ELSEVIER

Discrete Mathematics 223 (2000) 327–336

DISCRETE MATHEMATICS

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Heavy paths and cycles in weighted graphs ☆

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Received 6 October 1998; revised 2 October 1999; accepted 11 October 1999

Abstract

A weighted graph is a graph in which each edge is assigned a non-negative number, called the weight. The weight of a path (cycle) is the sum of the weights of its edges. The weighted degree of a vertex is the sum of the weights of the edges incident with the vertex. A usual (unweighted) graph can be considered as a weighted graph with constant weight 1. In this paper, it is proved that for a 2-connected weighted graph, if every vertex has weighted degree at least d, then for any given vertex y, either y is contained in a cycle with weight at least $2d$ or every heaviest cycle is a Hamilton cycle. This result is a common generalization of Grötschel's theorem and Bondy–Fan's theorem assuring the existence of a cycle with weight at least $2d$ on the same condition. Also, as a tool for proving this result, we show a result concerning heavy paths joining two specific vertices and passing through one given vertex. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 05C45; 05C38; 05C35

Keywords: Weighted graph; (Long, optimal, Hamilton) cycle; Weighted degree

1. Terminology and notation

We use Bondy and Murty [2] for terminology and notation not defined here and consider finite simple graphs only.

Let $G=(V,E)$ be a simple graph. G is called a *weighted graph* if each edge e is assigned a non-negative number $w(e)$, called the *weight* of e. For any subgraph H of $G, V(H)$ and $E(H)$ denote the sets of vertices and edges of H, respectively.

☆ This research was carried out while the first author was visiting the Faculty of Applied Mathematics, University of Twente.

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PII: S0012-365X(99)00413-6

The *weight* of H is defined by

$$w(H)=\sum_{e\in E(H)} w(e).$$

A cycle is called *optimal* if it is a cycle with maximum weight among all cycles of G. For each vertex $v\in V$, $N_H(v)$ denotes the set, and $d_H(v)$ the number, of vertices in H that are adjacent to v. We define the *weighted degree* of v in H by

$$d_H^w(v)=\sum_{h\in N_H(v)} w(vh).$$

When no confusion occurs, we will denote $N_G(v)$, $d_G(v)$ and $d_G^w(v)$ by $N(v)$, $d(v)$ and $d^w(v)$, respectively. An (x,z)-*path* is a path connecting the two vertices x and z. For a given vertex y of G, an (x,z)-path is called an (x,y,z)-*path* if it passes through the vertex y. A cycle is called a y-*cycle* if it passes through the vertex y. If x and z are two vertices on a path P, $P[x,z]$ denotes the segment of P from x to z. Let C be a cycle in G with a fixed orientation. For any two vertices x and z on C, by $C[x,z]$ we denote the segment of C from x to z determined by this orientation. If H is a subgraph of G, by $G-H$ we denote the induced subgraph $G[V(G)\backslash V(H)]$.

An unweighted graph can be regarded as a weighted graph in which each edge e is assigned weight $w(e)=1$. Thus, in an unweighted graph, $d^w(v)=d(v)$ for every vertex v, and an optimal cycle is simply a longest cycle.

2. Heavy paths in weighted graphs

The following two theorems are on the existence of long paths. It is easy to see that Theorem B generalizes Theorem A.

Theorem A (Erdős and Gallai [5]). *Let G be a 2-connected graph and d an integer. Let x and z be two distinct vertices of G. If $d(v)\geqslant d$ for all $v\in V(G)\backslash\{x,z\}$, then G contains an (x,z)-path of length at least d.*

Theorem B (Enomoto [4]). *Let G be a 2-connected graph and d an integer. Let x and z be two distinct vertices of G. Suppose that $d(v)\geqslant d$ for all $v\in V(G)\backslash\{x,z\}$.*

(1) *Then for any given vertex y of G, G contains an (x,y,z)-path of length at least d.*
(2) *If for some vertex $y\in V(G)\backslash\{x,z\}$, G contains no (x,y,z)-path of length more than d, then the connected component H_y of $G-x-z$ that contains y is isomorphic to K_{d-1} and $V(H_y)\subseteq N(x)\cap N(z)$. If $y\in\{x,z\}$, then the assertion holds for any connected component of $G-x-z$.*

Bondy and Fan generalized Theorem A to weighted graphs as follows:

Theorem 1 (Bondy and Fan [1]). *Let G be a 2-connected weighted graph and d a real number. Let x and z be two distinct vertices of G. If $d^w(v) \geqslant d$ for all $v \in V(G)\setminus\{x,z\}$, then G contains an (x,z)-path of weight at least d.*

In this section, we prove the following analogue of Theorem B for weighted graphs. This result also generalizes Theorem 1.

Theorem 2. *Let G be a 2-connected weighted graph and d a real number. Let x and z be two distinct vertices of G. Suppose that $d^w(v) \geqslant d$ for all $v \in V(G)\setminus\{x,z\}$.*

(1) *Then for any given vertex y of G, G contains an (x,y,z)-path of weight at least d.*
(2) *If $w(e) > 0$ for all $e \in E(G)$ and for some vertex $y \in V(G)\setminus\{x,z\}$, G contains no (x,y,z)-path of weight more than d, then* (a) *the connected component H_y of $G-x-z$ that contains y is complete*; (b) $V(H_y) \subseteq N(x) \cap N(z)$; (c) $w(xv)=\alpha_x$, $w(zv)=\alpha_z$ *for all $v \in V(H_y)$ and $w(uv)=\beta_y$ for all $u,v \in V(H_y)$ so that $\alpha_x + \beta_y(|V(H_y)|-1)+\alpha_z = d$. If $y \in \{x,z\}$, then the assertion holds for any connected component of $G-x-z$.*

Proof. If $y \in \{x,z\}$, then the result in (1) follows from Theorem 1; The assertions in (2) can be proved by choosing any connected component of $G-x-z$ as H_y in the following proof. So we may assume that $y \notin \{x,z\}$.

Let $|V(G)|=n$. We use induction on n. If $n=3$, let y be the third vertex other than x and z, then the path xyz is an (x,y,z)-path of weight $d^w(y) \geqslant d$.

Suppose now $n \geqslant 4$ and the theorem is true for all graphs on k vertices with $3 \leqslant k \leqslant n-1$. Let $G'=G-z$ be the graph obtained by deleting z from G. We consider two cases:

Case 1: *G' is 2-connected.*

(1) Since G is 2-connected, we can choose $z' \in N(z)\setminus\{x\}$ such that

$$w(zz')=\max\{w(zv)\colon v \in N(z)\setminus\{x\}\}.$$

Then for all $v \in V(G')\setminus\{x\}$,

$$d^w_{G'}(v)=d^w(v)-w(zv) \geqslant d-w(zz').$$

By the induction hypothesis, for any given vertex $y \in V(G')\setminus\{x\}$, G' contains an (x,y,z')-path Q of weight at least $d-w(zz')$. Then the path $P=Qz'z$ is an (x,y,z)-path of weight at least d.

(2) If for some vertex $y \in V(G)\setminus\{x,z\}$, G contains no (x,y,z)-path of weight more than d, then the maximum weight of an (x,y,z')-path in G' is exactly $d'=d-w(zz')$. Moreover, by the induction hypothesis, G' has the described structure. Let H'_y be the connected component of $G'-x-z'$ that contains y. (If $y=z'$, take any connected

component of $G'-x-z'$ as H'_y.) Thus, H'_y is complete, $V(H'_y)\subseteq N_{G'}(x)\cap N_{G'}(z')$ and G' is weighted so that

$$w(xv)=\alpha'_x,\quad w(z'v)=\alpha'_{z'}\quad \text{for all } v\in V(H'_y)$$

and

$$w(uv)=\beta'_y\quad \text{for all } u,v\in V(H'_y),$$

where

$$\alpha'_x+\beta'_y(|V(H'_y)|-1)+\alpha'_{z'}=d'.$$

If $v\in V(H'_y)$, then $d^w_{G'}(v)=d'$. Thus

$$w(zv)=d^w(v)-d^w_{G'}(v)\geqslant d-d'=w(zz').$$

Since $w(zz')>0$, we have that $zv\in E(G)$. Moreover, by the choice of z', it is clear that $w(zv)=w(zz')$ for all $v\in V(H'_y)$. It follows that any vertex in $V(H'_y)\cup\{z\}$ could have been selected as the vertex z'. This implies that $\alpha'_{z'}=\beta'_y$.

Suppose that there exists another connected component H^* of $G'-x-z'$. By the induction hypothesis, then there must be an (x,z')-path of weight at least $d-w(zz')$ in $G[V(H^*)\cup\{x,z'\}]$. On the other hand, there is a (z,y,z')-path of weight $w(zz')+\beta'_y|V(H'_y)|$ in $G[V(H'_y)\cup\{z,z'\}]$. Combining these two paths, we get an (x,y,z)-path of weight at least $d+\beta'_y\,|\,V(H'_y)|>d$, which contradicts the assumption. Hence $G-x-z=G[V(H'_y)\cup\{z'\}]$ and

$$\begin{aligned} w(xz') &= d^w(z')-w(zz')-\beta'_y\,|\,V(H'_y)| \\ &\geqslant d-w(zz')-\beta'_y\,|\,V(H'_y)| \\ &= d'-\beta'_y\,|\,V(H'_y)| \\ &= \alpha'_x+\beta'_y(|V(H'_y)|-1)+\alpha'_{z'}-\beta'_y\,|\,V(H'_y)| \\ &= \alpha'_x. \end{aligned}$$

Furthermore, by the assumption that G contains no (x,z)-path of weight more than d we know that $w(xz')\leqslant\alpha'_x$. So $xz'\in E(G)$ and $w(xz')=\alpha'_x$. Now let H_y denote the connected component of $G-x-z$ that contains y and set $\alpha_z=w(zz')$, $\alpha_x=\alpha'_x$ and $\beta_y=\beta'_y$. Then H_y is complete, $V(H_y)\subseteq N(x)\cap N(z)$ and G is weighted so that

$$w(xv)=\alpha_x,\qquad w(zv)=\alpha_z\quad \text{for all } v\in V(H_y)$$

and

$$w(uv)=\beta_y\quad \text{for all } u,v\in V(H_y),$$

where

$$\alpha_x+\beta_y(|V(H_y)|-1)+\alpha_z=d.$$

Case 2: *G' is not 2-connected.*

(1) Since G is 2-connected, G' must be connected. We shall frequently make use of the following claim.

Claim. *Suppose B is an end-block of G' and b is the unique cut-vertex of G' contained in B. Let B' be the subgraph of G induced by $V(B)\cup\{z\}$. Then for any given vertex y of B', B' contains a (b,y,z)-path P' of weight at least d.*

Proof. If $zb\in E(G)$, then B' is 2-connected and for all $v\in V(B')\backslash\{b,z\}$, we have

$$d^w_{B'}(v)=d^w(v)\geqslant d.$$

By the induction hypothesis, for any given vertex y of B', B' contains a (b,y,z)-path P' of weight at least d.

If $zb\notin E(G)$, add zb to B' and set $w(zb)=0$. Applying the induction hypothesis to the resulting graph, we know that for any given vertex y of B', the resulting graph contains a (b,y,z)-path of weight at least d. If $d>0$, then $P'\neq zb$, since $w(zb)=0$. If $d=0$, then we can choose P' in B' such that $P'\neq zb$, since all we need is that $w(P')\geqslant d$. This shows that we always have a (b,y,z)-path P' in B' of weight at least d.

Case 2.1: *y is contained in a block of G' with two or more cut-vertices.* Choose an end-block B in G' with cut-vertex b such that there is an (x,y,b)-path Q in $G'-(B-b)$. Let B' be the subgraph of G induced by $V(B)\cup\{z\}$. By the above claim, we have that there is a (b,z)-path P' in B' of weight at least d. Combining these two paths Q and P', we get an (x,y,z)-path of weight at least d.

Case 2.2: *y is contained in an end-block B of G' with a cut-vertex b and $x\notin V(B)$.* Let B' be the subgraph of G induced by $V(B)\cup\{z\}$. It is easy to see that there exists an (x,b)-path Q in $G'-(B-b)$. By the above claim we have that there is a (b,y,z)-path P' in B' of weight at least d. Combining these two paths Q and P', we get an (x,y,z)-path of weight at least d.

Case 2.3: *y and x are contained in an end-block B_1 of G'.* If x is the unique cut-vertex of B_1, let B_1' be the subgraph of G induced by $V(B_1)\cup\{z\}$. Then from the above claim we know that there is an (x,y,z)-path P_1' in B_1' of weight at least d. Otherwise, since G' has at least two distinct end-blocks, we can choose an end-block B_2 in G' other than B_1. Let b_2 be the unique cut-vertex of G' contained in B_2 and B_2' be the subgraph of G induced by $V(B_2)\cup\{z\}$. Then there is a (b_2,z)-path P_2' in B_2' of weight at least d by the above claim, and there is also an (x,y,b_2)-path Q in $G'-(B_2-b_2)$. Combining these two paths Q and P_2', we get an (x,y,z)-path of weight at least d.

(2) From the above proof, we need only consider the case in which y is contained in an end-block B_1 of G' with x as its unique cut-vertex. In this case, the result follows from the induction hypothesis by considering the graph $G[V(B_1)\cup\{z\}]$.

This completes the proof. □

3. Heavy cycles in weighted graphs

There are many results on the existence of long cycles. The following two theorems are known.

Theorem C (Dirac [3]). *Let G be a 2-connected graph and d an integer. If $d(v) \geqslant d$ for every vertex v in G, then G contains either a cycle of length at least $2d$ or a Hamilton cycle.*

Theorem D (Grötschel [6]). *Let G be a 2-connected graph and d an integer. If $d(v) \geqslant d$ for every vertex v in G, then for any given vertex y of G, G contains either a y-cycle of length at least $2d$ or a Hamilton cycle.*

It is clear that Theorem D is a generalization of Theorem C.

Bondy and Fan generalized Theorem C to weighted graphs as follows:

Theorem 3 (Bondy and Fan [1]). *Let G be a 2-connected weighted graph and d a real number. If $d^w(v) \geqslant d$ for every vertex v in G, then either G contains a cycle of weight at least $2d$ or every optimal cycle is a Hamilton cycle.*

The aim of this section is to give a generalization of Theorem D to weighted graphs.

Theorem 4. *Let G be a 2-connected weighted graph and d a real number. If $d^w(v) \geqslant d$ for every vertex v in G, then for any given vertex y of G, either G contains a y-cycle of weight at least $2d$ or every optimal cycle in G is a Hamilton cycle.*

This theorem also generalizes Theorem 3.

Before proving the above theorem, we need the following result.

Theorem 5. *Let C be an optimal cycle in a weighted graph G. Suppose that there is an (x, y, z)-path P in $G-C$ such that $|N_C(x)| \geqslant 1$, $|N_C(z)| \geqslant 1$ and $|N_C(x) \cup N_C(z)| \geqslant 2$. Define*

$$X = N_C(x) \backslash N_C(z), \quad Z = N_C(z) \backslash N_C(x) \quad \text{and} \quad Y = N_C(x) \cap N_C(z).$$

If $|Y| = 1$ and either $X = \emptyset$ or $Z = \emptyset$, then there exists a y-cycle C' in G such that

$$w(C') \geqslant \frac{w(C)}{2} + \min\{d_C^w(x), d_C^w(z)\} + w(P).$$

Otherwise, there exist l $(l \geqslant 4)$ y-cycles $C_1, C_2, \ldots, C_l$ in G such that

$$\sum_{i=1}^{l} w(C_i) \geqslant (l-2)w(C) + 2d_Y^w(x) + 2d_Y^w(z) + 4d_X^w(x) + 4d_Z^w(z) + lw(P).$$

Proof. If $|Y| = 1$ and either $X = \emptyset$ or $Z = \emptyset$, we have two cases. In the case $|Y| = 1$ and $X = \emptyset$, we can assume that $Y = \{a_1\}$ and $Z = \{a_2, \ldots, a_k\}$. Without loss of generality, we suppose that the segment $C[a_2, a_1]$ is of weight at least $w(C)/2$. So the cycle

$C' = xPza_2C[a_2,a_1]a_1x$ is a y-cycle of weight

$$w(C') \geqslant \frac{w(C)}{2} + w(xa_1) + w(za_2) + w(P)$$
$$\geqslant \frac{w(C)}{2} + \min\{d^w_C(x), d^w_C(z)\} + w(P).$$

The case $|Y| = 1$ and $Z = \emptyset$ can be discussed by the same argument.

Otherwise, let $A = X \cup Y \cup Z$ and suppose that $A = \{a_1, a_2, \ldots, a_k\}$, where a_i are in order around C. For each pair of vertices (a_i, a_{i+1}), we shall construct two new cycles from C by replacing the segment $C[a_i, a_{i+1}]$ with two (a_i, a_{i+1})-paths. These two paths are defined according to four cases:

(1) $a_i, a_{i+1} \in Y$. The two paths are

$$a_ixPza_{i+1} \text{ and } a_izPxa_{i+1}.$$

(2) $a_i \in Y$ and $a_{i+1} \in X$ or Z. The two paths are

$$a_izPxa_{i+1} \text{ and } a_ixa_{i+1}, \text{ or } a_ixPza_{i+1} \text{ and } a_iza_{i+1}.$$

If $a_{i+1} \in Y$ and $a_i \in X$ or Z, the paths are defined in the same way.

(3) $a_i \in X$ and $a_{i+1} \in Z$ or $a_i \in Z$ and $a_{i+1} \in X$. The two paths are two copies of

$$a_ixPza_{i+1} \quad \text{or} \quad a_izPxa_{i+1}.$$

(4) $a_i, a_{i+1} \in X$ or $a_i, a_{i+1} \in Z$. The two paths are two copies of

$$a_ixa_{i+1} \quad \text{or} \quad a_iza_{i+1}.$$

In each case, we have defined two paths to replace the segment $C[a_i, a_{i+1}]$ and hence formed two cycles. Since there are k pairs of vertices (a_i, a_{i+1}) $(i = 1, \ldots, k)$, we obtain $2k$ cycles. In these cycles, every edge of C is traversed $2k - 2$ times; every edge from x or z to Y is traversed twice, every edge from x to X is traversed four times and, similarly, every edge from z to Z is traversed four times. Now suppose that the path P is traversed l times (we determine l later). Then the weight sum of these $2k$ cycles is

$$2(k-1)w(C) + 2d^w_Y(x) + 2d^w_Y(z) + 4d^w_X(x) + 4d^w_Z(z) + lw(P).$$

Without loss of generality, we can denote the l cycles which pass through the path P (also pass through the vertex y) by $C_1, C_2, \ldots, C_l$. Since C is an optimal cycle, those $2k - l$ cycles other than $C_1, C_2, \ldots, C_l$ have weight at most $w(C)$. Hence, we get the following inequality:

$$\sum_{i=1}^{l} w(C_i) \geqslant (l-2)w(C) + 2d^w_Y(x) + 2d^w_Y(z) + 4d^w_X(x) + 4d^w_Z(z) + lw(P).$$

Now we determine l. If $|Y| \geqslant 2$, then it is not difficult to see that $l \geqslant 2|Y|$; if $|Y| = 1$, $X \neq \emptyset$, and $Z \neq \emptyset$, then $l \geqslant 4$; if $|Y| = 0$, then noting that $|N_C(x)| \geqslant 1$ and $|N_C(z)| \geqslant 1$, we have that $X \neq \emptyset$ and $Z \neq \emptyset$, and $l \geqslant 4$. Therefore for all the cases we have that $l \geqslant 4$. $\square$

Proof of Theorem 4. Suppose that there exists an optimal cycle C in G which is not a Hamilton cycle. From Theorem 3 we have that $w(C)\geqslant 2d$. If y is contained in the cycle C, then we are done. Otherwise, let H be the component of $G-C$ which contains y. We consider two cases:

Case 1: *H is nonseparable.*

Case 1.1: $V(H)=\{y\}$. Suppose that $N_C(y)=\{a_1,a_2,\dots,a_k\}(k\geqslant 2)$, where a_i are in order around C. For each pair of vertices (a_i,a_{i+1}), we shall construct a y-cycle C_i from C by replacing the segments $C[a_i,a_{i+1}]$ with the path $a_i y a_{i+1}$. Since there are k pairs of vertices $(a_i,a_{i+1})(i=1,2,\dots,k)$, we obtain k cycles, and,

$$\begin{aligned}\sum_{i=1}^{k} w(C_i) &= (k-1)w(C)+2d_C^w(y)\\ &\geqslant 2(k-1)d+2d\\ &= 2kd.\end{aligned}$$

Then, among these k cycles there must be a y-cycle C' with weight at least $2d$.

Case 1.2: $|V(H)|\geqslant 2$. Choose distinct vertices x and z in H such that

(1) $|N_C(x)|\geqslant 1$, $|N_C(z)|\geqslant 1$, and
(2) $d_C^w(x)\geqslant d_C^w(z)\geqslant d_C^w(v)$ for all $v\in V(H)\backslash\{x,z\}$.

Case 1.2.1: $|N_C(x)\cup N_C(z)|\geqslant 2$. By the choice of x and z, we have

$$d_H^w(v)=d^w(v)-d_C^w(v)\geqslant \max\{0,d-d_C^w(z)\}\quad \text{for all } v\in V(H)\backslash\{x\}.$$

If $|V(H)|=2$, it is easy to find an (x,y,z)-path P in H of weight at least $\max\{0,d-d_C^w(z)\}$. Otherwise, applying Theorem 2 to H, we can choose an (x,y,z)-path P in H such that

$$w(P)\geqslant \max\{0,d-d_C^w(z)\}.$$

Now denote $N_C(x)\backslash N_C(z), N_C(x)\cap N_C(z)$ and $N_C(z)\backslash N_C(x)$ by X,Y and Z, respectively. If $|Y|=1$ and $X=\emptyset$ or $Z=\emptyset$, then by Theorem 5 we know that there is a y-cycle C' in G such that

$$w(C')\geqslant \frac{w(C)}{2}+\min\{d_C^w(x),d_C^w(z)\}+w(P)\geqslant 2d.$$

Otherwise, from Theorem 5 we know that G contains $l(l\geqslant 4)$ y-cycles $C_1,C_2,\dots,C_l$ such that

$$\begin{aligned}\sum_{i=1}^{l} w(C_i) &\geqslant (l-2)w(C)+2d_Y^w(x)+2d_Y^w(z)+4d_X^w(x)+4d_Z^w(z)+lw(P)\\ &= (l-2)w(C)+2d_C^w(x)+2d_C^w(z)+2d_X^w(x)+2d_Z^w(z)+lw(P)\\ &= (l-2)w(C)+4d_C^w(z)+l\max\{0,d-d_C^w(z)\}\\ &\geqslant 2ld.\end{aligned}$$

Then, among these l y-cycles in G there must be one with weight at least $2d$.

Case 1.2.2: $N_C(x)=N_C(z)=\{a\}$. Since G is 2-connected, there exists a vertex $b\in V(C)\backslash\{a\}$ which is adjacent to some vertex $u\in V(H)\backslash\{x,z\}$. By the choice of x and z, we have

$$d_H^w(v)=d^w(v)-d_C^w(v)\geqslant d-d_C^w(x) \quad \text{for all } v\in V(H).$$

Applying Theorem 2 to H, we have an (x,y,u)-path Q in H of weight

$$w(Q)\geqslant d-d_C^w(x)=d-w(xa),$$

then the path $axQub$ is of weight at least d. It is easy to see that we can form a y-cycle of weight at least $2d$.

Case 2: *H is separable.*

Case 2.1: *y is contained in a block of H with two or more cut-vertices.* Let B_1 and B_2 be two distinct end-blocks of H, and let b_i be the unique cut-vertex of H contained in B_i $(i=1,2)$. For $i=1,2$, we choose $x_i\in V(B_i)\backslash\{b_i\}$ such that

(1) $|N_C(x_i)|\geqslant 1$, and
(2) $d_C^w(x_i)\geqslant d_C^w(v)$ for all $v\in V(B_i)\backslash\{b_i\}$.

It follows that

$$d_{B_i}^w(v)=d^w(v)-d_C^w(v)\geqslant \max\{0,d-d_C^w(x_i)\} \quad \text{for all } v\in V(B_i)\backslash\{b_i\},\ (i=1,2).$$

Applying Theorem 2 to B_i we obtain an (x_i,b_i)-path P_i in B_i of weight

$$w(P_i)\geqslant \max\{0,d-d_C^w(x_i)\}.$$

If $|N_C(x_1)\cup N_C(x_2)|\geqslant 2$, then let P be an (x_1,y,x_2)-path in H of maximum weight. Then

$$w(P)\geqslant w(P_1)+w(P_2)\geqslant \max\{0,d-\min\{d_C^w(x_1),d_C^w(x_2)\}\}.$$

Denote $N_C(x_1)\backslash N_C(x_2), N_C(x_2)\backslash N_C(x_1)$ and $N_C(x_1)\cap N_C(x_2)$ by X_1,X_2 and Y, respectively. If $|Y|=1$ and $X_1=\emptyset$ or $X_2=\emptyset$, then by Theorem 5 we know that there is a y-cycle C' in G such that

$$w(C')\geqslant \frac{w(C)}{2}+\min\{d_C^w(x_1),d_C^w(x_2)\}+w(P)\geqslant 2d.$$

Otherwise, from Theorem 5 we know that G contains $l(l\geqslant 4)$ y-cycles $C_1,C_2,\ldots,C_l$ such that

$$\begin{aligned}\sum_{i=1}^{l} w(C_i) &\geqslant (l-2)w(C)+2d_Y^w(x_1)+2d_Y^w(x_2)\\ &\quad +4d_{X_1}^w(x_1)+4d_{X_2}^w(x_2)+lw(P)\\ &\geqslant 2(l-2)d+4\min\{d_C^w(x_1),d_C^w(x_2)\}\\ &\quad +l\max\{0,d-\min\{d_C^w(x_1),d_C^w(x_2)\}\}\\ &\geqslant 2ld.\end{aligned}$$

So, among these l y-cycles there must be one with weight at least $2d$.

If $N_C(x_1) = N_C(x_2) = \{a\}$, let Q be a (b_1, y, b_2)-path in H. The weight of $ax_iP_ib_i$ is at least d, then the cycle $ax_1P_1b_1Qb_2P_2x_2a$ has weight at least $2d$.

Case 2.2: *y is contained in an end-block B_1 of H*. Choose another end-block B_2 of H and let b_i be the unique cut-vertex of H contained in B_i $(i=1,2)$. For $i=1,2$, choose $x_i \in V(B_i)\backslash\{b_i\}$ such that

(1) $|N_C(x_i)| \geqslant 1$, and
(2) $d_C^w(x_i) \geqslant d_C^w(v)$ for all $v \in V(B_i)\backslash\{b_i\}$.

Applying Theorem 2 to B_1 and B_2, we obtain an (x_1, y, b_1)-path P_1 in B_1 of weight at least $\max\{0, d - d_C^w(x_1)\}$, and an (x_2, b_2)-path P_2 in B_2 of weight at least $\max\{0, d - d_C^w(x_2)\}$. It is also easy to know that there is a (b_1, b_2)-path Q in $H - (B_1 - b_1) - (B_2 - b_2)$. So the path $P = P_1QP_2$ is an (x_1, y, x_2)-path with weight.

$$w(P) \geqslant w(P_1) + w(P_2) \geqslant \max\{0, d - \min\{d_C^w(x_1), d_C^w(x_2)\}\}.$$

If $|N_C(x_1) \cap N_C(x_2)| \geqslant 2$, using the similar argument in Case 2.1, we can get a y-cycle of weight at least $2d$.

If $N_C(x_1) = N_C(x_2) = \{a\}$, there exists a vertex $b \in V(C)\backslash\{a\}$ which is adjacent to some vertex $u \in V(H)\backslash\{x_1, x_2\}$.

If $u \in V(B_1)$ and $u = b_1$, the path $bb_1P_1x_1a$ is of weight at least d; If $u \neq b_1$, we can choose a (u, y, b_2)-path Q, then the path $P = buQb_2P_2x_2a$ is of weight at least d. So in both cases we can form a y-cycle of weight at least $2d$.

If $u \notin V(B_1)$, we can choose a (b_1, u)-path Q in $H - (B_1 - b_1)$, and therefore the path $P = ax_1P_1b_1Qub$ is of weight at least d. It is easy to form a y-cycle with weight at least $2d$.

The proof is now complete. □

Acknowledgements

The first author is very grateful to Professor C. Hoede for his hospitality and help during his visit to the University of Twente.

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