Detecting symmetry breaking bifurcations in the system describing the dynamics of coupled arrays of Josephson junctions

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Abstract

Symmetry breaking and symmetry increasing bifurcations in a system of globally coupled Josephson junctions which possesses S_n symmetry have been investigated by constructing a detective for the group S_n , the permutation group of n elements. In addition, with help of this detective several bifurcations are followed numerically in two parameters.

An algorithm to obtain a detective for any finite subgroup of $O(n)$ is given.

Keywords: S_n symmetry; Detective; Symmetry increasing bifurcation

1. Introduction

The main goal of this paper is to describe the symmetry properties of attractors that have been observed in numerical simulations of models describing the dynamics of arrays of Josephson junctions [1] coupled through various loads. Soon after the discovery of Josephson effect [14] researchers realized the potential importance of large arrays of Josephson junctions for a variety of technological applications. While their technological interest has been the primary motivation for research over the past two decades, Josephson junction arrays have more recently become an archetype in the field of dynamical systems theory. Recent papers [1,21,17,22,5,4] have mainly been concerned with properties and bifurcations of symmetric solutions.

We consider two sets of equations corresponding to

the *pure capacitive* and *pure resistive* loads. A common feature of the model equations is invariance under permutation of the individual junctions. In the case of the pure capacitive load, the equations take the form [1]:

$$\ddot{\xi}_k = \frac{1}{3 + \beta} I - \frac{1}{\beta} \left(\dot{\xi}_k + \sin(\xi_k) - \frac{3}{n(3 + \beta)} \sum_{j=1}^n [\dot{\xi}_j + \sin(\xi_j)] \right), \quad (1)$$

for $k = 1, \dots, n$, where β and I are parameters. In the model with the pure resistive load, the equations take the form [1]:

$$\ddot{\xi}_k = \frac{1}{\beta} \left(I - \dot{\xi}_k - \sin(\xi_k) - \frac{1}{n} \sum_{j=1}^n \dot{\xi}_j \right), \quad (2)$$

for $k = 1, \dots, n$, where β and I are parameters. Due to symmetric coupling, both systems possess S_n symmetry.

As has been observed, these systems show very rich dynamics, both in physical and numerical experiments [1,21,17,5,4]. In particular, they admit trajectories and attractors with different symmetries. This has significant consequences for applications, since the existence of a single trajectory or attractor that has less symmetry than the system itself necessarily implies the coexistence of symmetry-related states. This can lead to a phenomenon known as attractor crowding [17,22], where the array becomes increasingly sensitive to noise as the number of oscillators grows [20]. On the other hand, the coexistence of many attractors could be exploited as a memory element, provided one could reliably switch between attractors with known symmetry [18].

These various manifestations of the symmetry in the system of Josephson junctions show the importance of devising methods to compute numerically the symmetry and detect the changes of symmetry of attractors.

The problem of finding the symmetries can be relatively simply resolved for steady state solutions. Indeed, the symmetry group of the fixed point must be the isotropy subgroup of that point and its numerical computation is, in principle, a straightforward task. Although there are certain restrictions on the symmetry groups of periodic orbits [3], determination of the symmetries of a limit cycle in higher dimensions is often difficult. Finding the symmetries of an attractor in higher dimensions is even more complicated, particularly because it is difficult to visualize these symmetries. Even in three dimensions it is sometimes difficult to determine the exact symmetries of a cloud of points, which is what a chaotic attractor of a discrete dynamical system may resemble. For finite subgroups of $O(n)$ the concept of a *detective* has been introduced by E. Barany, M. Dellnitz and M. Golubitsky [6,9]. The idea is to transfer the question of determining the symmetry of a set (the attractor) in phase space to the problem of determining the symmetry of a point in some auxiliary space determined by the symmetry group of the dynamical system. This method works in the following way. First, the attractor is thickened

to an open set preserving the symmetries. Then a certain equivariant mapping, called an *observable*, is integrated with respect to the Lebesgue measure over the thickened attractor. The authors [6,9] have shown that generically this method gives the correct symmetry. An alternative approach is to replace integration with respect to the Lebesgue measure by ergodic sums [6].

In the present article an algorithm to obtain a detective for S_n has been developed and an explicit form of the detective is obtained. In fact the algorithm gives a detective for any finite subgroup of $O(n)$.

The paper describes numerical investigation of the systems (1), (2). With the help of detectives *spontaneous symmetry breaking* [8] has been detected. Although we are interested in special features of the dynamics of large arrays of Josephson junctions, the system consisting of three coupled oscillators already shows very rich dynamics.

For the system of three coupled Josephson junctions extended numerical experiments have been performed for both the pure resistive and pure capacitive loads. In the case of the pure capacitive load, stable periodic orbits with S_3 , Z_3 and Z_2 symmetry as well as non-symmetric trajectories have been observed. An area of the occurrence of Z_3 symmetric attractor has been explored. In the case of the pure resistive load, stable S_3 , Z_2 symmetric as well as non-symmetric trajectories and attractors have been observed. Several bifurcations are found: torus bifurcations, period doubling bifurcations, pitchfork bifurcations, saddle-node bifurcations for periodic orbits and symmetry increasing bifurcations for attractors. Numerical evidence has been obtained for two scenarios of symmetry increasing bifurcations: through an explosion and through a heteroclinic cycle. A search over the grid in the space of two parameters has been performed to find regions where the attractors with certain fixed symmetry exist.

Symmetry detection has been performed for arrays of four and five Josephson junctions and complex dynamics have been observed.

In addition, the detective technique allows us to follow branches in the space of two parameters for *soft* bifurcations such as period doubling and pitchfork bifurcations for limit cycles. The detective functions are

used to provide the AUTO [10] driver with suitable integral constraints.

The paper is organized as follows. In Section 2 an introduction to the detective method is given and the *detective* theorems are stated. In Sections 2.1 and 2.2 an explicit form of the detective for S_n is obtained using the theorems and then adapted for the system of Josephson junctions. In Section 2.3 the algorithm to obtain a detective for any finite subgroup of $O(n)$ is developed as a generalization of the result from the previous section.

Section 3 presents the results of the numerical simulations together with several examples of the symmetry increasing (breaking) bifurcations found using the detectives. In addition, with help of detectives, several bifurcations were followed in two parameter space using the software package AUTO.

The Appendix contains the proof of the Theorem 2.5 and comments on [9]. We note that Golubitsky and Nicol [13] independently proved a similar detective theorem which is applicable for attractors with non-trivial *instantaneous* symmetry.

2. Detectives and S_n symmetry

First we give a general introduction to and basic concepts of the *detective* method as introduced by E. Barany, M. Dellnitz and M. Golubitsky [6,9]. In this section we next obtain an explicit form of the detective using the group ring R_Γ as a representation space. The notation and definitions in the group theory are consistent with [16].

Following [6] we define:

Definition 2.1. An observable is a C^∞ Γ -equivariant mapping $\phi : \mathbb{R}^n \rightarrow W$, where W is some (finite-dimensional) representation space of Γ .

An observation is $K_\phi(A) = \int_A \phi(x) d\mu(x)$, where μ is the Lebesgue measure and A is an open bounded set.

One can obtain an observable by averaging over the group as stated in the following lemma:

Lemma 2.2. Let W be a representation space and $\hat{f} : \mathbb{R}^n \rightarrow W$ be a polynomial mapping, then $P\hat{f}(x) = (1/|\Gamma|) \sum_{\gamma \in \Gamma} \gamma^{-1} \hat{f}(\gamma x)$ is a projection operator and $\phi = P\hat{f}$ is a Γ -equivariant polynomial mapping.

Let A be an open bounded subset of $\mathbb{R}^n$ which satisfies the dichotomy $\gamma A = A$ or $\gamma A \cap A = \emptyset$, $\forall \gamma \in \Gamma$. Note that such a set A exists if it has the property that it is an open basin of an attractor.

Let $\Sigma(A) \subset \Gamma$ be a symmetry subgroup of the set A , that is $\forall \sigma \in \Sigma(A)$ and $\forall a \in A$, $\sigma a \in A$, and let $\Sigma_\phi(A) \subset \Gamma$ be the isotropy group of the point $K_\phi(A)$. Let $\mathcal{O}$ be an open dense set of a neighborhood of identity in the space of Γ -equivariant diffeomorphisms $\text{Diff}_\Gamma(\mathbb{R}^n)$. Using this notation we now define a *detective* as follows.

Definition 2.3. [6]. The observable ϕ is a detective if for each A there exists $\mathcal{O}$ such that for any diffeomorphism $\psi \in \mathcal{O}$, $\Sigma_\phi(\psi(A)) = \Sigma(A)$.

In other words, an observable is a detective if the isotropy subgroup of observation is generically equal to the symmetry subgroup of an attractor L containing in the open set A .

Let us denote the fixed point subspace of the group Σ in the space W by $\text{Fix}_W(\Sigma)$. Now suppose that W distinguishes all subgroups of Γ , that is, if

$$\dim \text{Fix}_W(\Delta) > \dim \text{Fix}_W(\Sigma)$$

for any subgroups $\Delta \subset \Sigma \subset \Gamma$, $\Delta \neq \Sigma$. This means that all subgroups of Γ are isotropy subgroups in the space W . An important example of a representation space which distinguishes all subgroups is the group ring R_Γ [16]. The group ring is the linear space consisting of all real valued functions on Γ . The action of $\gamma \in \Gamma$ on $v \in R_\Gamma$ is defined by $(\gamma \cdot v)(\delta) = v(\gamma^{-1}\delta)$ and the functions h_γ given by

$$h_\gamma(\sigma) = \begin{cases} 1 & \text{if } \sigma = \gamma, \\ 0 & \text{if } \sigma \neq \gamma, \end{cases} \quad (3)$$

form a natural basis in R_Γ .

Theorem 2.4. Let $\phi : \mathbb{R}^n \rightarrow W$ be a polynomial observable and let W distinguish all subgroups of Γ .

If the coordinate components of ϕ , $\{\phi_j, 1 \leq j \leq N = \dim W\}$, are linearly independent, then ϕ is a detective.

It should be noted that the formulation of Theorem 2.4 is slightly different from that given in [6], although conceptually they are similar. The linear independence of $\{\phi_j\}$ implies that $\text{span}(\phi(\mathbb{R}^n)) = W$. Hence, if W contains all irreducible subspaces W_j corresponding to all non-equivalent irreducible representations of Γ , ϕ is non-zero on W_j , which is exactly the condition required in [6]. In our applications, where the group ring is considered as a representation space, linear independence is easier to verify, because it does not require computations of irreducible subspaces.

As noted in [6], when $n \geq 3$, then the memory requirements for the integral test are very extensive and it is no longer possible to cover a whole region containing the attractor by a grid. An alternative method using ergodic sums is therefore often better in practice. When applying the ergodic sum method the following assumptions have been made. If for an attractor A , $A = \omega(x(0))$, of the Γ -equivariant dynamical system there is an open neighborhood $U \supset A$ in which all points have ω -limit sets contained in A , then the ergodic sum is defined by

$$K_\phi^E(A) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} \phi(f^j(x(0))).$$

For this definition of $K_\phi^E(A)$ to be useful, the right hand side must be essentially independent of $x(0)$. Required properties of the dynamical system are discussed in [9].

Let us denote the isotropy subgroup of $K_\phi^E(A)$ by $\Sigma_\phi^E(A)$. The set A is no longer necessarily an open set, e.g. an attractor can lie in a fixed point subspace. Hence the observable should satisfy an additional condition to be a detective.

Theorem 2.5. Let the polynomial mapping ϕ , the representation space W , the diffeomorphism ψ and the neighborhood $\mathcal{O}$ be as in Theorem 2.4. If, in addition, $\text{span}(\phi(\text{Fix}_{\mathbb{R}^n}(\Delta))) = \text{Fix}_W(\Delta)$ for any isotropy subgroup $\Delta \subset \Gamma$, then there exists $\mathcal{O}$ such that $\forall \psi \in \mathcal{O}$, $\Sigma(A) = \Sigma_\phi^E(\psi(A))$.

Theorem 2.5 is a strengthened version of Theorem 3.3 in [9]. In [13] Golubitsky and Nicol proved a similar general detective theorem. In the Appendix we show how the ideas in [9] are sufficient to prove Theorem 2.5.

2.1. A detective for S_n

Consider S_n , the group of permutations of n objects. Let $p : \mathbb{R}^n \rightarrow \mathbb{R}$ be the polynomial mapping given by $p(x_1, \dots, x_n) = x_1 x_2^2 \cdots x_{n-1}^{n-1}$. An explicit form of a detective for S_n is given in the following theorem.

Theorem 2.6. Let R_{S_n} be the group ring associated with S_n and let $\phi : \mathbb{R}^n \rightarrow R_{S_n}$ be defined by

$$\phi(x)[\gamma] = p(\gamma^{-1}x).$$

Then the mapping ϕ is a detective for S_n .

Proof. First we note that the mapping ϕ is S_n -equivariant by Lemma 2.2. Next we will prove that the mapping ϕ has linearly independent components. Consider the polynomial $p_n(x_1, \dots, x_n) = x_1 x_2^2 \cdots x_{n-1}^{n-1}$ and let $\sigma_i \in S_n$ act by permutation of coordinates in $\mathbb{R}^n$, then $\{p_n(\sigma_i x) : \sigma_i \in S_n\}$ is a set of $n!$ linearly independent monomial mappings. We prove this by induction. For $n = 2$ the monomials x_1 and x_2 are obviously linearly independent. Now we prove the statement for n assuming that the monomials $\{p_{n-1}(\sigma_i x) : \sigma_i \in S_{n-1}\}$ are linearly independent. Consider a linear combination $\sum \alpha_i p_n(\sigma_i x) = 0$, and substitute $x_n = 0$ into it, then taking out the common factor $\prod_{j=1}^{n-1} x_j$ we have a linear combination of $(n-1)!$ monomials $x \mapsto p_{n-1}(\sigma_i x)$. These monomials are linearly independent by the induction assumption and hence if its linear combination is zero then all the coefficients are zero. Subsequently taking $x_j = 0$ for all $j = 1, \dots, n-1$, we obtain that all α_i are zeros which proves the required linear independence of the $\{p_n(\sigma_i x)\}$. This implies that the mapping ϕ has linearly independent coordinate components. The representation space R_{S_n} distinguishes all subgroups of S_n . Hence by Theorem 2.4, the mapping ϕ is a detective. $\square$

In [6] a detective for the group D_n has been obtained. As an application, the system of three coupled cells is considered, such that each cell is described by two equations:

$$\begin{aligned}\dot{x}_j &= y_j + \delta x_j^2 y_j, \\ \dot{y}_j &= -x_j - (x_j^2 - \lambda) y_j \\ &\quad + \alpha(y_{j-1} - 2y_j + y_{j+1}) + \beta x_j y_j,\end{aligned}\quad (4)$$

where $j = 1, 2, 3$, $y_0 = y_3$ and $y_4 = y_1$ and α, β, δ and λ are parameters. This dynamical system possesses $D_3(S_3)$ symmetry, where elements of the symmetry group act by block permutation matrices.

As an outcome of the Theorem 2.6, taking $p(x) = x_1 x_2^2$ we obtain the following detective $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^6 = R_{D_3}$ for the system (4):

$$\phi(x) = (x_1 x_2^2, x_1 x_3^2, x_2 x_1^2, x_2 x_3^2, x_3 x_2^2, x_3 x_1^2),$$

which depends only upon the x component of each cell. We can also make use of the fact that each cell is described by two variables to lower the degree of the detective and decrease the number of elementary computations required to evaluate the detective in one point. The mapping $(x, y) \mapsto \hat{\phi}(x, y) = (x_1 y_2, x_1 y_3, x_2 y_1, x_2 y_3, x_3 y_2, x_3 y_1)$ is equivariant and spans $R_{D_3} = \mathbb{R}^6$ as the mapping ϕ does it, and hence $\hat{\phi}$ is a detective. Let us now compare the detective $\hat{\phi}(x)$ with the detective obtained in [6]. W , the space of 6×6 real symmetric matrices, has been taken as a representation space with $\gamma \in D_3$ acting by similarity transformation on that space:

$$T(\gamma) \cdot w = \gamma w \gamma^{-1},$$

$\forall w \in W$, where γ is the block permutation matrix. The mapping $\Phi : \mathbb{R}^6 \rightarrow W$,

$$\Phi(x, y) = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{pmatrix} \cdot (x_1, y_1, x_2, y_2, x_3, y_3),$$

has been proven to be a detective. We note that this approach can be generalized for any finite subgroup

of $O(n)$. Let V be an n -dimensional state space with the basis vectors $\{v_j\}$ and Γ be a finite subgroup of $O(n)$ with $T(\gamma)$ the action on V . If m is sufficiently large, then the symmetric part of the tensor product $V^{\otimes m} = \underbrace{V \otimes \dots \otimes V}_m$ contains all irreducible represen-

tation spaces of the group Γ [7]. The symmetric part of $V^{\otimes m}$ is isomorphic to the linear space W spanned by the basis vectors $\{v_{k_1} \otimes v_{k_2} \otimes \dots \otimes v_{k_m}\}_{1 \leq k_1 \leq k_2 \leq \dots \leq k_m \leq n}$. To obtain a detective we identify the basis vectors with the homogeneous vector polynomial $\Phi : \mathbb{R}^n \rightarrow W$: $\Phi^{(k_1 k_2 \dots k_m)} = x_{k_1} x_{k_2} \dots x_{k_m}$, $1 \leq k_1 \leq k_2 \leq \dots \leq k_m \leq n$ [19]. For $m = 2$ and $n = 6$ the polynomial tensor Φ is the detective obtained in [6]. The detective for D_6 , which corresponds to $m = 2$ and $n = 12$, has been used by Kroon and Stewart [15] to study a model for hexapodal gaits.

We notice the following relation between the detectives, $\hat{\phi}$ and Φ . These two mappings are equal on the irreducible subspaces. To illustrate this, consider the linear subspace of W spanned by the matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and note that it is an *alternating* representation space of the group $S_3(D_3)$, i.e. if γ in $S_3(D_3)$ is odd then $\gamma M \gamma^{-1} = -M$. The composition of the detective $\Phi(x, y)$ with the orthogonal projection, P_M , on the one dimensional subspace spanned by the vector $M \in \mathbb{R}^{(7 \times 6)/2}$ is the mapping $P_M \circ \Phi = (x_1 y_2 + x_3 y_1 + x_2 y_3 - x_2 y_1 - x_3 y_2 - x_1 y_3)$. This is equal to the composition of the detective $\hat{\phi}$ with the orthogonal projection on the one-dimensional subspace of the alternating representation in the group ring R_{S_3} . The subspace of the alternating representation is the span of the vector $\sum_{s_i \in S_3} \delta(s_i) s_i \in R_{S_3}$, where $\delta(s_i) = +1$ if s_i is an even permutation and $\delta(s_i) = -1$ if s_i is an odd permutation.

2.2. Detectives for the system of coupled Josephson junctions

Now we will apply Theorem 2.6 to the system of Josephson junctions, which is an S_n symmetric system, where the elements of S_n act as permutation matrices consisting of 2×2 blocks corresponding to the permutations of the individual junctions. The equations describing the model can be written in the following form:

$$\begin{aligned}\dot{\xi}_k &= \psi_k, \\ \dot{\psi}_k &= f_k(\xi, \psi),\end{aligned}$$

where $f_k(\xi, \psi)$ is 2π -periodic in ξ , $k = 1, \dots, n$. The phase space for each oscillator is the cylinder, whereas the domain of the polynomial detective obtained in the Theorem 2.6 is $\mathbb{R}^n$.

Let ϕ satisfy the conditions of Theorem 2.5. With ϕ we associate the mapping $\tilde{\phi} : \mathbb{R}^n \rightarrow R_{S_n} \oplus iR_{S_n}$ defined by $\tilde{\phi}(\xi_1, \dots, \xi_n) = \phi(e^{i\xi_1}, \dots, e^{i\xi_n})$. The right hand side makes sense because ϕ is a polynomial mapping. Then by the following proposition $\text{Re } \tilde{\phi}$ is a detective for the system of coupled Josephson junctions:

Proposition 2.7. $\text{span}\{\text{Re } \tilde{\phi}(\xi) : \xi_k \in \mathbb{R}\} = R_{S_n}$.

Proof. If we extend the domain of the mapping ϕ to $\mathbb{C}^n$ setting $\hat{\phi}(Z) = \phi(z_1, \dots, z_n)$, then $\hat{\phi}$ still clearly has linearly independent coordinate components and $\text{span}\{\text{Re } \hat{\phi}(Z)\} = R_{S_n}$. Now we take the mapping $\phi(e^{i\xi_1}, \dots, e^{i\xi_n})$, $\xi_k \in \mathbb{R}$ and suppose that its coordinate components $p_k(e^{i\xi_1}, \dots, e^{i\xi_n})$ are linearly dependent, that is some linear combination $q(e^{i\xi_1}, \dots, e^{i\xi_n}) = \sum \alpha_k p_k(e^{i\xi_1}, \dots, e^{i\xi_n})$ is identically zero and not all α_k are zero. The function $q(e^{i\xi_1}, \dots, e^{i\xi_n})$ is an analytical function of all its arguments, thus $q(z_1, e^{i\xi_2}, \dots, e^{i\xi_n})$ is identically zero for any $z_1 \in \mathbb{C}$ as an analytical continuation of the zero function. Subsequently applying the same argument to all arguments of the function q , we conclude that $q(z) = 0, \forall z \in \mathbb{C}^n$, which is a contradiction with linear independence of the coordinate components of $\hat{\phi}$. $\square$

Let r be an n -dimensional vector with the following

components: $r = (1, 2, \dots, n-1, 0)$ and define x^r as $x^r = x_1^{r_1} x_2^{r_2} \dots x_n^{r_n}$. Denote the identity element of S_n by e and denote the Euclidean scalar product in $\mathbb{R}^N$ by $\langle \cdot, \cdot \rangle$, then the functions

$$\begin{aligned}\phi_1(x) &= \begin{pmatrix} \psi^{er} \\ \psi^{\sigma_1^{-1}r} \\ \vdots \\ \psi^{\sigma_N^{-1}r} \end{pmatrix}, \\ \phi_2(\xi) &= \begin{pmatrix} \cos(\langle er, \xi \rangle) \\ \cos(\langle \sigma_1^{-1}r, \xi \rangle) \\ \vdots \\ \cos(\langle \sigma_N^{-1}r, \xi \rangle) \end{pmatrix}\end{aligned}$$

are detectives for S_n by Theorem 2.6 and Proposition 2.7. The polynomial detective ϕ_1 suffers from the fact that it is high order polynomial mapping and thus has extreme scaling behaviour near the origin. Using the detective ϕ_2 overcomes this to a certain extent. Note that these functions also satisfy the conditions of the Theorem 2.5 and can be applied for both integral and ergodic sum criteria.

2.3. Detectives for finite subgroups of $O(n)$

The detectives described in subsection 2.1 can be easily modified to the case of any finite subgroup of $O(n)$. Let $\Gamma \in O(n)$ be a group of finite order N and $\mathbb{R}^n$ be the state space. Denote the matrix representation of Γ on $\mathbb{R}^n$ by $T(\gamma), \gamma \in \Gamma$. Let $p : \mathbb{R}^n \rightarrow \mathbb{R}$ be a polynomial such that $\{p(T(\gamma) \cdot) : \gamma \in \Gamma\}$ is a linearly independent set, then an explicit form of a detective for Γ is given in the following theorem.

Theorem 2.8. Let R_Γ be the group ring associated with Γ and $\phi : \mathbb{R}^n \rightarrow R_\Gamma$ be defined by

$$\phi(x)[\gamma] = p(T(\gamma^{-1})x).$$

Then ϕ is a detective.

Proof. Follows from the Lemma 2.2, Theorem 2.4 and the linear independence of the polynomials $\{p(T(\gamma) \cdot) : \gamma \in \Gamma\}$. $\square$

The symmetry of the attractor is given by the distances to the fixed point subspaces of all subgroups of Γ . The structure of these subspaces can be viewed as follows. Let $\{\gamma_j\}$ be a subset of Γ and Σ be the subgroup of Γ generated by the elements $\{\gamma_j\}$. For a set of matrices $\{A_j : A_j \in \mathbb{R}^{N^2}\}$, $w \in \mathbb{C}^N$ is called a joint eigenvector if it is an eigenvector for any A_j with the same eigenvalue. Let $\{v_i\}$ be the set of joint eigenvectors of the matrices $\{L(\gamma_j)\}$ of the left regular representation in the natural basis corresponding to the eigenvalue 1, then the subspace of R_Γ spanned by the vectors $\{v_i\}$ is the fixed point subspace for the subgroup Σ . Hence to obtain the isotropy subgroup for the observation $K_\phi(A) \in R_\Gamma$, one should subsequently calculate for all elements $\gamma \in \Gamma$ the distance from the observation to the eigenspace corresponding to the eigenvalue 1 for the matrix of the left regular representation. Let us consider the left regular representation as a permutation of $N = n!$ objects. Then each element $\gamma \in \Gamma$ has a representation as a product of k cycles, $s_1 s_2 \cdots s_k$, while the length of the cycle s_i is denoted by l_{s_i} , $\sum_{i=1}^k l_{s_i} = N$. Let d denote the Euclidean distance from the observation to the fixed point subspace of the subgroup generated by the element γ and $\{w_j\}_{j=1, \dots, N}$ denote the components of the N -dimensional vector $K_\phi(A) \in R_\Gamma$. Then the eigenspace corresponding to the eigenvalue 1 consists of the vectors which have equal components for indices from each cycle in the cycle notation or, equivalently, are constant on each coset of the subgroup generated by γ and, hence the following proposition has been proved.

Proposition 2.9.

$$d^2 = \sum_{i=1}^k \sum_{j \in s_i} \left(w_j - \frac{\sum_{m \in s_i} w_m}{l_{s_i}} \right)^2.$$

We used the last formula while performing a search over the grid.

To construct integral boundary conditions an estimate has been used which does not require the complete list of elements of Γ . Let M be the order of the element γ . Then $\frac{1}{M} \sum_{j=1}^M \gamma^j \phi(x)$ is the orthogonal projection of the vector $\phi(x)$ on the fixed point subspace of the subgroup generated by γ . Let d again

be the distance from the observation to the fixed subspace of the subgroup generated by the element γ .

Proposition 2.10. Let $\phi(x)$ be a detective, then

$$d \leq \frac{M-1}{2} \left\| \int_A (\phi(x) - \phi(\gamma x)) d\mu(x) \right\|.$$

Proof. Using orthogonality of the projection we obtain

$$\begin{aligned} d &= \left\| \int_A \left(\phi(x) - \frac{1}{M} \sum_{j=1}^M \gamma^j \phi(x) \right) d\mu(x) \right\| \\ &= \frac{1}{M} \left\| \sum_{j=1}^{M-1} \left(\int_A (M-j) (\phi(\gamma^{j-1} x) - \phi(\gamma^j x)) d\mu(x) \right) \right\| \\ &\leq \frac{1}{M} \sum_{j=1}^{M-1} \left(\left\| \int_A (M-j) \gamma^{j-1} (\phi(x) - \phi(\gamma x)) d\mu(x) \right\| \right) \\ &\leq \frac{M-1}{2} \left\| \int_A (\phi(x) - \phi(\gamma x)) d\mu(x) \right\|. \end{aligned}$$

□

The right hand side of the inequality is an integral which can be exploited in the program AUTO.

3. Results of numerical experiments

In this section we consider two sets of equations describing the dynamics of globally coupled Josephson junctions for the cases of the pure resistive and pure capacitive loads. Assuming the pure resistive load, the equations are:

$$\begin{aligned} \dot{\xi}_k &= \psi_k, \\ \dot{\psi}_k &= \frac{1}{\beta} \left(I - \psi_k - \sin(\xi_k) - \frac{1}{n} \sum_{j=1}^n \psi_j \right), \end{aligned} \quad (5)$$

where β and I are parameters, $k = 1, n$. For $n = 3$, extensive experiments were performed. First we de-

Fig. 1. Symmetry types.

scribe the methodology which is similar to the one used in [6] and results of a search over a particularly interesting two-parameter region.

In the notation above I and β covered the region $0.8 \leq I \leq 1.8$ and $0.6 \leq \beta \leq 3.58$. The search was conducted *quasi-statically* both in the parameter space and with respect to the initial data. For $I \geq 1.8$, the S_3 -symmetric periodic orbits are globally attracting [1], thus the initial parameter values were taken to be $(I, \beta) = (1.8, 0.6)$, then β was held fixed while I was decremented by 0.005 until it reached its minimum value of 0.8. Then β was incremented by 0.005 with I again taken to be its maximum value 1.8 and then I was decremented with steps equal to 0.005 until its minimum value was reached. This procedure was repeated until whole region of interest was covered by the grid. For each pair of the parameter values, the map was iterated 10000 times to eliminate transients. The initial conditions corresponding to $(I, \beta) = (1.8, 0.6)$ were taken arbitrarily, but they were subsequently chosen quasi-statically in the following sense. When convergence to an attractor occurred, the final value was taken to be the initial condition for the transient of the next set of parameters. In order to avoid being artificially caught in invariant subspaces, the value was perturbed by adding a small random number ($O(10^{-4})$) to each component.

The results are summarized in Fig. 1. This is the parameter plane with I along the horizontal axis and β along the vertical axis. One can see three grades of the grey scale on the figure. The color of each pixel corresponds to a certain detected symmetry of attrac-

tors obtained as a result of numerical simulations with the value of the parameters corresponding to the coordinates of the pixel. The correspondence between the colors and distances, d_{S_i} , to the fixed point subspaces of the subgroups of S_3 in Fig. 1 is

- white: $d_{S_3} \leq 0.05$;
- grey: $d_{Z_2} \leq 0.05$ and $d_{S_3} > 0.05$;
- dark-grey: trivial symmetry (all distances greater than 0.05).

In performing numerical experiment the main difficulty is to decide when the distance between the observation and fixed point subspace is truly zero. To obtain independent indicators of the symmetry of the attractors we used an equivariant projection of the phase space onto an invariant subspace that renders the symmetry of the attractor visible and we performed bifurcation analysis of the system using AUTO [10] in order to relate changes of symmetries and bifurcations of solutions.

The dotted and dashed lines in Fig. 1 represent branches of the period doubling bifurcations (PD) and pitchfork bifurcations (BF). These curves were obtained using software package AUTO. Integral boundary conditions were constructed along the lines of Proposition 2.10 taking the norm $\left\| \int_A \phi(x) - \phi(\sigma x) d\mu(x) \right\|$ be equal to some small number (0.1 in our case). Such an application of the detective method to bifurcation analysis offers an advantage over standard path-following procedures applied to systems with symmetry.

Fig. 2 represents the dependence of the distances to fixed point subspaces of four subgroups of S_3 generated by the elements (12), (13), (23), (123) depending on I , taken $\beta = 1.59$.

For $I \geq 1.683$, the attractor is the S_3 -symmetric trajectory corresponding to in-phase oscillations of all three junctions. At $I = 1.682$ the symmetric periodic solution undergoes period doubling bifurcation, which leads to the change in symmetry. For $1.565 \leq I \leq 1.682$, the attractor is essentially Z_2 symmetric. One can hardly see from that diagram that the periodic Z_2 symmetric periodic orbit undergoes Hopf bifurcation at $I = 1.58$, because this bifurcation does not lead to a change in symmetry [11].

Fig. 2. Distances between the observation and the fixed point subspaces of various subgroups of S_3 ($\beta = 1.59$, $0.95 \leq I \leq 1.7$).

For $I = 1.565$, a symmetry increasing bifurcation takes place. In order to visualize the symmetry of the attractor we project the 6-dimensional phase space onto V_2 , a 2-dimensional irreducible subspace for S_3 (remember that the phase space $\mathbb{R}^6$ admits the following decomposition into irreducible subspaces: $\mathbb{R}^6 = 2V_1 \oplus 2V_2$, $\dim V_1 = 1$). Numerically computed orbits for the value of parameters close to the symmetry increasing bifurcation are shown in Figs. 3, 4, 5. The orbits are projected onto V_2 . The projection of the Z_2 symmetric attractor is shown in Fig. 3 for $I = 1.566$. In Fig. 4 three Z_2 symmetric conjugate attractors are shown for the same value of I . For $I = 1.565$, three Z_2 symmetric conjugate attractors collide and the resulting attractor is the S_3 symmetric one shown in Fig. 5. The estimate of the correlation dimension of this attractor is 2.8, and this is numerical evidence of its chaotic structure. Fig. 6a shows distances to fixed

Fig. 3. Z_2 -symmetric attractor.

Fig. 4. Three conjugate Z_2 -symmetric attractors. Fig. 5. S_3 -symmetric chaotic attractor.Table 1
Symmetries of the system

I	Instantaneous symmetry	Average symmetry
0.9–1.13	(23), (34), (24)	–
1.13–1.18	(23)	(14), (23)
1.18–1.19	(23)	–
1.19–1.22	(23)	(14), (23)
1.22–1.32	(14), (23)	–
1.32–1.38	–	(14), (34)
1.38–1.45	–	(14)
1.45–1.51	(14), (34)	–
1.51–1.53	(12), (34)	(23)
1.53–1.56	(12), (34)	–
1.56–1.69	(12), (34)	(23)

point subspaces corresponding to four subgroups of S_3 near $I = 1.565$. When the parameter I is changed from 1.566 to 1.565 the distances corresponding to the subgroups generated by elements (12), (23), (231) jump to a value close to zero.

An S_3 symmetric attractor exists for the range of parameter $1.485 \leq I \leq 1.565$. For $I = 1.565$, it undergoes a symmetry breaking bifurcation represented in Fig. 6b, and for $I = 1.46$, there exist three conjugate Z_2 -symmetric limit cycles in the fixed point subspaces.

For the value of parameters $1.224 \leq \beta \leq 1.309$, there exists a Z_2 symmetric attractor.

Similar experiments were performed for the vertical cross-section in the parameter plane. Now we fix $I = 1.55$ and vary β from 0.6 to 2.6. See Fig. 7.

Next we consider a larger array. For $n = 4$, there are 16 generators of all possible subgroups of S_4 .

In Fig. 8 all sixteen distances to the fixed point subspaces are represented for $\beta = 1.59$ and $0.9 \leq I \leq 1.7$.

All observed average and instantaneous symmetries are given in Table 1.

For large n , memory and time requirements in general do not allow performance of a search over the grid of two parameters, moreover it is difficult to represent symmetries in terms of distances corresponding to all possible subgroups. For these reasons, we changed our strategy for $n = 5$. First we took a particular trajectory and found its symmetries. Next we performed numerical simulations, quasi-statically changing the parameters and initial data and observing the distances to

Fig. 6. (a) Symmetry increasing bifurcation. (b) Symmetry breaking bifurcation.

Fig. 7. Distances to the fixed point subspaces ($0.6 \leq \beta \leq 2.6$, $I = 1.55$).

the fixed point subspaces corresponding to the symmetries of the entire trajectory as well as distances to the fixed point subspaces corresponding to the possible symmetry increasing bifurcations.

In Fig. 9 there are the distances to the fixed point

subspaces of the subgroups of S_5 generated by the following elements of S_5 : $\{(13), (24), (25), (12), (15)\}$. The values of the parameters are: $\beta = 1.6$, $0.9 \leq I \leq 1.65$.

One can see that for the whole range of parameters

Fig. 8. Distances to the fixed point subspaces ($0.9 \leq I \leq 1.7$, $\beta = 1.59$).

$0.9 \leq I \leq 1.38$ the attractor lies in the fixed point subspaces of subgroups containing the element (13). The *normalizer* of the group Z_2 in S_5 , the largest subgroup of S_5 which leaves the subspace $\text{Fix}_{\mathbb{R}^{10}}(Z_2)$ invariant, is the subgroup $S_3 \times Z_2 \subset S_5$. This implies

that the possible average symmetry subgroup is a subgroup of $S_3 \times Z_2/Z_2$ (see [13]). The group $S_3 \times Z_2/Z_2$ (see [13]), is generated by the following elements: $\{(24), (25), (45), (245)\}$ and is isomorphic to S_3 .

The distances corresponding to these elements are

Fig. 9. Distances to the fixed point subspaces ($\beta = 1.6$, $0.9 \leq I \leq 1.65$). Fig. 10. Distances to the fixed point subspaces ($\beta = 1.6$, $0.9 \leq I \leq 1.65$).

Fig. 11. Symmetry types.

shown in Fig. 10.

A number of experiments has been performed for the case of the pure capacitive load. The equations read:

$$\dot{\xi}_k = \psi_k, \quad (6)$$

$$\dot{\psi}_k = \frac{1}{3 + \beta} I - \frac{1}{\beta} \left(\psi_k + \sin(\xi_k) - \frac{3}{n(3 + \beta)} \sum_{j=1}^n [\psi_j + \sin(\xi_j)] \right), \quad (7)$$

for $k = 1, n$. For $n = 3$, a search over the grid of the parameters was performed. The parameters I and β covered the region $0.8 \leq I \leq 1.5$, $0.1 \leq \beta \leq 0.6$.

The grid with symmetries of attractors represented by different grades of the grey scale is plotted in Fig. 11.

Fig. 12. Distances between the observation and the fixed point subspaces of various subgroup of S_3 ($\beta = 0.2$, $0.85 \leq I \leq 1.25$).

Fig. 13. Non-symmetric periodic orbit ($I = 0.973$, $\beta = 0.2$). Fig. 15. Z_3 -symmetric attractor ($I = 0.9729$, $\beta = 0.2$). Fig. 14. Symmetry increasing bifurcation ($I = 0.97294$, $\beta = 0.2$). Fig. 16. Z_3 -symmetric attractor ($I = 0.96$, $\beta = 0.2$).

The color of each pixel corresponds to a certain detected symmetry of attractors obtained as a result of numerical simulations with the value of the parameters corresponding to the coordinates of the pixel. The correspondence between the colors and distances, d_S , to the fixed point subspaces of the subgroups of S_3 in Fig. 11 is

- white – $d_{S_3} \leq 0.05$;
- light-grey – $d_{Z_3} \leq 0.05$ and $d_{S_3} > 0.05$;
- dark-grey – trivial symmetry;

- black – $d_{Z_2} \leq 0.05$ and $d_{S_3} > 0.05$ (there are only a few such pixels in the grid).

Fig. 12 represents the dependence of the distances to the fixed point subspaces of four subgroups of S_3 depending on I , taking $\beta = 0.2$.

For $I \geq 1.21$, the attractor is the S_3 -symmetric trajectory corresponding to in-phase oscillations of all three junctions. At $I = 1.21$ it undergoes the fixed point bifurcation which leads to the change in symmetry. For $1.01 \leq I \leq 1.206$, the attractor is essen-

Fig. 17. Time series representation of the trajectory shown in Fig. 14.

tially Z_3 -symmetric. For $1.98 \leq I \leq 1.206$, the attractor is a Z_3 symmetric periodic orbit (POM [1]). Starting from $I = 1.098$ it undergoes a sequence of period doubling bifurcations. For $0.973 \leq I \leq 1.009$, the attractor is a non-symmetric periodic orbit, which is shown in Fig. 13. For $I = 0.97294$, it undergoes the saddle-node bifurcation for limit cycles (an additional Floquet multiplier becomes 1) which is detected numerically using AUTO. For $0.955 \leq I \leq 0.973$, a Z_3 -symmetric attractor has been observed. For $I = 0.973$, the near-zero distances in the detective method suggest that a symmetry increasing bifurcation takes place resulting in an S_3 symmetric attractor; but a closer look at the distances suggests only Z_3 , because, although there is a change in the distances that occurs at this parameter value, the distance d_{Z_3} is significantly smaller than others. This conclusion is confirmed by looking at the shape of the attractor in V_2 (Figs. 14, 15, 16). In Fig. 14 one can clearly see the structure which looks like three conjugate periodic orbits (each of them looks like periodic orbit in Fig. 13) connected by a trajectory. This can be considered as numerical evidence of the symmetry increasing bifurcation through the heteroclinic connection for periodic orbits. In Fig. 17 the solution shown in Fig. 14 is represented as a time series with time along the horizontal axis. The graph represents as a function of time the

Fig. 18. Z_3 -symmetric attractor ($I = 0.9553$, $\beta = 0.2$).

distance between the trajectory for $I = 0.97294$, $\beta = 0.2$ and the three conjugate limit cycles that exist for $I = 0.973$, $\beta = 0.2$. The distance close to 1 means that the trajectory visited a small neighborhood of the first conjugate cycle, the distance close to 2 means that the trajectory visited a small neighborhood of the second conjugate cycle and the distance close to 3 means that the trajectory was close to the third conjugate cycle. It is clear from the figure that the trajectory spends

Fig. 19. Z_3 -symmetric orbit ($I = 0.955$, $\beta = 0.2$).

Fig. 20. (a) Symmetry increasing bifurcation. (b) Symmetry breaking bifurcation.

most of the time near these cycles, jumping from one to another. The estimate of the correlation dimension of this attractor is 2.

In Fig. 18 the Z_3 symmetric attractor is shown for $I = 0.9553$, $\beta = 0.2$.

For $0.942 \leq I \leq 0.955$, there exists Z_3 symmetric periodic orbit which appears through the saddle-node bifurcation. It is shown in Fig. 19.

Figs. 20a and 20b show the change of distances between the observations and the fixed point subspaces corresponding to the all possible subgroups of S_3 .

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Appendix. Comments and the proof of Theorem 2.5

Theorem 3.3 [9] assumes existence of points with the trivial isotropy in the attractor. If this assumption is violated, then the detective can detect wrong symmetry. Consider the following example. Suppose that $\tilde{\phi}$, a candidate for a detective for S_n , can be represented as a product of some mapping ϕ , that satisfies the assumptions of Theorem 3.3 [9] and the invariant polynomial $p(x) = \prod_{i,j=1;i < j}^n (x_i - x_j)^2$, $\tilde{\phi} = p \cdot \phi$. $\tilde{\phi}$ also satisfies the conditions of the theorem. Note that $\tilde{\phi}$ vanishes on all fixed point spaces, so it cannot be a detective for any attractor completely contained in a fixed point space.

Proof of Theorem 2.5. If the attractor A contains points with trivial symmetry then the proof in [9] is correct. For the attractor A completely contained in a fixed point space of some subgroup $\Xi \subset \Gamma$ similar ideas can be exploited.

It is sufficient to require, in addition, that for any isotropy subgroup $\Delta \subset \Gamma$,

$$\text{span}\{\phi(\text{Fix}_{\mathbb{R}^n}(\Delta))\} = \text{Fix}_W(\Delta). \quad (8)$$

Let $P : W \rightarrow \text{Fix}_W(\Sigma(A))$ be the orthogonal projection. Let us introduce subspace $S_\phi^\epsilon \subset W$ as follows: $S_\phi^\epsilon = \text{span}\{P(D\phi)_x(X(x))\}$ for $x \in A^\epsilon$ and $X \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, where $A^\epsilon = \{x : x \in \text{Fix}_W(\Xi), d(x, A) < \epsilon\}$. By (8) $\text{span}\{D(\phi)_x(X(x)), x \in A^\epsilon\} = \text{Fix}_W(\Xi) \supset \text{Fix}_W(\Sigma(A))$ because A^ϵ is an open set in $\text{Fix}_{\mathbb{R}^n}(\Xi)$ and ϕ is a polynomial mapping. This implies that $S_\phi^\epsilon = \text{Fix}_W(\Sigma(A))$ and is sufficient to complete the proof as has been done in [9]. $\square$

Now we consider an algorithm to obtain a near identity diffeomorphism which maps each point $x_i \in A^\epsilon$ to the corresponding point $a_i \in A$. Let x_i be such that the set $\{P(D\phi)_{x_i}(X(x_i))\}_{i=1,N}$ forms a basis for W , $\dim W = N$, and let $\{x_i\}_{i=1,k}$ be a basis for $\text{Fix}_W(\Xi)$. Take an open ball $B(x_i)$ centered in x_i for all $\{x_i\}_{i=1,k}$ which satisfies

- (i) $\gamma B(x_i) = B(x_i)$, $\forall \gamma \in \Xi$;
- (ii) $\gamma B(x_i) \cap B(x_j) = \emptyset$, $\forall \gamma \in \Gamma$ for $j \neq i$;
- (iii) $\gamma B(x_i) \cap B(x_i) = \emptyset$, $\forall \gamma \in \Gamma - \Xi$.

That is possible since x_i are not on the same group

orbit and have minimal isotropy. Now we take a point $a_j \in A$ in $B(x_j)$ and define a near identity diffeomorphism as the γ -equivariant extension, namely $\psi(x_i) = a_i$, $\psi(\gamma x_i) = \gamma \psi(x_i) = \gamma a_i$, ψ maps $B(x_i)$ onto an open neighborhood V of a_i , a_j and all points of V have minimal isotropy. The above setting implies that ψ can be chosen a smooth function.

Using this construction we obtain an open dense subset of a neighborhood of identity in $\text{Diff}_\Gamma(\mathbb{R}^n)$. That set is open because any mapping ψ sufficiently close to the identity should remain $\{P(D\phi)_{\psi(x_i)}(X(\psi(x_i)))\}$ to be a basis and the superposition $\phi \circ \psi$ maps x_i into a neighborhood V of a_i and it is dense because for any set $\{x_i\}_{i=1,k}$ its small perturbation is a basis for $\text{Fix}_W(\Xi)$ keeping disjoint orbits and minimal isotropy.

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