The linear-viscoelastic behaviour of a dispersion of transversely rigid spherical capsules

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Abstract: A rheological model has been derived for the linear-viscoelastic behaviour of a dispersion of transversely rigid spherical capsules. The model incorporates finite thickness of the elastic shell of the capsules, anisotropy of the mechanical properties of the interface and finite volume fraction. The dynamic viscosity of the dispersion is calculated. The influence of the microstructural parameters is considered and the results are compared with those of other models. The model shows that finite thickness of the shell can strongly influence the relaxation times.

Key words: Capsule, transversely rigid shell, dispersion, complex viscosity, linear viscoelasticity

Notations

a	– particle radius
A	– 6×6 matrix
b	– radius of the cell
B	– 6×6 matrix
C	– 6×1 matrix
D	– rate of strain tensor
e	– unit vector
E	– Young's modulus
f	– correction factor
F_λ	– force vector
g_λ	– scalar quantity defined in eq. (20)
G	– constant proportional to the applied rate of strain
h	– shell thickness
k	– particle index
L	– $(= h/2a_m)$ relative shell thickness
m_λ	– resultant surface moment per unit surface
M_λ	– resultant bending moment per unit length
$M_{\lambda\mu}$	– resultant twisting moment per unit length
n	– normal vector
N_λ	– resultant normal force per unit length
$N_{\lambda\mu}$	– resultant shear force per unit length
p	– pressure
p_0	– equilibrium pressure
q_λ	– resultant loading force per unit surface
Q_λ	– resultant shear force per unit length
r	– spherical coordinate
$\mathbf{r}$	– position vector
R	– $(= b/a_m)$ relative cell radius
R_λ	– radius of curvature
s	– displacement vector
s_λ	– component of s
S	– area

t	– time
$\mathbf{T}$	– stress tensor in the fluid
$T_{rr}, T_{\phi r}$	– components of $\mathbf{T}$
$\mathbf{u}$	– velocity vector
u_r, u_ϕ	– spherical components of $\mathbf{u}$
$\mathbf{u}^0$	– velocity vector at the sample boundary
V	– $(= \eta^{(i)}/\eta^{(e)})$ viscosity ratio
V_c	– cell volume
∂V_c	– cell surface
V_p	– particle volume
∂V_p	– particle surface
V_s	– sample volume
∂V_s	– sample surface
$X_1, \dots, X_6$	– functions of R, L, V and v
$\mathbf{Y}$	– 6×1 matrix
$Y_1, \dots, Y_6$	– components of $\mathbf{Y}$
Z	– $(= 2\omega\eta^{(e)}a_m/(Eh))$
α, β	– curvilinear surface coordinates
$\gamma_{\lambda\mu}$	– strain component
ε_λ	– strain component
ζ	– distance along the normal to the middle surface
η	– viscosity
η^*	– $(= \eta' - i\eta'')$ complex viscosity
η_{spec}^*	– $(= (\eta^* - \eta^{(e)})/\eta^{(e)} = \eta'_{\text{spec}} - i\eta''_{\text{spec}})$ specific complex viscosity
θ	– spherical coordinate
θ_λ	– strain component
κ	– surface dilatational modulus
K_λ	– strain component
μ	– surface shear modulus
ν	– Poisson ratio
σ_λ	– normal force per unit surface
$\tau_{\lambda\mu}$	– shear force per unit surface
ϕ	– spherical coordinate

- Φ – volume concentration
 ω – angular frequency

Subscripts

- ext – including shell volume
 int – excluding shell volume
 l – longest
 m – middle surface
 s – shortest
 α, β, ζ – component in α, β, ζ direction

Superscripts

- e – external fluid
 i – internal fluid
 ζ – quantity at a distance ζ from the middle surface

1. Introduction

That part of the rheology of emulsions which focuses on the influence of the mechanical properties of the almost spherical interface between dispersed and continuous phase has attracted increasing interest over the last few years. A few reasonably well defined model systems are available to study interfaces experimentally. Dynamic shear experiments have been applied for the investigation of ionic and nonionic (micro-) emulsions [1, 2]. In particular non-linear properties of capsule dispersions with cross-linked polymeric interfaces [3] have been studied. Potentially spherical vesicles would seem a suitable subject for further study.

On the theoretical side several models [4–9] are available. A survey, arranged with aid of the param-

eters volume fraction ϕ and relative shell thickness L , of these models as far as the complex viscosity is concerned is given in table 1.

The subject of the present work was to develop a theoretical model for a capsule dispersion that incorporates finite thickness, anisotropy of the mechanical properties of the interface and finite volume fractions and which would allow us to interpret the results of dynamic shear measurements for instance of vesicles.

The considered capsules consist of a Newtonian fluid with viscosity $\eta^{(i)}$ enclosed in a spherical transversely rigid shell of thickness h . A capsule is surrounded by a Newtonian fluid with viscosity $\eta^{(e)}$ (see figure 1). The shell is infinitely stiff in a direction perpendicular to the surface and mechanically isotropic in a tangential direction. The mechanical properties in the shell are characterized by the Young's modulus E and the Poisson ratio ν . Transversely rigid shells simplify the relationship between forces and deformations in the hydrodynamic description of the system. It is assumed

Fig. 1. Schematic picture of the capsule

Table 1. Survey of theoretical models

volume fraction \ shell thickness	$L = 0^*)$	$L \leq 0.05$	L arbitrary
Φ finite	Oosterbroek [7] – $\omega \geq 0$	Present work – $\omega \geq 0$ – transversely rigid	
$\Phi \rightarrow 0$	Oldroyd [4] – $\omega \geq 0$ Barthès-Biesel and Chhim [8] – $\omega \rightarrow 0$ – incompressible – isotropic		Sakanishi and Takano [5] Brunn [6] – $\omega \geq 0$ – incompressible – isotropic Takano and Sakanishi [9] – $\omega \geq 0$ – incompressible – anisotropic

*) With $L = 0$ the limit described in section 5.2 is meant.

to be applicable up to $L = h/(2a_m) = 0.05$ [10], where a_m is the radius of the so-called middle surface of the shell. To be able to account for hydrodynamic interactions a cell model has been used. At the cell surface a harmonically oscillating pure straining flow is applied. The flow is so weak that the capsules remain almost spherical. For the formulation of the boundary conditions at the shell the occurring stresses and strains for small ellipsoidal deformations are considered. The complete set of six boundary conditions (at the cell surface and the shell) makes it possible to solve the six unknowns. One of these determines the complex viscosity.

The results are derived numerically and presented graphically. From the present work it is possible to deduce results for other models [4–8] as far as they overlap.

Some striking results can be interpreted by considering the influence of the parameters and the frequency on the mechanics of an oscillating straining process.

2. Formal calculation of the complex viscosity

2.1 Volume averaged stress tensor

For the calculation of the complex viscosity η^* we will use the general definition:

$$\langle \bar{\mathbf{T}} \rangle \equiv 2\eta^* \langle \mathbf{D} \rangle. \quad (1)$$

In (1) the deviatoric part $\langle \bar{\mathbf{T}} \rangle$ of the volume-averaged stress $\langle \mathbf{T} \rangle$ may be evaluated by surface integrals over the particles [11]

$$\langle \bar{\mathbf{T}} \rangle = 2\eta \langle \mathbf{D} \rangle + \frac{1}{V_s} \sum_k \int_{\partial V_s} \{ \mathbf{T} \cdot \mathbf{n} \mathbf{r} - \eta (\mathbf{u} \mathbf{n} + \mathbf{n} \mathbf{u}) - \frac{1}{3} (\mathbf{r} \cdot \mathbf{T} \cdot \mathbf{n}) \mathbf{1} \} dS. \quad (2)$$

The volume averaged rate of strain tensor $\langle \mathbf{D} \rangle$ may be related to the macroscopic velocity at the sample boundaries

$$\langle \mathbf{D} \rangle = \frac{1}{2V_s} \int_{\partial V_s} (\mathbf{u}^0 \mathbf{n} + \mathbf{n} \mathbf{u}^0) dS. \quad (3)$$

In principle this method gives a complete constitutive equation, but for spherical particles η^* turns out to be just a scalar function [12].

2.2 Statistical cell model

To account for the hydrodynamic interaction between deformable spheres a statistical cell model will be used. In this model it is assumed that the averaged hydrodynamic effect of neighbouring particles may be expressed in boundary conditions that apply at a cer-

tain distance b from each particle. This distance can be regarded as the hydrodynamic interaction radius. The concentration dependence of properties calculated with this model is governed by an assumed relation between b and the concentration. This is a semi-empirical relation.

In the low-concentration region a relatively simple expression for b was proposed by Simha [13]:

$$b = f \cdot a \cdot \Phi^{-1/3}. \quad (4)$$

Thomas [14] compared the results of Simha's calculations with high-shear steady-state viscosity data on suspensions of rigid spheres. He found a good agreement with experimental data if $f = 1.111$ for $\Phi < 0.15$. Although the statistical cell model is not selfconsistent theoretically [15] it is very useful for describing the concentration dependence.

At $r = b$ we shall use Simha's boundary conditions [12], i.e. the velocity at $r = b$ is equal to the velocity of a homogeneous sample at that place. We only consider the pure straining part of the flow field and neglect the rotational part of it since the particles are assumed to remain almost spherical in the flow. For convenience we will take the harmonically oscillating flow at the boundaries of the sample and cell to be identical with the flow field used by Oldroyd [4, 16]. Complex notation will be used.

$$\mathbf{u}^0 = G \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \mathbf{r} \exp(i\omega t) \text{ for } \begin{cases} \mathbf{r} \text{ at } \partial V_c, & (5a) \\ \mathbf{r} \text{ at } \partial V_s. & (5b) \end{cases}$$

2.3 Hydrodynamics

Both internal and external fluid are treated as incompressible Newtonian fluids:

$$\nabla \cdot \mathbf{u} = 0, \quad (6)$$

$$\mathbf{T} = -p \mathbf{1} + 2\eta \mathbf{D}. \quad (7)$$

The Reynolds number is taken to be sufficiently small for the creeping-motion equation to be valid.

$$\eta \nabla^2 \mathbf{u} = \nabla p. \quad (8)$$

Lamb [17] has given the solution of (6–8). The formulae with the symmetry of (5) are most conveniently expressed in spherical coordinates [4]

$$u_r = \left\{ \frac{1}{7} A r^3 + \frac{1}{2} B r^{-2} + 2 C r - 3 D r^{-4} \right\} \cdot \left\{ \frac{3}{2} \cos^2 \phi - \frac{1}{2} \right\} \{ \exp(i\omega t) \}, \quad (9)$$

$$u_\phi = \left\{ \frac{5}{42} Ar^3 + Cr + Dr^{-4} \right\} \{-3 \cos \phi \sin \phi\} \cdot \{\exp(i\omega t)\}, \quad (10)$$

$$p = \eta \{Ar^2 + Br^{-3}\} \left\{ \frac{3}{2} \cos^2 \phi - \frac{1}{2} \right\} \cdot \{\exp(i\omega t)\} + p_0. \quad (11)$$

The components of the stress tensor that are relevant to the calculation of η^* are given by

$$T_{rr} = \eta \left\{ -\frac{1}{7} Ar^2 - 3Br^{-3} + 4C + 24Dr^{-5} \right\} \cdot \left\{ \frac{3}{2} \cos^2 \phi - \frac{1}{2} \right\} \{\exp(i\omega t)\} - p_0, \quad (12)$$

$$T_{r\phi} = \eta \left\{ \frac{8}{21} Ar^2 + \frac{1}{2} Br^{-3} + 2C - 8Dr^{-5} \right\} \cdot \{-3 \cos \phi \sin \phi\} \{\exp(i\omega t)\}. \quad (13)$$

2.4 Formal solution of η^*

It is now possible to express η^* formally in terms of the constant $B^{(e)}$ of the flow field in the external fluid and the applied rate of strain [7]. The flow field and the pressure field are given as a function of 8 unknown parameters. If there are no net forces acting on the particles and inertia effects are neglected, the choice of the origin has no effect on the evaluation of the integral in (1). Thus it is permissible to choose a different origin for each particle. Using eqs. (9, 10, 12, 13) we find

$$\langle \bar{T} \rangle = 2\eta^{(e)} \langle D \rangle - \frac{1}{V_s} \sum_k \frac{2\pi\eta^{(e)} B^{(e)}}{3} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \exp(i\omega t). \quad (14)$$

The specific viscosity of the dispersion may now be expressed in $B^{(e)}/G$, using (1, 5b, 14)

$$\eta_{\text{spec}}^* = \frac{\eta^* - \eta^{(e)}}{\eta^{(e)}} = -\frac{1}{4} \Phi a^{-3} \frac{B^{(e)}}{G}. \quad (15)$$

Thus η^* can be solved by finding the value of $B^{(e)}/G$. This can be done by formulating a set of eight linear homogeneous equations in the parameters $A^{(i)}$, $B^{(i)}$, $C^{(i)}$, $D^{(i)}$, $A^{(e)}$, $B^{(e)}$, $C^{(e)}$ and $D^{(e)}$ from which $B^{(e)}/G$ can be solved. Two equations follow from the continuity of the velocity at the cell boundary, and another two from the finiteness of the velocity at $r=0$.

The continuity of the velocity and the forces at the boundaries of the shell (in principle eight equations) are reduced to four equations (see sections 4.3 and 4.4).

To formulate equations describing the continuity of the forces across the shell it is necessary to look more accurately to the mechanics of transversely rigid axisymmetric shells.

3. Mechanics of transversely rigid axisymmetric shells

3.1 Forces and moments in shells

For the description of the mechanics of transversely rigid axisymmetric shells we will closely follow the shell theory described by Seide [10].

A shell of constant thickness is defined by the equation of its middle surface and the magnitude of the thickness. A system of orthogonal curvilinear coordinates for points inside the shell is defined by the coordinates α and β corresponding to the lines of curvature on the middle surface and the distance ζ along the normal to the middle surface (see figure 2).

We will first summarize the differential equations of the three-dimensional theory of elasticity for the region between the bounding surfaces of the shell. To obtain equilibrium equations for stresses in the shell, we consider a small element in the shell bounded by two surfaces parallel to the middle surface of the shell, a distance $d\zeta$ apart, and four adjacent surfaces perpendicular to the middle surface with normals along the lines of curvature. As is usual in small deformation theory, we assume that the effects of geometry changes of the element on equilibrium of forces are negligible. The surface forces on the element of the shell are shown in figure 2.

On three perpendicular adjacent surfaces of the element, dS_α , dS_β and dS_ζ the forces $\mathbf{F}_\alpha^{(\zeta)}$, $\mathbf{F}_\beta^{(\zeta)}$ and $\mathbf{F}_\zeta^{(\zeta)}$ act, e.g.

$$\mathbf{F}_\alpha^{(\zeta)} = [(\sigma_\alpha)^{(\zeta)} \mathbf{e}_\alpha + \tau_{\alpha\beta}^{(\zeta)} \mathbf{e}_\beta + \tau_{\alpha\zeta}^{(\zeta)} \mathbf{e}_\zeta] g_\beta^{(\zeta)} d\beta d\zeta.$$

Fig. 2. Surface forces acting on the shell element

On the three opponent perpendicular surfaces, the forces are given by

$$\mathbf{F}_\alpha^{(\zeta)} + \frac{\partial \mathbf{F}_\alpha^{(\zeta)}}{\partial \alpha} d\alpha, \quad \mathbf{F}_\beta^{(\zeta)} + \frac{\partial \mathbf{F}_\beta^{(\zeta)}}{\partial \beta} d\beta, \quad \mathbf{F}_\zeta^{(\zeta)} + \frac{\partial \mathbf{F}_\zeta^{(\zeta)}}{\partial \zeta} d\zeta.$$

When we assume body forces to be absent, we can formulate the vector equation of equilibrium by letting the sum of all the forces on the element be equal to zero. We then obtain

$$\frac{\partial \mathbf{F}_\alpha^{(\zeta)}}{\partial \alpha} d\alpha + \frac{\partial \mathbf{F}_\beta^{(\zeta)}}{\partial \beta} d\beta + \frac{\partial \mathbf{F}_\zeta^{(\zeta)}}{\partial \zeta} d\zeta = 0. \quad (16)$$

This leads to the following equations for the equilibrium of forces

$$\begin{aligned} & \frac{\partial}{\partial \alpha} \left[g_\beta \left(1 + \frac{\zeta}{R_\beta} \right) \sigma_\alpha^{(\zeta)} \right] + \frac{\partial}{\partial \beta} \left[g_\alpha \left(1 + \frac{\zeta}{R_\alpha} \right) \tau_{\alpha\beta}^{(\zeta)} \right] \\ & + \frac{\partial}{\partial \zeta} \left[g_\alpha g_\beta \left(1 + \frac{\zeta}{R_\alpha} \right) \left(1 + \frac{\zeta}{R_\beta} \right) \tau_{\alpha\zeta}^{(\zeta)} \right] \\ & + \frac{\partial g_\alpha}{\partial \beta} \left(1 + \frac{\zeta}{R_\beta} \right) \tau_{\alpha\beta}^{(\zeta)} - \frac{\partial g_\beta}{\partial \alpha} \left(1 + \frac{\zeta}{R_\alpha} \right) \sigma_\beta^{(\zeta)} \\ & + \frac{g_\alpha g_\beta}{R_\alpha} \left(1 + \frac{\zeta}{R_\beta} \right) \tau_{\alpha\zeta}^{(\zeta)} = 0, \end{aligned} \quad (17)$$

$$\begin{aligned} & \frac{\partial}{\partial \alpha} \left[g_\beta \left(1 + \frac{\zeta}{R_\beta} \right) \tau_{\alpha\beta}^{(\zeta)} \right] + \frac{\partial}{\partial \beta} \left[g_\alpha \left(1 + \frac{\zeta}{R_\alpha} \right) \sigma_\beta^{(\zeta)} \right] \\ & + \frac{\partial}{\partial \zeta} \left[g_\alpha g_\beta \left(1 + \frac{\zeta}{R_\alpha} \right) \left(1 + \frac{\zeta}{R_\beta} \right) \tau_{\beta\zeta}^{(\zeta)} \right] \\ & + \frac{\partial g_\beta}{\partial \alpha} \left(1 + \frac{\zeta}{R_\alpha} \right) \tau_{\alpha\beta}^{(\zeta)} - \frac{\partial g_\alpha}{\partial \beta} \left(1 + \frac{\zeta}{R_\beta} \right) \sigma_\alpha^{(\zeta)} \\ & + \frac{g_\alpha g_\beta}{R_\beta} \left(1 + \frac{\zeta}{R_\alpha} \right) \tau_{\beta\zeta}^{(\zeta)} = 0, \end{aligned} \quad (18)$$

$$\begin{aligned} & \frac{\partial}{\partial \alpha} \left[g_\beta \left(1 + \frac{\zeta}{R_\beta} \right) \tau_{\alpha\zeta}^{(\zeta)} \right] + \frac{\partial}{\partial \beta} \left[g_\alpha \left(1 + \frac{\zeta}{R_\alpha} \right) \tau_{\beta\zeta}^{(\zeta)} \right] \\ & + \frac{\partial}{\partial \zeta} \left[g_\alpha g_\beta \left(1 + \frac{\zeta}{R_\alpha} \right) \left(1 + \frac{\zeta}{R_\beta} \right) \sigma_\zeta^{(\zeta)} \right] \\ & - \frac{g_\alpha g_\beta}{R_\alpha} \left(1 + \frac{\zeta}{R_\beta} \right) \sigma_\alpha^{(\zeta)} - \frac{g_\alpha g_\beta}{R_\beta} \left(1 + \frac{\zeta}{R_\alpha} \right) \sigma_\beta^{(\zeta)} = 0, \end{aligned} \quad (19)$$

in which the scalar quantities g_λ and the radii of curvature R_λ are defined by

$$g_\lambda = \left(\frac{\partial \mathbf{r}}{\partial \lambda} \cdot \frac{\partial \mathbf{r}}{\partial \lambda} \right)^{1/2}, \quad (20)$$

$$\frac{1}{R_\lambda} = -\frac{1}{g_\lambda^2} \mathbf{e}_\lambda \cdot \frac{\partial^2 \mathbf{r}}{\partial \lambda^2}. \quad (21)$$

Equilibrium of moments about three orthogonal axes has been satisfied automatically by taking the magnitudes of shear stresses on adjacent faces to be equal i.e.

$$\tau_{\alpha\beta} = \tau_{\beta\alpha}, \quad (22)$$

$$\tau_{\alpha\zeta} = \tau_{\zeta\alpha}, \quad (23)$$

$$\tau_{\beta\zeta} = \tau_{\zeta\beta}. \quad (24)$$

3.2 Constitutive equations of shells

The constitutive equations give relations between the stresses and strains in materials. In the linear-elastic shell theory besides these equations usually also strain-displacement relations are given. Assuming the shell thickness to be constant during the deformation, we can describe the strain displacement relations by the following six equations:

$$\varepsilon_\alpha^{(\zeta)} = \frac{1}{g_\alpha \left(1 + \frac{\zeta}{R_\alpha} \right)} \left[\frac{\partial s_\alpha^{(\zeta)}}{\partial \alpha} + \frac{1}{g_\beta} \frac{\partial g_\alpha}{\partial \beta} s_\beta^{(\zeta)} + \frac{g_\alpha}{R_\alpha} s_\zeta^{(\zeta)} \right], \quad (25)$$

$$\varepsilon_\beta^{(\zeta)} = \frac{1}{g_\beta \left(1 + \frac{\zeta}{R_\beta} \right)} \left[\frac{\partial s_\beta^{(\zeta)}}{\partial \beta} + \frac{1}{g_\alpha} \frac{\partial g_\beta}{\partial \alpha} s_\alpha^{(\zeta)} + \frac{g_\beta}{R_\beta} s_\zeta^{(\zeta)} \right], \quad (26)$$

$$\varepsilon_\zeta^{(\zeta)} = \frac{\partial s_\zeta^{(\zeta)}}{\partial \zeta}, \quad (27)$$

$$\begin{aligned} \gamma_{\alpha\beta}^{(\zeta)} = & \frac{1}{g_\alpha \left(1 + \frac{\zeta}{R_\alpha} \right)} \left[\frac{\partial s_\beta^{(\zeta)}}{\partial \alpha} - \frac{1}{g_\beta} \frac{\partial g_\alpha}{\partial \beta} s_\alpha^{(\zeta)} \right] \\ & + \frac{1}{g_\beta \left(1 + \frac{\zeta}{R_\beta} \right)} \left[\frac{\partial s_\alpha^{(\zeta)}}{\partial \beta} - \frac{1}{g_\alpha} \frac{\partial g_\beta}{\partial \alpha} s_\beta^{(\zeta)} \right], \end{aligned} \quad (28)$$

$$\gamma_{\alpha\zeta}^{(\zeta)} = \frac{\partial s_\alpha^{(\zeta)}}{\partial \zeta} + \frac{1}{g_\alpha \left(1 + \frac{\zeta}{R_\alpha} \right)} \left[\frac{\partial s_\zeta^{(\zeta)}}{\partial \alpha} - \frac{g_\alpha}{R_\alpha} s_\alpha^{(\zeta)} \right], \quad (29)$$

$$\gamma_{\beta\zeta}^{(\zeta)} = \frac{\partial s_\beta^{(\zeta)}}{\partial \zeta} + \frac{1}{g_\beta \left(1 + \frac{\zeta}{R_\beta} \right)} \left[\frac{\partial s_\zeta^{(\zeta)}}{\partial \beta} - \frac{g_\beta}{R_\beta} s_\beta^{(\zeta)} \right], \quad (30)$$

in which the displacement vector $\mathbf{s}$ is given by

$$\mathbf{s} = s_\alpha \mathbf{e}_\alpha + s_\beta \mathbf{e}_\beta + s_\zeta \mathbf{e}_\zeta. \quad (31)$$

Since the shell material is assumed to be linear-elastic and isotropic in layers parallel to the middle surface,

we can write the following stress-strain relations.

$$\varepsilon_\alpha^{(\zeta)} = \frac{1}{E} [\sigma_\alpha^{(\zeta)} - \nu \sigma_\beta^{(\zeta)}] - \frac{\nu_\zeta}{E_\zeta} \sigma_\zeta^{(\zeta)}, \quad (32)$$

$$\varepsilon_\beta^{(\zeta)} = \frac{1}{E} [\sigma_\beta^{(\zeta)} - \nu \sigma_\alpha^{(\zeta)}] - \frac{\nu_\zeta}{E_\zeta} \sigma_\zeta^{(\zeta)}, \quad (33)$$

$$\varepsilon_\zeta^{(\zeta)} = \frac{1}{E_\zeta} \{ \sigma_\zeta^{(\zeta)} - \nu_\zeta [\sigma_\alpha^{(\zeta)} + \sigma_\beta^{(\zeta)}] \}, \quad (34)$$

$$\gamma_{\alpha\beta}^{(\zeta)} = \frac{2(1+\nu)}{E} \tau_{\alpha\beta}^{(\zeta)}, \quad (35)$$

$$\gamma_{\alpha\zeta}^{(\zeta)} = \frac{2(1+\nu_\zeta)}{E_\zeta} \tau_{\alpha\zeta}^{(\zeta)}, \quad (36)$$

$$\gamma_{\beta\zeta}^{(\zeta)} = \frac{2(1+\nu_\zeta)}{E_\zeta} \tau_{\beta\zeta}^{(\zeta)}. \quad (37)$$

If $E = E_\zeta$ and $\nu = \nu_\zeta$ we deal with the conventional Young's modulus and Poisson ratio, respectively.

3.3 Assumptions for transversely rigid shells

In most cases, solutions of the equations of three-dimensional elasticity theory are extremely difficult to obtain so that approximations to the theory are desirable. The so-called theory of transversely rigid shells is an approximation which replaces the three-dimensional problem by a two-dimensional problem which is somewhat easier to handle. The assumptions underlying this theory are as follows:

The distributions of stresses and displacements in a shell composed of a hypothetical material which has infinite resistance to extension in the direction normal to the middle surface and infinite resistance to shearing in planes normal to the middle surface are an adequate representation of the corresponding distributions in a shell composed of an isotropic material.

These assumptions for transversely rigid shells lead to

$$\nu_\zeta = \text{finite}, \quad (38)$$

$$E_\zeta = \infty. \quad (39)$$

This leads to a reduction of the stress-strain relations (32–37):

$$\varepsilon_\alpha^{(\zeta)} = \frac{1}{E} [\sigma_\alpha^{(\zeta)} - \nu \sigma_\beta^{(\zeta)}], \quad (40)$$

$$\varepsilon_\beta^{(\zeta)} = \frac{1}{E} [\sigma_\beta^{(\zeta)} - \nu \sigma_\alpha^{(\zeta)}], \quad (41)$$

$$\gamma_{\alpha\beta}^{(\zeta)} = \frac{2(1+\nu)}{E} \tau_{\alpha\beta}^{(\zeta)}, \quad (42)$$

$$\varepsilon_\zeta^{(\zeta)} = \gamma_{\alpha\zeta}^{(\zeta)} = \gamma_{\beta\zeta}^{(\zeta)} = 0. \quad (43)$$

These stress-strain relations correspond to those implied by the assumptions for the deformation attributed to Love and Kirchhoff [18] which can be described as follows:

1. Straight lines normal to the undeformed middle surface remain straight and normal to the deformed middle surface and do not change in length.
2. The normal stresses acting on surfaces parallel to the middle surface may be neglected in comparison with the other stresses.

From these deformation assumptions, it can be derived that the tangential displacements are linear across the shell and that the normal displacements are constant. As a consequence of these assumptions the strain-displacement relations reduce to the following equations expressed in the deformation of the middle surface:

$$\varepsilon_\alpha^{(\zeta)} = \frac{1}{g_\alpha \left(1 + \frac{\zeta}{R_\alpha}\right)} \left[\frac{\partial s_\alpha}{\partial \alpha} + \frac{1}{g_\beta} \frac{\partial g_\alpha}{\partial \beta} s_\beta + \frac{g_\alpha}{R_\alpha} s_\zeta - \zeta \left[\frac{\partial \theta_\alpha}{\partial \alpha} + \frac{1}{g_\beta} \frac{\partial g_\alpha}{\partial \beta} \theta_\beta \right] \right], \quad (44)$$

$$\varepsilon_\beta^{(\zeta)} = \frac{1}{g_\beta \left(1 + \frac{\zeta}{R_\beta}\right)} \left[\frac{\partial s_\beta}{\partial \beta} + \frac{1}{g_\alpha} \frac{\partial g_\beta}{\partial \alpha} s_\alpha + \frac{g_\beta}{R_\beta} s_\zeta - \zeta \left[\frac{\partial \theta_\beta}{\partial \beta} + \frac{1}{g_\alpha} \frac{\partial g_\beta}{\partial \alpha} \theta_\alpha \right] \right], \quad (45)$$

$$\gamma_{\alpha\beta}^{(\zeta)} = \frac{1}{g_\alpha \left(1 + \frac{\zeta}{R_\alpha}\right)} \left[\frac{\partial s_\beta}{\partial \alpha} - \frac{1}{g_\beta} \frac{\partial g_\alpha}{\partial \beta} s_\alpha - \zeta \left[\frac{\partial \theta_\beta}{\partial \alpha} - \frac{1}{g_\beta} \frac{\partial g_\alpha}{\partial \beta} \theta_\alpha \right] \right] + \frac{1}{g_\beta \left(1 + \frac{\zeta}{R_\beta}\right)} \left[\frac{\partial s_\alpha}{\partial \beta} - \frac{1}{g_\alpha} \frac{\partial g_\beta}{\partial \alpha} s_\beta - \zeta \left[\frac{\partial \theta_\alpha}{\partial \beta} - \frac{1}{g_\alpha} \frac{\partial g_\beta}{\partial \alpha} \theta_\beta \right] \right], \quad (46)$$

$$\varepsilon_\zeta^{(\zeta)} = \gamma_{\alpha\zeta}^{(\zeta)} = \gamma_{\beta\zeta}^{(\zeta)} = 0, \quad (47)$$

in which

$$\theta_\alpha = \frac{1}{g_\alpha} \left[\frac{\partial s_\zeta}{\partial \alpha} - \frac{g_\alpha}{R_\alpha} s_\alpha \right], \quad (48)$$

$$\theta_\beta = \frac{1}{g_\beta} \left[\frac{\partial s_\zeta}{\partial \beta} - \frac{g_\beta}{R_\beta} s_\beta \right]. \quad (49)$$

3.4 Stress resultants and equilibrium conditions

In a straightforward approach we could express the stresses $\sigma_\alpha^{(\zeta)}$, $\sigma_\beta^{(\zeta)}$ and $\tau_{\alpha\beta}^{(\zeta)}$ in terms of the middle surface

displacements by means of eqs. (40–42, 44–46). The equilibrium conditions (2–4) together with the known values of $\sigma_\zeta^{(\zeta)}$, $\tau_{\alpha\zeta}^{(\zeta)}$ and $\tau_{\beta\zeta}^{(\zeta)}$ on the inner and outer parallel bounding surfaces defined by $\zeta = \pm h/2$ make it possible to solve $\sigma_\zeta^{(\zeta)}$, $\tau_{\alpha\zeta}^{(\zeta)}$ and $\tau_{\beta\zeta}^{(\zeta)}$ in terms of the middle surface displacements.

However, it is possible to avoid the consideration of the detailed distributions of stresses in the shell and only deal with the known values of these stresses on the surfaces bounding the shell by integrating the equations of equilibrium over the wall thickness. Stress resultants, expressed as moments and forces per unit length of middle surface, act on the sides of a shell element bounded by the parallel surfaces $\zeta = \pm h/2$ and planes normal to the middle surface. The sides of this element will be taken in the directions of adjacent lines of curvature. The shell element in figure 2 is only a small portion of this overall shell element.

The force components per unit length of the middle surface can be defined by integrating the appropriate stresses over the cross section on which they act and dividing them by the differential length of the surface line of curvature, see figure 3. Bending and twisting moment per unit middle surface length are defined similarly, see figure 4.

Fig. 3. Resultant force components acting on the shell element

Fig. 4. Resultant moment components acting on the shell element

By integrating the equilibrium conditions (2–4) over the shell thickness we find:

$$\frac{\partial}{\partial \alpha} (g_\beta N_\alpha) + \frac{\partial}{\partial \beta} (g_\alpha N_{\beta\alpha}) + \frac{\partial g_\alpha}{\partial \beta} N_{\alpha\beta} - \frac{\partial g_\beta}{\partial \alpha} N_\beta + g_\alpha g_\beta \left(\frac{Q_\alpha}{R_\alpha} + q_\alpha \right) = 0, \quad (50)$$

$$\frac{\partial}{\partial \beta} (g_\alpha N_\beta) + \frac{\partial}{\partial \alpha} (g_\beta N_{\alpha\beta}) + \frac{\partial g_\beta}{\partial \alpha} N_{\beta\alpha} - \frac{\partial g_\alpha}{\partial \beta} N_\alpha + g_\alpha g_\beta \left(\frac{Q_\beta}{R_\beta} + q_\beta \right) = 0, \quad (51)$$

$$\frac{\partial}{\partial \alpha} (g_\beta Q_\alpha) + \frac{\partial}{\partial \beta} (g_\alpha Q_\beta) - g_\alpha g_\beta \left(\frac{N_\alpha}{R_\alpha} + \frac{N_\beta}{R_\beta} - q_\zeta \right) = 0, \quad (52)$$

where q_α , q_β and q_ζ are the resultants of surface forces per unit middle surface area acting on the shell element, see figure 5.

The equilibrium equations for the resulting moments can be obtained in a similar way:

$$\frac{\partial}{\partial \alpha} (g_\beta M_\alpha) + \frac{\partial}{\partial \beta} (g_\alpha M_{\beta\alpha}) + \frac{\partial g_\alpha}{\partial \beta} M_{\alpha\beta} - \frac{\partial g_\beta}{\partial \alpha} M_\beta - g_\alpha g_\beta (Q_\alpha - m_\alpha) = 0, \quad (53)$$

$$\frac{\partial}{\partial \beta} (g_\alpha M_\beta) + \frac{\partial}{\partial \alpha} (g_\beta M_{\alpha\beta}) + \frac{\partial g_\beta}{\partial \alpha} M_{\beta\alpha} - \frac{\partial g_\alpha}{\partial \beta} M_\alpha - g_\alpha g_\beta (Q_\beta - m_\beta) = 0, \quad (54)$$

$$N_{\alpha\beta} - N_{\beta\alpha} - \frac{M_{\beta\alpha}}{R_\beta} + \frac{M_{\alpha\beta}}{R_\alpha} = 0, \quad (55)$$

where m_α and m_β are the resultant moments of surface forces per unit middle surface area, see figure 6.

3.5 Results for axisymmetric shells

The theory of transversely rigid shells will be used now for the special case of small axisymmetric deformations of spherical shells, and describe the sphere in spherical coordinates r , ϕ , θ . The radius of the

Fig. 5. Resultant external loading force components

Fig. 6. Resultant external loading moment components

middle surface of the shell is a_m . We choose the symmetry-axis to be in the $\phi = 0$ direction. Thus the following forces are zero

$$\tau_{\theta r}^{(\zeta)} = \tau_{\phi \theta}^{(\zeta)} = 0. \quad (56)$$

This leads to:

$$N_{\theta \phi} = N_{\phi \theta} = Q_{\theta} = M_{\phi \theta} = M_{\theta \phi} = q_{\theta} = m_{\theta} = 0. \quad (57)$$

The chosen geometry leads to

$$R_{\phi} = R_{\theta} = g_{\phi} = a_m, \quad (58)$$

$$g_{\theta} = a_m \sin \phi. \quad (59)$$

The equilibrium equations (50–55) then become for the axisymmetric case:

$$\frac{\partial}{\partial \phi} (N_{\phi} \sin \phi) - N_{\theta} \cos \phi + (Q_{\phi} + q_{\phi} a_m) \sin \phi = 0, \quad (60)$$

$$\frac{\partial}{\partial \phi} (Q_{\phi} \sin \phi) - (N_{\phi} + N_{\theta} - q_r a_m) \sin \phi = 0, \quad (61)$$

$$\frac{\partial}{\partial \phi} (M_{\phi} \sin \phi) - M_{\theta} \cos \phi - (Q_{\phi} - m_{\phi}) a_m \sin \phi = 0, \quad (62)$$

where q_{ϕ} , q_r and m_{ϕ} are reduced to:

$$q_{\phi} = \left(1 + \frac{h}{2a_m}\right)^2 T_{\phi r}^{(h/2)} - \left(1 - \frac{h}{2a_m}\right)^2 T_{\phi r}^{(-h/2)}, \quad (63)$$

$$q_r = \left(1 + \frac{h}{2a_m}\right)^2 T_{rr}^{(h/2)} - \left(1 - \frac{h}{2a_m}\right)^2 T_{rr}^{(-h/2)}, \quad (64)$$

$$m_{\phi} = \frac{h}{2} \left(1 + \frac{h}{2a_m}\right)^2 T_{\phi r}^{(h/2)} + \frac{h}{2} \left(1 - \frac{h}{2a_m}\right)^2 T_{\phi r}^{(-h/2)}. \quad (65)$$

The stress-strain relations can be described as

$$N_{\phi} = \frac{Eh}{1-\nu^2} (\varepsilon_{\phi} + \nu \varepsilon_{\theta}), \quad (66)$$

$$N_{\theta} = \frac{Eh}{1-\nu^2} (\varepsilon_{\theta} + \nu \varepsilon_{\phi}), \quad (67)$$

$$M_{\phi} = \frac{Eh^3}{12(1-\nu^2)} (K_{\phi} + \nu K_{\theta}), \quad (68)$$

$$M_{\theta} = \frac{Eh^3}{12(1-\nu^2)} (K_{\theta} + \nu K_{\phi}). \quad (69)$$

The strain-deformation relations in this configuration are

$$\varepsilon_{\phi} = \frac{1}{a_m} \left(\frac{ds_{\phi}}{d\phi} + s_r \right), \quad (70)$$

$$\varepsilon_{\theta} = \frac{1}{a_m} (s_{\phi} \cot \phi + s_r), \quad (71)$$

$$K_{\phi} = -\frac{1}{a_m^2} \left(\frac{d^2 s_r}{d\phi^2} - \frac{ds_{\phi}}{d\phi} \right), \quad (72)$$

$$K_{\theta} = -\frac{1}{a_m^2} \left(\frac{ds_r}{d\phi} - s_{\phi} \right) \cot \phi. \quad (73)$$

The eqs. (60–73) make it possible to express the external loading force components q_r and q_{ϕ} and the external loading moment m_{ϕ} in terms of the displacements of the middle surface.

These relations will be used in section 4.4 to derive the equations for the continuity of the forces across the shell.

4. Boundary conditions for harmonically oscillating pure straining flow

4.1 Velocity cell boundary conditions

As was stated before in section 2.4, we have to formulate a set of eight linear homogeneous equations in the parameters $A^{(i)}$, $B^{(i)}$, $C^{(i)}$, $D^{(i)}$, $A^{(e)}$, $B^{(e)}$, $C^{(e)}$ and $D^{(e)}$ to solve the specific viscosity. The first equations can be derived from the continuity of the velocity at the cell boundary. Outside the cell volume the flow field is taken to be the harmonically oscillating flow used by Oldroyd [4, 16]. This flow field is described by eq. (5). Inside the cell volume the flow field can be described by eqs. (9) and (10), using the parameters of external flow fields. Continuity of the velocity at the cell boundary leads to

$$\frac{1}{7} b^2 A^{(e)} + \frac{1}{2} b^{-3} B^{(e)} + 2 C^{(e)} - 3 b^{-5} D^{(e)} = 2 G, \quad (74)$$

$$\frac{5}{14} b^2 A^{(e)} + 3 C^{(e)} + 3 b^{-5} D^{(e)} = 3 G. \quad (75)$$

4.2 Finite velocity conditions

The next two equations are simply derived from the requirement of finite velocity at $r = 0$:

$$B^{(i)} = 0, \quad (76)$$

$$D^{(i)} = 0. \quad (77)$$

4.3 Continuity of the velocity across the shell

We can derive two equations from the continuity condition for the velocity across the shell. The velocity is given by eqs. (9) and (10) using the internal parameters for the inner region, $A^{(i)}$, $B^{(i)}$, $C^{(i)}$ and $D^{(i)}$ and using the external parameters $A^{(e)}$, $B^{(e)}$, $C^{(e)}$ and $D^{(e)}$ for the velocity in the outer region. These velocities can be related by relations from the shell theory.

Seide [10] gives also a relation between the velocity of the middle surface of the shell and the velocity of a point in the shell wall at a distance ζ from the middle surface. These relations can be written as follows:

$$u_r^{(m)} = u_r^{(\zeta)}, \quad (78)$$

$$u_\phi^{(m)} = \frac{R_\phi}{R_\phi + \zeta} u_\phi^{(\zeta)} + \frac{\zeta}{\zeta + R_\phi} \frac{\partial u_r^{(\zeta)}}{\partial \phi}. \quad (79)$$

Assuming the no-slip condition to be valid the velocities $u_\phi^{(\pm h/2)}$, $u_r^{(\pm h/2)}$ at the boundary surfaces of the shell are known in terms of $A^{(i)}$, ... and $A^{(e)}$, ... Using eqs. (78) and (79) at $\zeta = \pm h/2$ we can not only obtain the velocities at the middle surface in terms of the flow field of the inner region, but also in terms of the flow field of the outer region. By the requirement of the continuity of the velocity at the middle surface we thus can obtain two relations:

$$\begin{aligned} & \frac{1}{7} A^{(i)} \left(a_m - \frac{h}{2} \right)^3 + \frac{1}{2} B^{(i)} \left(a_m - \frac{h}{2} \right)^{-2} \\ & + 2 C^{(i)} \left(a_m - \frac{h}{2} \right) - 3 D^{(i)} \left(a_m - \frac{h}{2} \right)^{-4} \\ & = \frac{1}{7} A^{(e)} \left(a_m + \frac{h}{2} \right)^3 + \frac{1}{2} B^{(e)} \left(a_m + \frac{h}{2} \right)^{-2} \\ & + 2 C^{(e)} \left(a_m + \frac{h}{2} \right) - 3 D^{(e)} \left(a_m + \frac{h}{2} \right)^{-4}, \end{aligned} \quad (80)$$

$$\begin{aligned} & \frac{a_m}{a_m - h/2} \left\{ \frac{5}{42} A^{(i)} \left(a_m - \frac{h}{2} \right)^3 + C^{(i)} \left(a_m - \frac{h}{2} \right) \right. \\ & \left. + D^{(i)} \left(a_m - \frac{h}{2} \right)^{-4} \right\} \end{aligned}$$

$$\begin{aligned} & - \frac{h/2}{a_m - h/2} \left\{ \frac{1}{7} A^{(i)} \left(a_m - \frac{h}{2} \right)^3 + \frac{1}{2} B^{(i)} \left(a_m - \frac{h}{2} \right)^{-2} \right. \\ & \left. + 2 C^{(i)} \left(a_m - \frac{h}{2} \right) - 3 D^{(i)} \left(a_m - \frac{h}{2} \right)^{-4} \right\} \\ & = \frac{a_m}{a_m + h/2} \left\{ \frac{5}{42} A^{(e)} \left(a_m + \frac{h}{2} \right)^3 + C^{(e)} \left(a_m + \frac{h}{2} \right) \right. \\ & \left. + D^{(e)} \left(a_m + \frac{h}{2} \right)^{-4} \right\} \\ & + \frac{h/2}{a_m + h/2} \left\{ \frac{1}{7} A^{(e)} \left(a_m + \frac{h}{2} \right)^3 + \frac{1}{2} B^{(e)} \left(a_m + \frac{h}{2} \right)^{-2} \right. \\ & \left. + 2 C^{(e)} \left(a_m + \frac{h}{2} \right) - 3 D^{(e)} \left(a_m + \frac{h}{2} \right)^{-4} \right\}. \end{aligned} \quad (81)$$

4.4 Equilibrium of the forces in the shell

In section 3.5 we have written the relations which make it possible to obtain two identities describing the equilibrium of the forces on the shell. Using eqs. (63) and (64) we can write the external loading force components in terms of a jump of components of the stress tensor:

$$q_r = T_{rr}^{(e)}|_{r=a_m+h/2} - T_{rr}^{(i)}|_{r=a_m-h/2}, \quad (84)$$

$$q_\phi = T_{r\phi}^{(e)}|_{r=a_m+h/2} - T_{r\phi}^{(i)}|_{r=a_m-h/2}. \quad (83)$$

Substituting eqs. (12) and (13) we can write q_r and q_ϕ in terms of the coefficients of the flow field.

Alternative expressions for q_r and q_ϕ in terms of these coefficients can be derived from the eqs. (60–62).

$$q_r = -\frac{1}{a_m} \left\{ \frac{\partial Q_\phi}{\partial \phi} + \cot \phi Q_\phi - N_\phi - N_\theta \right\}, \quad (84)$$

$$q_\phi = \frac{1}{a_m} \left\{ -\frac{\partial N_\phi}{\partial \phi} - \cot \phi N_\phi + \cot \phi N_\theta - Q_\phi \right\} \quad (85)$$

in which

$$Q_\phi = \frac{1}{a_m} \left\{ \frac{\partial M_\phi}{\partial \phi} + \cot \phi M_\phi - \cot \phi M_\theta \right\} + m_\phi. \quad (86)$$

m_ϕ can be written in terms of the coefficients of the flow field by using eq. (65):

$$\begin{aligned} m_\phi &= \frac{h}{2} \left(1 + \frac{h}{2a_m} \right)^2 T_{r\phi}^{(e)} \Big|_{r=a_m+h/2} \\ &+ \frac{h}{2} \left(1 - \frac{h}{2a_m} \right)^2 T_{r\phi}^{(i)} \Big|_{r=a_m-h/2} \end{aligned} \quad (87)$$

and substituting eqs. (12) and (13).

The resultant normal forces N_ϕ and N_θ and the resultant bending moments M_ϕ and M_θ can be written in terms of the coefficients of the flow field by using eqs. (66–73) and substituting the solutions of the flow field, eqs. (9) and (10).

The expressions for q_r and q_ϕ derived from the forces on the shell caused by the fluid flow have to be equal to the expressions derived from the deformation of the shell. After lengthy calculations this leads to the following identity using q_r :

$$\begin{aligned}
 & \frac{1}{2a_m^2} \eta^{(e)} \left\{ -\frac{1}{7} \left(a_m + \frac{h}{2} \right)^4 A^{(e)} - 3 \left(a_m + \frac{h}{2} \right)^{-1} B^{(e)} \right. \\
 & \quad \left. + 4 \left(a_m + \frac{h}{2} \right)^2 C^{(e)} + 24 \left(a_m + \frac{h}{2} \right)^{-3} D^{(e)} \right\} \\
 & - \frac{1}{2a_m^2} \eta^{(i)} \left\{ -\frac{1}{7} \left(a_m - \frac{h}{2} \right)^4 A^{(i)} - 3 \left(a_m - \frac{h}{2} \right)^{-1} B^{(i)} \right. \\
 & \quad \left. + 4 \left(a_m - \frac{h}{2} \right)^2 C^{(i)} + 24 \left(a_m - \frac{h}{2} \right)^{-3} D^{(i)} \right\} \\
 & = \frac{E h^3 (5 + \nu)}{(i \omega) 12 a_m^4 (1 - \nu^2)} \left\{ \frac{a_m}{14} \left(a_m - \frac{h}{2} \right)^2 A^{(i)} \right. \\
 & \quad \left. + \frac{3}{2} a_m \left(a_m - \frac{h}{2} \right)^{-3} B^{(i)} \right. \\
 & \quad \left. + 3 a_m C^{(i)} - 12 a_m \left(a_m - \frac{h}{2} \right)^{-5} D^{(i)} \right\} \\
 & + \frac{E h (1 + \nu)}{(i \omega) a_m^2 (1 - \nu^2)} \\
 & \quad \left\{ -\frac{(3 a_m - 2 h)}{14} \left(a_m - \frac{h}{2} \right)^2 A^{(i)} \right. \\
 & \quad \left. + \frac{1}{2} (a_m + 2 h) \left(a_m - \frac{h}{2} \right)^{-3} B^{(i)} - (a_m - 2 h) C^{(i)} \right. \\
 & \quad \left. - 6 \left(a_m + \frac{h}{2} \right) \left(a_m - \frac{h}{2} \right)^{-5} D^{(i)} \right\} \\
 & + \frac{3 h}{2 a_m^3} \eta^{(i)} \left\{ \frac{8}{21} \left(a_m - \frac{h}{2} \right)^4 A^{(i)} + \frac{1}{2} \left(a_m - \frac{h}{2} \right)^{-1} B^{(i)} \right. \\
 & \quad \left. + 2 \left(a_m - \frac{h}{2} \right)^2 C^{(i)} - 8 \left(a_m - \frac{h}{2} \right)^{-3} D^{(i)} \right\} \\
 & + \frac{3 h}{2 a_m^3} \eta^{(e)} \left\{ \frac{8}{21} \left(a_m + \frac{h}{2} \right)^4 A^{(e)} + \frac{1}{2} \left(a_m + \frac{h}{2} \right)^{-1} B^{(e)} \right. \\
 & \quad \left. + 2 \left(a_m + \frac{h}{2} \right)^2 C^{(e)} - 8 \left(a_m + \frac{h}{2} \right)^{-3} D^{(e)} \right\} \quad (88)
 \end{aligned}$$

and the identity using q_ϕ :

$$\begin{aligned}
 & \frac{1}{a_m^2} \eta^{(e)} \left\{ \frac{8}{21} \left(a_m + \frac{h}{2} \right)^4 A^{(e)} + \frac{1}{2} \left(a_m + \frac{h}{2} \right)^{-1} B^{(e)} \right. \\
 & \quad \left. + 2 \left(a_m + \frac{h}{2} \right)^2 C^{(e)} - 8 \left(a_m + \frac{h}{2} \right)^{-3} D^{(e)} \right\} \\
 & - \frac{1}{a_m^2} \eta^{(i)} \left\{ \frac{8}{21} \left(a_m - \frac{h}{2} \right)^4 A^{(i)} + \frac{1}{2} \left(a_m - \frac{h}{2} \right)^{-1} B^{(i)} \right. \\
 & \quad \left. + 2 \left(a_m - \frac{h}{2} \right)^2 C^{(i)} - 8 \left(a_m - \frac{h}{2} \right)^{-3} D^{(i)} \right\} \\
 & = \frac{E h (5 + \nu)}{(i \omega) a_m^2 (1 - \nu^2)} \left\{ \frac{5 a_m - 3 h}{42} \left(a_m - \frac{h}{2} \right)^2 A^{(i)} \right. \\
 & \quad \left. - \frac{h}{4} \left(a_m - \frac{h}{2} \right)^{-3} B^{(i)} + (a_m - h) C^{(i)} \right. \\
 & \quad \left. + \left(a_m + \frac{3}{2} h \right) \left(a_m - \frac{h}{2} \right)^{-5} D^{(i)} \right\} \\
 & + \frac{E h (1 + \nu)}{(i \omega) a_m^2 (1 - \nu^2)} \left\{ -\frac{1}{7} \left(a_m - \frac{h}{2} \right)^3 A^{(i)} \right. \\
 & \quad \left. - \frac{1}{2} \left(a_m - \frac{h}{2} \right)^{-2} B^{(i)} - 2 \left(a_m - \frac{h}{2} \right) C^{(i)} \right. \\
 & \quad \left. + 3 \left(a_m - \frac{h}{2} \right)^{-4} D^{(i)} \right\} \\
 & + \frac{E h^3 (5 + \nu)}{(i \omega) 12 a_m^4 (1 - \nu^2)} \left\{ -\frac{a_m}{42} \left(a_m - \frac{h}{2} \right)^2 A^{(i)} \right. \\
 & \quad \left. - \frac{1}{2} a_m \left(a_m - \frac{h}{2} \right)^{-3} B^{(i)} \right. \\
 & \quad \left. - a_m C^{(i)} + 4 a_m \left(a_m - \frac{h}{2} \right)^{-5} D^{(i)} \right\} \\
 & + \frac{h}{2 a_m^3} \eta^{(i)} \left\{ -\frac{8}{21} \left(a_m - \frac{h}{2} \right)^4 A^{(i)} \right. \\
 & \quad \left. - \frac{1}{2} \left(a_m - \frac{h}{2} \right)^{-1} B^{(i)} - 2 \left(a_m - \frac{h}{2} \right)^2 C^{(i)} \right. \\
 & \quad \left. + 8 \left(a_m - \frac{h}{2} \right)^{-3} D^{(i)} \right\} \\
 & + \frac{h}{2 a_m^3} \eta^{(e)} \left\{ -\frac{8}{21} \left(a_m + \frac{h}{2} \right)^4 A^{(e)} \right. \\
 & \quad \left. - \frac{1}{2} \left(a_m + \frac{h}{2} \right)^{-1} B^{(e)} - 2 \left(a_m + \frac{h}{2} \right)^2 C^{(e)} \right. \\
 & \quad \left. + 8 \left(a_m + \frac{h}{2} \right)^{-3} D^{(e)} \right\}. \quad (89)
 \end{aligned}$$

5. Solution and results

5.1 Solution of η_{spec}^*

As was shown in section 2.4, we can express the specific complex viscosity of the dispersion in terms of the constant $B^{(e)}/G$. This constant is derived from the six boundary conditions derived in section 3. Therefore we will introduce dimensionless quantities and use a matrix notation. This leads to a notation with six unknowns and five parameters defined as

$$Y_1 = A^{(i)} a_m^2 G^{-1}, \quad (90)$$

$$Y_2 = C^{(i)} G^{-1}, \quad (91)$$

$$Y_3 = A^{(e)} a_m^2 G^{-1}, \quad (92)$$

$$Y_4 = B^{(e)} a_m^{-3} G^{-1}, \quad (93)$$

$$Y_5 = C^{(e)} G^{-1}, \quad (94)$$

$$Y_6 = D^{(e)} a_m^{-5} G^{-1}, \quad (95)$$

$$R = b a_m^{-1}, \quad (96)$$

$$L = h (2 a_m)^{-1}, \quad (97)$$

$$V = \eta^{(i)} \cdot (\eta^{(e)})^{-1}, \quad (98)$$

$$Z = 2 \omega \eta^{(e)} a_m (E h)^{-1}, \quad (99)$$

and the Poisson ratio ν .

Instead of the parameter R , which is very useful in the description of the mechanics, we can also use the volume fraction Φ_{ext} which is more useful in the graphical representation of the results.

$$\Phi_{\text{ext}} = f^3 (1+L)^3 R^{-3}. \quad (100)$$

Here Φ_{ext} is the volume fraction of the particles including the shell.

Substituting these quantities in the six boundary conditions we obtain:

$$\frac{1}{7} R^2 Y_3 + \frac{1}{2} R^{-3} Y_4 + 2 Y_5 - 3 R^{-5} Y_6 = 2, \quad (101)$$

$$\frac{5}{14} R^2 Y_3 + 3 Y_5 + 3 R^{-5} Y_6 = 3, \quad (102)$$

$$\begin{aligned} & \frac{1}{7} (1-L)^3 Y_1 + 2 (1-L) Y_2 - \frac{1}{7} (1+L)^3 Y_3 \\ & - \frac{1}{2} (1+L)^{-2} Y_4 - 2 (1+L) Y_5 + 3 (1+L)^{-4} Y_6 = 0, \end{aligned} \quad (103)$$

$$\begin{aligned} & \frac{1}{42} (5-6L) (1-L)^2 Y_1 + (1-2L) Y_2 \\ & - \frac{1}{42} (5+6L) (1+L)^2 Y_3 - \frac{1}{2} L (1+L)^{-3} Y_4 \\ & - (1+2L) Y_5 - (1-3L) (1+L)^{-5} Y_6 = 0, \end{aligned} \quad (104)$$

$$\begin{aligned} & \left\{ \left(-\frac{1}{14} + \frac{8}{7} L \right) (1-L)^4 V - i \frac{1}{21} \frac{5+\nu}{1-\nu^2} L^2 (1-L)^2 Z^{-1} \right. \\ & \left. + i \frac{1}{7} \frac{1+\nu}{1-\nu^2} (3-4L) (1-L)^2 Z^{-1} \right\} Y_1 \\ & + \left\{ (2+6L) (1-L)^2 V - i 2 \frac{5+\nu}{1-\nu^2} L^2 Z^{-1} \right. \\ & \left. + i 2 \frac{1+\nu}{1-\nu^2} (1-4L) Z^{-1} \right\} Y_2 \\ & + \left(\frac{1}{14} + \frac{8}{7} L \right) (1+L)^4 Y_3 + \frac{3}{2} Y_4 \\ & + (-2+3L) (1+L)^2 Y_5 - (12+24L) (1+L)^{-3} Y_6 = 0, \end{aligned} \quad (105)$$

$$\begin{aligned} & \left\{ \frac{8}{21} (1-L)^5 V + i \frac{1}{63} \frac{5+\nu}{1-\nu^2} (L^2 + 18L - 5) \right. \\ & \left. \cdot (1-L)^2 Z^{-1} + i \frac{2}{7} \frac{1+\nu}{1-\nu^2} (1-L)^3 Z^{-1} \right\} Y_1 \\ & + \left\{ 2 (1-L)^3 V + i \frac{2}{3} \frac{5+\nu}{1-\nu^2} (L^2 + 6L - 3) Z^{-1} \right. \\ & \left. + i 4 \frac{1+\nu}{1-\nu^2} (1-L) Z^{-1} \right\} Y_2 - \frac{8}{21} (1+L)^5 Y_3 \\ & - \frac{1}{2} Y_4 - 2 (1+L)^3 Y_5 + 8 (1+L)^{-2} Y_6 = 0. \end{aligned} \quad (106)$$

The specific complex viscosity of the dispersion can be written as (see eq. (15))

$$\eta_{\text{spec}}^* = -\frac{1}{4} f^3 R^{-3} Y_4. \quad (107)$$

Y_4 can be solved most easily by using the matrix notation of the equations (101–106):

$$\mathbf{A} \cdot \mathbf{Y} = \mathbf{C}, \quad (108)$$

where $\mathbf{A}$ is a 6×6 matrix and $\mathbf{Y}$ and $\mathbf{C}$ are 6×1 matrices. Y_4 can be solved by Kramers rule:

$$Y_4 = \frac{\det(\mathbf{B})}{\det(\mathbf{A})}, \quad (109)$$

where the matrix $\mathbf{B}$ can be derived from the matrices $\mathbf{A}$ and $\mathbf{C}$ by substituting $\mathbf{C}$ in the fourth column of the matrix $\mathbf{A}$. One can solve Y_4 analytically with the symbolic computer language Reduce II [19, 20].

5.2 Analytical solution

The analytical solution of the quantity Y_4 can be written with the quantity Z , a dimensionless angular

frequency, as the running variable, in the following way

$$Y_4 = \frac{X_1(Zi)^2 + X_2(Zi) + X_3}{X_4(Zi)^2 + X_5(Zi) + X_6}, \quad (110)$$

where $X_1 - X_6$ are functions of R , L , V and ν . The analytical solutions of these functions could be given, but are rather lengthy.

The form of this solution is in accordance with the expectations on the basis of the form of general linear viscoelastic equations with some additional considerations. Axelrad [21] demonstrated that assuming the validity of the principle of passivity four categories of linear constitutive equations can be identified. Taking into account that for a dispersion of capsules as considered $\lim_{\omega \rightarrow 0} \omega \eta'' = 0$ and $\lim_{\omega \rightarrow \infty} \eta' \neq 0$ only one category

remains. The specific viscosity η_{spec}^* of (107) with Y_4 as given in (110) belongs to this category. It is characterized by two retardation and two relaxation times.

In general the analytical solution of η^* is too cumbersome to be manageable. Only the case $L = 0$ gives manageable results. To be able to find the limit for $L \rightarrow 0$ while the interface obtains two-dimensional elastic properties, the specific dynamic viscosity has been expressed in Z . Thus it is possible to take $L = 0$ in (101–106) while in Z (see eq. (99)) Eh is retained as a two-dimensional elasticity. If in the following pages the case $L = 0$ is considered, the preceding condition is meant.

To obtain some insight into the general peculiarities of the model η_{spec}^* was calculated numerically.

5.3 Numerical results

In this section some numerical results of this model will be discussed. These results are given graphically with Z as the running variable and η_{spec}' and η_{spec}'' as the rheological functions considered. They are defined by:

$$\eta_{\text{spec}}^* = \eta_{\text{spec}}' - i \eta_{\text{spec}}'' \quad (111)$$

The correction factor f defined in eq. (4) is given the value 1.111 (see section 2.2). In all examples ν is chosen such that the two transitions in η_{spec}^* are separated clearly. The maximum value of L is taken to be 0.1 to show clearly the influence of L . However one should bear in mind that this shell theory is assumed to be valid only for $L \leq 0.05$.

An example of the influence of the parameter V is given in figure 7. The most remarkable effect of an increase of V is an increase of the level of the specific viscosity at higher frequencies. Besides it results in a shift of the relaxation times to longer times.

Fig. 7. Influence of the parameter V on the specific viscosity of a dispersion. Parameter values: $\Phi = 0.10$, $L = 0$, $\nu = 0.999$, for $V = 10$, $V = 1$, $V = 0.1$

Fig. 8. Influence of the parameter ν on the specific viscosity of a dispersion. Parameter values: $\Phi = 0.10$, $L = 0$, $V = 1$, for $\nu = 0.999$, $\nu = 0.99$, $\nu = 0.9$

Fig. 9. Influence of the parameter Φ on the specific viscosity of a dispersion. Parameter values: $L = 0$, $V = 1$, $\nu = 0.999$, for $\Phi = 0.10$, $\Phi = 0.05$, $\Phi = 0.025$

Fig. 11. Influence of the parameter L on the specific viscosity of a dispersion. Parameter values: $\Phi_{\text{int}} = 0.1$, $V = 1$, $\nu = 0.999$, for $L = 0$, $L = 0.01$, $L = 0.1$. Here Φ_{int} is the volume of the liquid enclosed within the shell

Fig. 10. Influence of the parameter L on the specific viscosity of a dispersion. Parameter values: $\Phi_{\text{ext}} = 0.10$, $V = 1$, $\nu = 0.999$, for $L = 0$, $L = 0.01$, $L = 0.1$

Fig. 12. Influence of the parameter ν on the specific viscosity of a dispersion with $L = 0.1$. Other parameter values: $\Phi_{\text{ext}} = 0.1$, $V = 1$, for $\nu = 0.999$ and $\nu = 0.99$

An example of the influence of the parameter ν is given in figure 8. This parameter does not influence the level of the specific viscosity at high or low frequencies. Its influence on the relaxation times is significant. The shortest relaxation time decreases by about one decade if the value $1 - \nu$ decreases by one decade. The longest relaxation time does not change much. Only at higher values of the parameter L does it decrease significantly if the value $1 - \nu$ decreases and for $L = 0.1$ both the shortest and the longest relaxation time shift by about the same factor if $1 - \nu$ changes (see figure 12).

The influence of the volume fraction Φ is visible as a change in magnitude of η^* over the whole frequency range, see figure 9. The relaxation times are slightly influenced by the volume fraction. An increase of the volume fraction results in an increase of the relaxation times. The longest relaxation time appears (numerically) to be more sensitive than the shortest one.

The influence of L is the most remarkable result of this model. From the numerical results it appears that an increase of L has a very significant effect on the relaxation times and on the level of the specific viscosity between the two relaxation times. There is also a small increasing effect on the level of the specific viscosity at high frequencies. Examples are given in figure 10, 11 and 12. The longest relaxation time decreases appreciably with increasing shell thickness. The influence on the shortest relaxation time is much smaller (not visible in the given figure but numerically it is significant). The level of the specific viscosity between the two relaxation times increases according to the increase in the shell thickness.

6. Discussion

6.1 Comparison with the literature

The results obtained can be compared with those from the models given in the literature. The present model is for $L = 0$ identical to Oosterbroek's model [7] if in his model the interfacial tension and the interfacial viscosities are taken to be zero. The two-dimensional parameters κ (surface dilatational modulus) and μ (surface shear modulus) are given by

$$\kappa = \frac{EL}{1 - \nu} a_m \quad (112)$$

and

$$\mu = \frac{EL}{1 + \nu} a_m. \quad (113)$$

The identity is self-evident if one considers the six boundary conditions (101–106) with $L = 0$ and the

corresponding conditions in [7]. It has been proved [7] that for $\Phi \rightarrow 0$ Oldroyd's model follows from Oosterbroek's model.

A special case of Oldroyd's model should be comparable with the complex viscosity deducible from the linear part of the constitutive equations for a capsule dispersion derived by Barthès-Biesel and Chhim [8]. They studied capsules with an incompressible, homogeneous and isotropic shell of finite thickness characterized by a shear modulus. The validity of their constitutive equation is restricted to a slow varying flow. Thus the deducible complex viscosity is valid for low frequencies. In first order of the frequency, thickness and volume fraction they find

$$\eta' = \eta^{(e)} \left(1 + \frac{5}{2} \Phi\right) \quad (114)$$

and

$$\eta'' = \frac{95}{4} \eta^{(e)} \omega \frac{a \eta^{(e)}}{E h} \Phi. \quad (115)$$

This result can be derived from Oldroyd's model as follows. From the analysis of the present work it appeared that for the transition from three dimensions to two dimensions (112) and (113) must be used. To incorporate the three-dimensional incompressibility, as assumed by Barthès-Biesel and Chhim, obviously $\nu = \frac{1}{2}$ should be taken instead of $\nu = 1$ for two-dimensional incompressibility. Indeed then (114) and (115) follow from [4].

Comparison with the work of Sakanishi and Takano [5] and Brunn [6] is also possible if explicit formulae have been given. They studied the same kind of capsules as Barthès-Biesel and Chhim did and derived in first order of volume fraction the complex viscosity and the linear constitutive equation for the capsule dispersion, respectively. In general four relaxation times are found. This can be expected because at each shell surface two force balance equations apply. For $L \ll 1$ two relaxation times are expected to be equal to those found by Oldroyd. It appears that the two longest relaxation times can be identified as those occurring for a two-dimensional shell. This can be demonstrated by comparing the relaxation times found by Oldroyd, again using (112, 113) and $\nu = \frac{1}{2}$, with those explicitly given in [5] and [6] for $h \rightarrow 0$. The times are given by

$$\tau_{l,s} = \frac{3a\eta_0}{5Eh} \left(1 \mp \sqrt{1 - \frac{96\eta_0}{25(24\eta^{(e)} + 19\eta^{(i)})}}\right)^{-1} \quad (116)$$

with

$$\eta_0 = \frac{(3\eta^{(e)} + 2\eta^{(i)})(16\eta^{(e)} + 19\eta^{(i)})}{(24\eta^{(e)} + 19\eta^{(i)})}. \quad (117)$$

Brunn [6] gave the same times. Sakanishi and Takano [5] gave the limit cases $\eta^{(e)}/\eta^{(i)} \rightarrow 0$ and $\eta^{(e)}/\eta^{(i)} \rightarrow \infty$. The physical origin of the other two times, which are for $L < 0.05$ at least a factor 100 shorter, should be due to the three-dimensionality of the membrane.

In a more recent paper Takano and Sakanishi [9] studied capsules with shells characterized by anisotropic elastic properties. For these capsules also two relaxation times are found. They are difficult to compare with the results of the present work not only because numerically derived results are difficult to compare but their relaxation effects (so-called negative relaxations) are not compatible with the principle of passivity [21].

6.2 Interpretation of the results

In the following some striking aspects of the present results are interpreted.

The ratio V influences η^* in different ways (see figure 7). The relaxation times become longer and the magnitude of the transition decreases if V increases. A relaxation time can be written as a positive constant multiplied by an effective viscosity and divided by an effective elasticity. In this case the effective viscosity is a monotonically increasing function of $\eta^{(i)}$ and of $\eta^{(e)}$ [7]. The effective elasticity is either $E/(1-\nu)$ or $E/(1+\nu)$. The first fact causes an increase of the relaxation times with increasing V . At sufficiently high frequencies the stresses acting at each side of the shell become much larger than the difference between them due to the elastic deformation as the deformation of the shell cannot become larger than the overall applied strain. As a consequence the high frequency limit of η' can be described by a model in which the interfacial mechanical properties are negligible (see e.g. [22]). For $V \sim O(1)$ the limit will be almost proportional to the volume fraction. For $V \rightarrow \infty$ $\eta'(\omega \rightarrow \infty)$ will become equal to $\eta'(\omega \rightarrow 0)$ while the maximum of η'' decreases.

In figure 8 the effect of ν is shown. Since for $L = 0$ the longest relaxation time τ_l is proportional to μ^{-1} and the shortest τ_s to κ^{-1} [7] according to eqs. (112) and (113) τ_l changes with $(1+\nu)$ and τ_s with $(1-\nu)$.

The influence of the volume fraction is shown in figure 9. The viscosity $\eta'(\omega = 0)$ equals Simha's steady-state viscosity for solid spheres (see also [7]). The explanation is as follows: For $\omega \rightarrow 0$ the viscous stresses in the forces equilibrium conditions become zero. As a consequence the jump in the viscous stresses due to the elastic deformation of the shell should also vanish. Since E is finite the deformation of the shell becomes zero and the capsule dispersion becomes indistinguishable from a suspension of solid spheres. It is not

surprising that this correspondence was also found for finite L within numerical accuracy. Since $\eta'(\omega = 0)$ increases strongly in a non-linear way with Φ the magnitude of the transitions should increase strongly.

Finite L has an effect on the relaxation times and the magnitude of η^* (figures 10, 11 and 12). The difference between figure 10 and 11 is mainly seen in the magnitude difference at a fixed frequency. It can be explained largely by the influence of the volume fraction. In figure 10 Φ_{ext} includes the shell volume while in figure 11 Φ_{int} is just the volume fraction of the dispersed phase enclosed in the shells. Changing L if Φ_{int} is constant implies changing Φ_{ext} , so $\eta'(\omega \rightarrow 0)$ increases (see the preceding comment on figure 9).

The main influence of L itself for a fixed value of ν is a large shift of τ_l and a small shift of τ_s . This can be understood by consideration of the occurring forces and ensuing deformation. After some calculations starting from eqs. (84) and (85) it follows that the angle-independent elastic part in the force equilibria can be written as:

$$q_r = \kappa g_A(a, L) + (3\kappa + 2\mu) L^2 g_B(a, L) \quad (118)$$

for the T_{rr} jump condition and

$$q_\phi = \kappa g_A(a, L) + \frac{1}{3} (3\kappa + 2\mu) L^2 g_B(a, L) + \mu (g_A(a, L) - g_B(a, L)) \quad (119)$$

for the $T_{r\phi}$ jump condition.

Here g_A and g_B are functions which can be deduced from eqs. (88) and (89). As already stated the elastic deformation of the capsule will vanish for $\omega \rightarrow 0$ and hence $g_A(\omega \rightarrow 0) = 0$ and $g_B(\omega \rightarrow 0) = 0$. For $L = 0$ and $\nu \sim 1$ it has been demonstrated [7] that τ_l is (mainly) determined by μ . This can be realized if $q_r(a, 0) \sim 0$ and $q_\phi(a, 0) \sim -\mu g_B(a, 0)$ is left. Figure 8 confirms that τ_l is hardly influenced by κ . For finite but small $L (< 0.1)$ one can assume that $g_A(a, L) \sim g_A(a, 0)$. For $\omega \rightarrow 0$ again $q_r \rightarrow 0$. Assuming that also for finite L the condition for τ_l is still such that $q_r \sim 0$ then

$$g_\phi \sim -\frac{2}{3} (3\kappa + 2\mu) L^2 g_B(a, L) + \mu (g_A(a, L) - g_B(a, L)).$$

In figures 10 and 11 the maximum value of $L = 0.1$ and $\nu = 0.999$. Then $\frac{2}{3} (3\kappa + 2\mu) L^2 \sim 2\kappa L^2 \sim 40\mu$. This would imply that τ_l for $L = 0.1$ is mainly due to κL^2 and no longer to μ if $|g_B(a, L)|$ is of the same order of magnitude as $|g_A(a, L)|$. The shift of τ_l between $L = 0$ and $L = 0.1$ in figures 10 and 11 is in agreement with this interpretation.

A stronger test for this type of interpretation follows using the fact that as observed τ_s is hardly influenced by L . This is plausible by consideration of (118) and

(119). Assuming that $g_A(a, 0) \sim g_A(a, L)$, $g_B(a, 0) \sim g_B(a, L)$, $|g_A| \sim |g_B|$, $\nu \sim 1$, $\kappa \gg \mu$ and L maximal 0.1, $q_r \sim \kappa g_A(a, L)$ and $q_\phi \sim \kappa g_A(a, L)$. Thus on a log scale no appreciable shift of τ_s due to a change in L is to be expected. Combining the interpretations of τ_l and τ_s for fixed a , L and Φ , and if one expects that if $1 - \nu$ changes by some factor (e.g. 0.01 to 0.001) τ_l and τ_s should change by about the same factor in the same direction. This is confirmed numerically in figure 12. It may be concluded that τ_s is almost independent of L and that τ_l decreases strongly as a consequence of resistance against bending in the case of finite thickness. This reasoning is only valid if $\nu \sim 1$, as otherwise a combination of effects can be expected.

It may be interesting to note that the origin of the shift of τ_l is the influence of the coefficient of flexural rigidity $Eh^3/(12(1-\nu^2))$ in (88) and (89). Kléman [23] has pointed out that the energy connected with the pertinent deformation is of the same form as the splay energy (energy of curvature) given by De Gennes [24] and Helfrich [25]. Apparently complex viscosity measurements, e.g. on vesicles, might be a tool for the investigation of a curvature effect.

In conclusion it can be said that the present model shows that finite thickness of the shell can strongly influence the relaxation times. The model can be used for the interpretation of the linear viscoelastic behaviour of a dispersion of hydrodynamically interacting capsules at finite volume concentration. Its significance with respect to other available models is explained.

Acknowledgement

Thanks are due to Mr. R. Tropper for his preliminary studies on this subject.

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(Received April 18, 1984)

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