsional vibrations due to an asymmetric payload. Since our goal in this article is to describe a new design methodology for flexible beams, we shall consider the simplified model, that of bending vibrations. This is possible by considering a symmetric payload, as we shall do in the next section. In this case the bending and torsional vibrations will be decoupled.

2.2. The motor

We shall start with the equations of the rotation motor as were given in Sakawa and Luo (1989). Thus

$$J_m \ddot{\theta}_0 + \mu \dot{\theta}_0 = \tau(t) + \beta(t).$$

Here J_m is the moment of inertia of the motor, μ

is the viscous friction coefficient, τ is the torque developed by the motor, and

$$\beta(t) = (EI) \frac{\partial^2 \phi}{\partial r^2}(t, 0) \quad (2.4)$$

is the bending moment of the beam which works as a reaction on the motor shaft.

Since the motor is a DC one having an amplifier of speed-feedback type,

$$\tau(t) = k(V_{\text{ref}}(t) - k_f \dot{\theta}_0(t)),$$

where V_{ref} is the input-speed reference voltage to the amplifier, and k, k_f are gain constants.

In many motors the coefficients J_m and μ are extremely small and can be neglected. Then the equation of the motor becomes

$$\dot{\theta}_0 = \frac{1}{k_f} \left(V_{\text{ref}}(t) + \frac{1}{k} \beta(t) \right). \quad (2.5)$$

This is an integrator with output the angle of the motor shaft θ_0 , and with two inputs: a reference input V_{ref} , and the reaction input β as described in (2.4).

Remark 2.4. This simplification does not represent any loss of generality to our design method. For the motor we work with to implement our design at the Control Engineering Laboratory of the University of Twente, this simplified form of the motor dynamics is quite realistic, see Kruse *et al.* (1989).

3. THE CLOSED LOOP

We shall begin by describing briefly how the closed loop works. We want to position the tip of the beam by an appropriate positioning of the motor shaft. As a matter of fact, in steady state the angle of the tip of the beam to the ox axis in Fig. 1 is equal to the angle of the motor shaft. However, the transient response is affected by the feedback generated by the reaction of the bending moment on the shaft of the motor, see (2.5) and (2.4).

Due to uncertainties in the parameters of the plant (i.e. in the flexural rigidity of the beam and in the payload) and due to the accuracy required in positioning the tip of the beam, we have to use a feedback loop. A simple approach (which makes sense in view of the steady state property mentioned above) is to base the feedback loop on the measurement of the angle of the motor shaft. A more complex possibility is to add strain gauges as well to close the feedback loop. We settle for the first case since we want to obtain maximum of benefit from a feedback loop based on single and relatively simple measurement. Our approach is justified by the fact that in many

FIG. 2. The closed loop.

practical situations there are quite severe constraints imposed on measurements.

It is useful to have a clear representation of the closed loop, with its two interlaced feedback loops, see Fig. 2. Here B is the plant (beam) described by (2.2), with input $\theta_0(t)$ (the angle of the motor shaft), and two outputs: $\phi(t, R)$, the arc described by the tip of the beam which is the output to be controlled, and $\frac{\partial^2 \phi}{\partial r^2}(t, 0)$ which, as already pointed out, is an additional output appearing as a result of the bending of the beam. M is the motor described by (2.5), with output $\theta_0(t)$, and two inputs: a voltage $V_{ref}(t)$, and a second one due to the reaction of the bending of the beam on the motor shaft. This second input is related via (2.4) to the second output of the plant, thus generating the "reaction" feedback mentioned before. Finally, the elements of the loop which should be designed in order to achieve the desired behavior of the closed loop are G , the compensator (which generates the input reference voltage to the motor) and F , the prefilter; $c(t)$ is the command input (e.g. a reference).

We shall proceed by deducing the closed loop transfer function from $c(t)$ to $\phi(t, r)$ for an arbitrary $r \in [0, R]$, in the presence of a new input: the inertial force generated by a symmetric payload (see Remark 2.3) of mass m , located at the point r_t on the beam. Then, by invoking (2.2), we can write

$$\Sigma(\phi(t, r)) = m \frac{d^2 \phi_{t, r_t}}{dt^2} \delta_r(t), \quad (3.1)$$

where δ is the Dirac delta function. Applying the Laplace transform in the time variable we obtain

$$\Phi(s, r) = \Theta_0(s)P_{0r}(s) + H\Phi(s, r_t)P_{r,r}(s). \quad (3.2)$$

As usual, we denoted the transform variable by s and used capital letters for the transformed functions. We have considered the boundary condition $\theta_0(t)$ as an independent input. Therefore, by superposition of inputs, we have that $P_{r_1 r_2}$ is the transfer function from an input located at r_1 to an output evaluated at r_2 ; when $r_1 = 0$ the transfer function is from the boundary

condition $\Theta_0(s)$, seen as an input, to the output at the point r_2 . Also, $H = -ms^2$; the minus sign shows that the inertial force due to the payload is acting as a reaction on the beam.

Remark 3.1. When $r_1 \neq 0$, $P_{r_1 r_2}$ can be computed, as extended by Theorem 2.2 of Coddington and Levinson (1955), as Green's function for the boundary-value problem (2.2), (2.3), but with θ_0 replaced by 0. When $r_1 = 0$, $P_{r_1 r_2}$ is simply the solution of the boundary-value problem (2.2), (2.3). In both situations s is a parameter and the resulting function is nonrational in s , as opposed to the lumped case where the transfer functions are rational. The actual computation, though standard, may be somehow elaborate. For our design purpose it is enough, however, to have $P_{r_1 r_2}$ computed numerically.

The feedback loops are described by the following relation:

$$\Theta_0 = ((CF - \Theta_0)G + (EI/k)\Phi^{(2)}(0))M, \quad (3.3)$$

see also Fig. 2. Here we have dropped the s argument, as we shall do from now on.

From (3.2) we obtain, by setting $r = r_t$,

$$\Phi(r_t) = \Theta_0 P_{0r_t} + H\Phi(r_t)P_{r_t r_t},$$

and by differentiating twice with respect to r (and then setting $r = 0$)

$$\Phi^{(2)}(0) = \Theta_0 P_{00}^{(2)} + H\Phi(r_t)P_{r_t 0}^{(2)}.$$

A straightforward (but lengthy) computation to eliminate Θ_0 , $\Phi(r_t)$ and $\Phi^{(2)}(0)$ from the last two equations as well as (3.2) and (3.3), leads to the following result.

$$\Phi(r) = T_C C, \quad (3.4)$$

where

$$T_C = F \frac{L}{1 + L} D_1, \quad (3.5)$$

and

$$L = GP_{\text{eqv}}. \quad (3.6)$$

Here

$$P_{\text{eqv}} = \frac{M}{1 - C_1},$$

$$M = 1/(k_t s) \text{ see (2.5),} \quad (3.7)$$

$$D_1 = P_{0r} + \frac{HP_{r_t}P_{0r_t}}{1 - HP_{r_t r_t}},$$

and

$$C_1 = (EI/k)M \left(P_{00}^{(2)} + P_{r_t 0}^{(2)} \frac{HP_{0r_t}}{1 - HP_{r_t r_t}} \right).$$

Thus, in relation (3.4) T_C is the closed loop transfer function from the command input c

applied at the root of the beam to the output (arc) at the point r on the beam, and L from (3.6) is the loop transmission corresponding to an equivalent subplant P_{eqv} . Also, a brief computation shows that D_1 represents the response of the beam from a step-angle input at the root of the beam to the point r on the beam, when there is no "reaction" feedback, see (2.4).

Remark 3.2. Let us mention that while computing (3.4) we canceled the factor $1 + GM$ from the numerator and denominator of both T_C and T_U . Thus, in designing G we should ensure that $1 + GM$ has no zeros in the right half of the complex plane.

Remark 3.3. Assume that the movement of the beam is generated by the motor, but in open loop. Then a similar computation (with the command input being the reference-voltage V_{ref} and the disturbance input being zero) leads to the open loop transfer function

$$P_0 = P_{eqv} D_1. \quad (3.8)$$

A comparison with formula (3.5) shows that the feedback loop acts only on the subplant P_{eqv} leaving the other subplant D_1 unaffected. One could try to use more complex feedback structures, see for instance Kelemen *et al.* (1990), to influence the other part of the plant as well. We believe, however, that only a spatially distributed feedback loop, which is not implementable in practice, can bring the entire plant under control.

4. THE DESIGN PROBLEM

The design problem we shall consider in this article concerns the achievement of a desired response to the command input of the flexible structure described previously. We shall not treat the disturbance attenuation problem. This can be solved along the lines presented in Kelemen *et al.* (1989).

We shall formulate now our design problem which calls for both robust stability of the closed loop, and quantitative performance specifications to be satisfied at the point $r = r_c$ (where the payload is located) on the controlled beam.

Problem. Given the range of uncertainty for the payload, $0 \text{ [kg]} \leq m \leq 0.5 \text{ [kg]}$, and for the flexural elasticity of the beam, $20 \text{ [Nm}^2\text{]} \leq EI \leq 25 \text{ [Nm}^2\text{]}$, find (if possible) a real rational, strictly proper fixed compensator G and prefilter F , such that the family indexed by m and EI of closed loop transfer functions T_C (with $r = r_l$)

(i) is stable

(ii) satisfies the stability margin condition given

$$\text{by } 20 \ln \left| \frac{L}{1+L} \right| \leq 2.3 \text{ dB}$$

(iii) satisfies the frequency-domain quantitative specifications

$$20 \ln \left| \frac{0.98}{1+s} \right| \leq 20 \ln |T_C(s)| \leq 20 \ln \left| \frac{1.02}{1+s/1.4} \right|, \quad s = j\omega. \quad (4.1)$$

The considered ranges of the uncertain parameters are physically realistic. (Note that the allowed variation of EI is quite small since, usually, it can be measured with reasonable accuracy.) As for other data relating to the beam, we use values identified in the flexible arm used in Kruse *et al.* (1989). The "fixed" data of the problem are: the length of the beam $R = 1.9 \text{ [m]}$, the mass density per unit length of the beam $\rho = 0.648 \text{ [kg m}^{-1}\text{]}$, the damping factor $\varepsilon = 0.00138 \text{ [s]}$, and the motor constants (Section 2.2) $\frac{1}{k_f} = 2.58 \text{ [rad sec}^{-1} \text{ V}^{-1}\text{]}$, and $\frac{1}{k} = 0.025 \text{ [V N}^{-1} \text{ m}^{-2}\text{]}$. Also, since the payload is not placed exactly on the tip of the beam, we have considered the location of the payload at $r_c = 0.99R$. We expressed the design specifications (ii) and (iii) in dB since it is the appropriate way for our design method.

By stability we mean that all the poles should be in the left half of the complex plane, bounded away from the imaginary axis (i.e. exponential stability). This can be checked by the Nyquist criterion. Due to formula (3.5) the closed loop is stable if we design the feedback loop in a stable way, and if the factor D_1 (unchanged by feedback) is stable. This explains why, in (ii),

the stability margin is imposed on $\frac{L}{1+L}$ and not

on T_C . We recall, see Horowitz and Sidi (1972),

that the peak value of $\left| \frac{L}{1+L} \right|$ is related to the

damping factor of the closed loop system, that is to the attenuation of vibrations in which we are interested.

We explain now the way the specifications in (iii) are chosen. They embody a frequency-domain representation of the following requirement: the family (indexed by m and EI) of step responses of the tip of the (controlled) beam should evolve in time in a tube with boundaries given by the step responses of the first order (lumped) systems with transfer functions given in the left and right hand sides of (iii). Admittedly, this translation of specifications from time domain to frequency is not rigorous. But, as it is

well-known, a precise translation procedure is still an unsolved problem. We based, however, our choice of specifications on an intuitive idea in Horowitz (1976), i.e. that one can achieve a certain change in time domain for the system with an appropriate change in the corresponding frequency response. The computer simulations and the laboratory experiments confirm this intuition.

The specifications we have adopted are rather tough, and we do not expect them to be satisfied over a large range of frequencies. What we expect is to satisfy specifications in a certain frequency range, thereby securing a reasonable satisfaction of the time domain specifications as well.

Finally, one can see that the specifications (iii) leave a small margin at frequency zero, corresponding to a narrow margin (1.02 vs 0.98) in the steady state. It is true that there is no uncertainty at frequency zero in the case of our plant (for H in (3.2) and the characteristic roots of the differential equation for computing $P_{r/2}$, see Remark 3.1, do not depend on m and EI at $s = 0$). However, if we want to satisfy specifications on a reasonable (i.e. not very small) range of frequency it is better to leave such a margin.

5. THE DESIGN PROCEDURE

The design methodology is an extension of the quantitative feedback theory (QFT), as it is presented in Horowitz and Sidi (1972), to our distributed case. The method makes use of the logarithmic complex plane, i.e. the Nichols Chart, for translating the specifications on the closed loop into bounds on the (open) loop transmission. This is accomplished by taking advantage of the fact that the Nichols Chart has two systems of coordinates imprinted on it: a rectangular one for the magnitude, in dB, and the phase, in degrees, of a complex number H , and a curved one for the magnitude and phase of the complex number $\frac{H}{1+H}$.

The compensator G and the prefilter F are designed separately. By G we assure that the variation in $|T_C|$ (when the parameters go over their uncertainty range) is within the maximum variation allowed in (iii). By F we assure that the actual spread of (the family) $|T_C|$ is between the bounds given in (iii).

Design of G

Step 1. Build the (uncertainty) templates of the plant.

That is, one has to find the locus in the Nichols Chart, for each ω , of $P_{eqv}(j\omega, m, EI)$, when m and EI vary over their range of uncertainty, i.e. $0 \leq m \leq 0.5$ and $20 \leq EI \leq 25$. In Fig. 3 some templates are displayed. Actually, only a few templates are enough for performing the design.

FIG. 3(a). Template for $\omega = 1$.

FIG. 3(b). Template for $\omega = 3$.

FIG. 3(c). Template for $\omega = 4$.

(d)

FIG. 3(d). Template for $\omega = 7$.

Step 2. Find bounds on a nominal loop transmission L .

We have chosen as nominal point ($m = 0.5$, $EI = 20$), represented by a thick point in Fig. 3. However, the design does not depend on this choice. Indeed, as we shall see by using the templates, by satisfying bounds for the nominal loop transmission we are assured that the variation of T_C due to the parameter uncertainty is within the prespecified limits.

Let us consider T_C (see (3.5)) as a map from the set $\mathcal{P}$ of plants denoted by P_{eqv} and indexed by m and EI , to the complex plane,

$$T_C(P_{eqv}) = F \frac{L}{1+L} D_1.$$

Then, if Δ is the difference operator,

$$\Delta \ln T_C = \Delta \ln \left(\frac{L}{1+L} D_1 \right). \quad (5.1)$$

F disappears since it is a constant (i.e. it has no uncertain parameters), hence it is canceled in the difference operation given by Δ . Satisfying (iii) (from Section 4) over $\mathcal{P}$ will result, via (5.1), in bounds (on the nominal) $L(j\omega)$ for a certain range of frequencies. Actually, the bounds are obtained by manipulating the templates in the Nichols Chart. Typical bounds are shown in Fig. 4. The admissible regions are the shaded ones.

More precisely, we shall find the bounds by solving a modified QFT problem, which resembles the classical (lumped) one: modified in the sense that the term D_1 will be incorporated in the specifications, but the problem will not become a genuine lumped one due to the nonrational P_{eqv} . Thus by (5.1), (iii) is satisfied at a given frequency if

$$\Delta \left| \frac{L}{1+L} \right|_{dB} \leq T_{\max} - T_{\min} - (D_{\max} - D_{\min}), \quad (5.2)$$

FIG. 4. Nichols chart design of G .

over all the parameter uncertainty range. Here $T_{\max}$, $T_{\min}$ are the functions appearing at the right hand side and left hand side of the inequalities (4.1), respectively, and $D_{\max}$, $D_{\min}$ are the maximum and minimum value of $|D_1|_{dB}$ over the parameter uncertainty range. Also, the right hand side of (5.2) represents the actual specifications on $\frac{L}{1+L}$.

From here we can see that a necessary condition to satisfy (iii) is that

$$D_{\max} - D_{\min} \leq T_{\max} - T_{\min}. \quad (5.3)$$

Numerical results summarized in Table 1 below show that for $\omega = 1.5$ the inequality (5.3) is still satisfied, however for $\omega = 2$ it is not. But even for $\omega = 1.5$ the margin is so small (i.e.

TABLE 1. PARAMETER SPECIFICATIONS FOR DESIGN OF F

ω [rad sec ⁻¹]	0.01	0.1	1.0	1.5	2.0
$T_{\max}$	0.171	0.149	-1.618	-3.148	-4.657
$T_{\min}$	-0.175	-0.218	-3.185	-5.294	-7.165
$D_{\max}$	5.488	5.495	6.341	7.526	9.484
$D_{\min}$	5.487	5.490	5.759	6.107	6.615
$T'_{\max}$	≈ 0	-0.035	-1.752	-2.314	-2.462
$T'_{\min}$	≈ 0	-0.042	-2.111	-2.920	-3.359
Specs: $\frac{L}{1+L}$	0.345	0.362	0.985	0.727	-0.355
Specs: F	-5.316	-5.350	-6.206	-8.360	-11.670
	-5.662	-5.690	-6.833	-8.481	-10.421

$\frac{L}{1+L}$ has to be so insensitive to parameter uncertainty) that it requires an extremely high gain feedback. (We recall that this is not practical since it greatly amplifies the sensor noise at the input of the plant.) That is why we chose to satisfy (iii) on the smaller frequency range, $0 \leq \omega \leq 1$.

Remark 5.1. We stress that the limitation in the feedback loop capabilities given by (5.3) does not appear in the lumped case where $D_1 = 1$. They appear in our case because we want to control the behavior of a plant distributed in space, by using a point (i.e. not distributed in space) feedback loop.

The remaining requirements of the design problem, (i) and (ii), refer to stability. The Nyquist stability criterion can be traced back on the Nichols Chart (the critical point being now 0 dB and -180 degrees). To apply this criterion we have to know the number of unstable poles of the plant; by frequency-domain simulations we conclude that the family of plants P_{eqv} is stable.

As a matter of fact we have not only to stabilize, but to achieve a given stability margin (ii), in a robust way with respect to (m, EI) . Therefore we shall use again the (uncertainty) templates even for frequencies larger than $\omega = 1$, which was the limit for achieving performance specifications (iii). This time we shall use them for achieving robust stabilization, i.e. with the stability specification (ii), obtaining more bounds to be satisfied by the nominal loop transmission. In Fig. 4 a bound can be seen corresponding to $\omega = 7$.

Step 3. Shape the loop transmission, by means of the compensator G , to satisfy both the performance and stability bounds obtained at Step 2. This is done by a usual rational function compensation procedure.

The QFT methodology enables a suboptimal design, in the sense of reducing the cost of feedback, i.e. the effect of the sensor noise at the input of the plant (or roughly speaking, the bandwidth of the closed loop). However, the closer the suboptimal design is to an optimal one the more complex structure is needed from the compensator. Since in our case the frequency range for achieving performance specifications is very small it does not seem necessary to try to satisfy it with a compensator of complicated structure.

Therefore, our first approximation for a compensator was a second order system. To satisfy the stability bounds as well, we had to consider also a first order numerator for the

compensator. By shaping the loop transmission L , we found

$$G = \frac{1.66(s + 1.33)}{(s + 4)(1 + s/20)}.$$

One may check that the zeros of $1 + GM$ are in the left half of the complex plane, see Remark 3.2.

Let us note that design methods by loop shaping have re-emerged within other recent approaches to feedback control, see Glover and MacFarlane (1990).

Remark 5.2. As can be seen from Fig. 4, for $\omega = 3$ and $\omega = 4$ our design is not far from the corresponding stability bounds. We could afford this closeness since we worked with the exact, i.e. the non-reduced model of the plant, as the one provided in Section 2. This shows the usefulness of the exact model of the plant to the robust stabilization of distributed systems.

Design of F

The prefilter F should be designed in such a way that (iii) of Section 4 will be satisfied. Therefore,

$$T_{\min} \leq |F|_{\text{dB}} + \left| \frac{L}{1+L} \right|_{\text{dB}} + |D_1|_{\text{dB}} \leq T_{\max},$$

over all the uncertainty range. Hence,

$$T_{\min} - T'_{\min} - D_{\min} \leq |F|_{\text{dB}} \leq T_{\max} - T'_{\max} - D_{\max}, \quad (5.4)$$

where $T_{\max}$, $T_{\min}$, $D_{\max}$ and $D_{\min}$ are from (5.2), and $T'_{\max}$, $T'_{\min}$ are the maximum and minimum values over the range of parameter uncertainty of $\left| \frac{L}{1+L} \right|_{\text{dB}}$.

Since G has already been designed, L can be readily computed. With the numerical results from Table 1 and another rational function compensation procedure, we obtained the prefilter

$$F = \frac{0.54}{1 + s/1.57},$$

which completes the design of the feedback loop. In Table 1, at the specifications for F , the first line corresponds to the right hand side of (5.4) and the second line to the left hand side of (5.4). Let us note that from $\omega = 2$ on it is not possible to satisfy specifications anymore, because they are contradictory. Also, $D_{\max}$, $T'_{\max}$ correspond to $m = 0.5$, $EI = 20$, and $D_{\min}$, $T'_{\min}$ correspond to $m = 0$, $EI = 25$.

The designed feedback loop solves the problem of Section 4; specification (iii) is satisfied for $0 \leq \omega \leq 1$.

Remark 5.3. It is interesting to note and study, mathematically, the “morphogenesis” of the templates (Fig. 3) as the frequency increases: from a very small strip they grow to large, curved ones, and finally shrink back to zero. Such a study could shed some light on the physical phenomena involved in manufacturing flexible beams.

Remark 5.4. The gain of the compensator we designed is relatively small. In this way we have prevented the non-minimum-phase property of the equivalent plant in (3.7) from damaging the stability of the closed loop in (3.5). (The non-minimum-phase property is a consequence of the non-zero distance between the location of the control action, i.e. the root of the beam, and the location of the controlled point, i.e. the tip of the beam, compare (Kelemen *et al.*, 1990).) The price to be paid is a relatively slow time response. This, however, helps in keeping the behavior of the plant (beam) close to the linear approximation we have adopted in this work, see Remark 2.2.

Time domain simulations

In Fig. 5 we have collected time responses to a unitary step input of the feedback system we have designed in this section. The coordinate y denotes the arc described by the tip of the beam. We have represented responses corresponding to various (and quite different) combinations of the parameters m and EI belonging to the admissible uncertainty range. In each figure we

FIG. 5(a). Time response for $m = 0.5$, $EI = 20$.

FIG. 5(b). Time response for $m = 0.5$, $EI = 25$.

FIG. 5(c). Time response for $m = 0$, $EI = 20$.

FIG. 5(d). Time response for $m = 0$, $EI = 25$.

FIG. 5(e). Time response for $m = 0$, $EI = 22.3$.

FIG. 5(f). Time response for $m = 0.25$, $EI = 22.3$.

FIG. 5(g). Time response for $m = 0.5$, $EI = 22.3$.

have represented also the specifications corresponding to (iii) in Section 4, i.e. the time responses to a unitary step of the systems with transfer functions appearing in the left, respectively right hand side of (4.1). (These are the curves which settle to 0.98 and 1.02, respectively.)

Let us note that the specifications are satisfied, at least for larger t (i.e. >4 [s]), for all the cases presented. In fact, this is true for the worst case which corresponds to the maximal payload and minimal flexural rigidity, where after an initial oscillation the behavior is quite good. For the remaining cases the situation is much better, and when the payload is zero it is even very good. This is quite a good performance if we remember that the specifications in the frequency domain were satisfied only in a very small range, i.e. for $\omega \in [0, 1]$. A possible explanation is in the combination of the frequency-domain (quantitative) design procedure with the Final Value Theorem (which relates the behavior of the time response for large t to that of the frequency response at small ω).

As a matter of fact we have shown that the frequency-domain specifications cannot be satisfied for ω larger than 2. This suggests that one should not expect, in the time domain, to do considerably better than what is shown in Fig. 5.

A closer look at the time responses shows the following.

- (a) The "final values" seem to be the same. This has two causes: in steady state our plant is insensitive to parameter variations (see the end of Section 4); and the motor is a pure integrator. But, in view of (5.3) it is reasonable to leave some margin in the specifications (which is very small in our case), even at frequency zero.
- (b) For a given flexural rigidity (EI) of the beam, the closed loop system behaves better (smaller oscillations, smoother evolution) when the payload is smaller.
- (c) When the payload is given, the closed loop system behaves better when the flexural rigidity is higher.

These remarks, compatible with the physical intuition, could have some practical value in designing the beam because they could help settle contradictory requirements. High speed and low energy consumption call for increased elasticity; but a too high elasticity could render the control of the beam impractical, if not altogether impossible. To evaluate the effort required to control economically the beam, some estimates of the dependence of the solution of (2.2)–(2.5) (or in frequency domain of P_0 from

(3.8)) in terms of m and EI could be of great help. These ideas deserve a separate study which will not be performed in the present article.

One final point to be mentioned is that this designed feedback loop has been successfully implemented on an actual flexible robot arm in the Control Engineering Laboratory, University of Twente. The details of laboratory experiments can be found in Kelemen *et al.* (1992).

6. CONCLUSIONS

In this article we have solved the following problem: achieving (to the extent possible) quantitative specifications for the movement of a flexible robot arm carrying a payload. This has been done by a quantitative feedback methodology for designing a feedback loop to control the flexible arm, and to compensate for the uncertainty in the payload and flexural rigidity of the beam. The same design procedure has clarified some limitations in the feedback loop capabilities.

The feedback loop is based on a single measurement, the angle of the motor shaft. Since this type of constraint is encountered in many practical situations, our technique shows how to achieve benefit of feedback under these conditions. In the design we have used the non-reduced model of the plant, which proved to be useful in the robust stabilization of the closed loop. The time domain simulations confirm the qualities of the design procedure presented in this work.

At the same time we have pointed out some interesting new problems: a mathematical study of the way in which the payload and flexural rigidity of the beam influences the dynamical properties of the plant and closed loop, and alternative feedback structures for controlling the beam.

Finally, we plan to apply our design methodology to more complex plants (e.g. involving torsional vibrations as well) in order to compare it with some existing procedures mentioned above. This will call for a multivariable design methodology.

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