Bifurcation of periodic orbits near a frequency maximum in near-integrable driven oscillators with friction

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1. Introduction

Bifurcation of periodic orbits due to Hamiltonian perturbations of integrable systems in one degree of freedom with monotone frequency as function of the energy is well known [1]. In particular in monotone twist mappings the appearance of the so-called Poincaré-Birkhoff chains is readily obtained via an elegant geometric proof. Here we consider the effect of perturbations on nonmonotone integrable twist systems whose frequency has a maximum (or minimum) as a function of the energy. We allow for a dissipative perturbation and a time periodic Hamiltonian forcing, such that the system area is contracting with constant Jacobian. Apart from the parameter for this friction there are two other bifurcation parameters in the problem describing the variation of the frequency and of the amplitude of the driving term respectively. The main goal of this paper is to investigate analytically the bifurcation of periodic orbits near this maximum, both for differential equations and for mappings.

The resulting phenomena were observed earlier in the Hénon mapping [2]. Figure 1, taken from this reference, displays a typical sequence of phase portraits for an area preserving system, showing in particular the subsequent bifurcation of two chains of saddle center pairs. Observe that after bifurcation the homoclinic and heteroclinic connections reorder to form two standard Poincaré-Birkhoff chains. Furthermore it is important to note that the numerically determined rotation number is nonmonotone if one moves away from the central fixed point.

Such phase portraits already appeared in [3], where the emphasis was on the reconnection process. The bifurcation process was investigated in detail [4] for an explicitly given mapping, typical for the present bifurcations. There is also a similarity with the Arnold webs that arise in certain perturbed nontwist systems [5, 6].

The set up of this paper is as follows. In Section 2 the problem is formulated. Furthermore the so-called bifurcation equations and a central existence result are derived. This is done for mappings expressed in terms of a generating function, using action angle variables. The effect of the dissipation is included in this generating function formalism. The resulting class of systems is called semi-symplectic elsewhere [7]. It is shown that the same results apply for continuous systems. In Section 3 the bifurcation equations are analyzed to obtain the bifurcation diagram, i.e. the boundaries in parameter space where bifurcations occur. In the final Section 4 we consider a particle in a potential well, subject to friction and periodic forcing. It is shown how to define a suitable set of action angle variables and from there on how the generating function of the Poincaré map (on which the analysis in Section 2 is based) can be found. Thus we obtain a definite interpretation of the bifurcation parameters used in Section 2 in terms of friction, frequency and driving amplitude.

In a more general setting this problem has been studied earlier [8]. There are two reasons why we cannot simply use these results. Firstly, the authors consider a two parameter problem, while we have good reasons to have three parameters as the analysis will show. Secondly, they impose a hypothesis which is not satisfied in our case.¹ We carry out a slightly more detailed perturbation analysis. In [2] a one-parameter problem with the same degeneracy was investigated.

2. Existence of periodic orbits

The bifurcation problem is formulated for a general class of near integrable dissipative mappings expressed in terms of a generating function. The main existence result for periodic orbits is proved and the bifurcation equations are derived. It is shown that the analogous problem for differential equations leads to the same results.

¹ Actually, this is their hypothesis H6. Their results, however, can be proved avoiding this hypothesis.

Figure 1

Left row: first quadrant of phase portraits of the Henonmapping $x' = 2Cx + 2x^2 - y$, $y' = x$ for decreasing values of C (taken from [2]); C changes only slightly: from -0.306280 (upper) to -0.306299 (lower). Right row: the numerically determined rotation number ϱ of trajectories starting at the line $x = y$ for the corresponding values of C . Observe the subsequent bifurcation of two chains of saddle center orbits with period 10 and rotation number $3/10$. At the end ordinary Poincaré-Birkhoff chains have been formed. At the same time the height of the rotation curve increases from below $3/10$ to above.

2.1. Formulation of the problem, existence

Consider mappings that are expressed with a generating function $F(\phi_n, a_{n+1})$ according to

$$(1 - \delta)a_n = \partial F / \partial \phi_n, \quad \phi_{n+1} = \partial F / \partial a_{n+1}. \quad (2.1)$$

Here a and ϕ have to be considered as action angle variables, fixed action describing a circle in a plane or around a cylinder. Provided that a_{n+1} can be solved from the first relation this is a two-dimensional dynamical system, with Jacobian equal to $1 - \delta$ as one readily verifies. This way of prescribing a mapping is well-known for symplectic ones [9]. In fact it is a natural way to prescribe discrete time maps that represent Hamiltonian flows, which are generated by the action integral [10]. This formalism can easily be extended to systems with friction, as is made clear in Section 4 (Lemma 4.1).

The sufficient invertibility condition, i.e. $\partial^2 F / \partial a_{n+1} \partial \phi_n$ is definite, is certainly satisfied by the generating function

$$F(a_{n+1}, \phi_n) = \phi_n a_{n+1} + N(a_{n+1}) + \mu M(a_{n+1}) + \varepsilon G(a_{n+1}, \phi_n), \quad (2.2a)$$

provided F is smooth and ε is sufficiently small. Here N is a function whose derivative, N' , has a maximum, say at $a_{n+1} = 1$, that equals p/q with p and q relatively prime. In fact we write

$$N'(a) = p/q - Q(a) \quad \text{with} \quad Q(a) = (a - 1)^2/2 + O(|a - 1|^3). \quad (2.2b)$$

The derivative of M in (2.2a) at 1 is unequal to zero and is chosen to be positive in a region of interest,

$$M'(a) > 0 \quad \text{for} \quad 1 - x_0 \leq \sqrt{a} \leq 1 + x_0, \quad M'(1) = 1. \quad (2.2c)$$

Note that x_0 is not necessarily small. G depends periodically on ϕ with period one and has zero average. Observe that for $\varepsilon = \delta = 0$ the transformation (2.1) is integrable and that the maximum of the frequency curve shifts through p/q if μ passes through zero.

For the generating function (2.2a) the mapping (2.1) reads

$$\begin{aligned} (1 - \delta)a_n &= a_{n+1} + \varepsilon \partial G / \partial \phi_n, \\ \phi_{n+1} &= \phi_n + N'(a_{n+1}) + \mu M'(a_{n+1}) + \varepsilon \partial G / \partial a_{n+1} \end{aligned} \quad (2.3)$$

which for our purpose can be formulated more conveniently as follows. Let a bold symbol denote an element of the space of infinite rows of real numbers with n th element given by $[\mathbf{a}]_n = a_n$ etc. Then with the definition of the shift operator S and the vector $\mathbf{e}$

$$[S\mathbf{f}]_n = \text{def} [\mathbf{f}]_{n+1}, \quad [\mathbf{e}]_n = 1, \quad (2.4)$$

Figure 2

a) Representation of the phase space in Euclidean variables $q = a^{1/2} \cos \phi$, $p = a^{1/2} \sin \phi$. The region of interest is bounded by circles with radius $\sqrt{(1+x_0)}$ and $\sqrt{(1-x_0)}$ respectively. b) Sketch of the frequency $N'(a)$ of the unperturbed problem and the perturbation $M'(a)$ along a radial direction in a). If μ increases the frequency maximum shifts vertically.

(2.3) can be written as

$$(1 - S)\mathbf{a} = \delta \mathbf{a} + \varepsilon \mathbf{G}_\phi(S\mathbf{a}, \phi), \quad (2.5a)$$

$$(S - 1)\phi = \varepsilon p/q - \mathbf{Q}(S\mathbf{a}) + \varepsilon \mathbf{G}_a(S\mathbf{a}, \phi) + \mu \mathbf{M}'(S\mathbf{a}). \quad (2.5b)$$

Again bold capitals are used to indicate an element of the infinite row space according to the rule

$$[\mathbf{G}_\phi(\mathbf{a}, \boldsymbol{\beta})]_i = \text{def } G_\phi([\mathbf{a}]_i, [\boldsymbol{\beta}]_i), \quad (2.6)$$

and subscripts denote partial differentiation.

We are looking for periodic orbits with rotation number p/q , i.e. $a_{n+q} = a_n$ and $\phi_{n+q} = \phi_n + 2\pi p$, near the top of the frequency curve. Such orbits can be written as

$$\phi = \boldsymbol{\sigma} + \gamma \mathbf{e} + \boldsymbol{\psi}, \quad \mathbf{a} = \mathbf{e} + x\mathbf{e} + \mathbf{b}, \quad \text{with } [\boldsymbol{\sigma}]_n \stackrel{\text{def}}{=} np/q. \quad (2.7a)$$

Here γ and x are any real numbers restricted by $0 \leq \gamma < 1$ and $|x| \leq x_0$ respectively (cf. (2.2c)). Vectors $\mathbf{b}$ and $\boldsymbol{\psi}$ are small in some sense and satisfy

$$\psi_{n+q} = \psi_n, \quad b_{n+q} = b_n. \quad (2.7b)$$

Uniqueness of the decompositions (2.7a) is guaranteed by the requirement

$$\sum_{n=1}^q \psi_n = \sum_{n=1}^q b_n = 0. \quad (2.7c)$$

Remark. From (2.7b) it is clear that we are dealing with the period q subset of the space of infinite rows. This is a finite-dimensional space

which is isomorphic to R^q . Then using the standard Euclidean inner product, (2.7c) means that $\mathbf{b}$ and $\boldsymbol{\psi}$ are orthogonal to $\mathbf{e}$.

Remark. One readily verifies that for $\varepsilon = 0$ the system has periodic solutions only if $\delta = 0$. These periodic orbits are $\mathbf{a} = (1+x)\mathbf{e}$ and $\boldsymbol{\phi} = \boldsymbol{\sigma} + \gamma\mathbf{e}$ with x and γ free, while μ and x are related by $Q(1+x) = \mu M'(1+x)$. Because of (2.2b,c) this implies that in the δ - μ plane periodic orbits occur only on the positive μ -axis. For nonzero ε this region where periodic orbits occur unfolds with growing ε , as will be shown in Section 3.

After substitution of these expressions in (2.5) and some mathematical analysis given in the next subsection one can prove the existence result.

Theorem 2.1. Consider the range $|x| \leq x_0$ and $0 \leq \gamma < 1$. Assume sufficient differentiability of the generating function and $M'(1+x)$ definite (cf. (2.2c)). Then for $|\varepsilon|$ sufficiently small and all x and γ in this range, there exist functions $\mathbf{b}^*$, $\boldsymbol{\psi}^*$, δ^* and μ^* such that the equations (2.3) have unique p/q orbits of the form

$$\begin{aligned}\boldsymbol{\phi}(\varepsilon, x, \gamma) &= \boldsymbol{\sigma} + \gamma\mathbf{e} + \boldsymbol{\psi}^*(\varepsilon, \mu^*, \delta^*, x, \gamma), \\ \mathbf{a}(\varepsilon, x, \gamma) &= \mathbf{e} + x\mathbf{e} + \mathbf{b}^*(\varepsilon, \mu^*, \delta^*, x, \gamma)\end{aligned}\tag{2.8a}$$

$\mathbf{b}^*$ and $\boldsymbol{\psi}^*$ satisfying (2.14) and μ^* and δ^* given by (2.17) below. The solutions satisfy

$$S\boldsymbol{\phi}(\gamma) = \boldsymbol{\phi}(\gamma + p/q), \quad S\mathbf{a}(\gamma) = \mathbf{a}(\gamma + p/q).\tag{2.8b}$$

Proof. (2.8a) follows directly from the two Lemmas below. To prove (2.8b) one uses (2.7), which yields

$$S\boldsymbol{\phi} = \boldsymbol{\sigma} + (\gamma + p/q)\mathbf{e} + S\boldsymbol{\psi}, \quad S\mathbf{a} = \mathbf{e} + x\mathbf{e} + S\mathbf{b}, \quad \text{with } S\boldsymbol{\psi} \perp \mathbf{e}, \quad S\mathbf{b} \perp \mathbf{e},\tag{2.9}$$

and the fact that the equations (2.5) commute with S , which follows from the definition (2.6). The relation (2.8b) then follows from the uniqueness of the solutions (2.8a). $\square$

This Theorem states that for given ε in a range sufficiently small and for each point in the annulus in the phase space (Fig. 2), there is precisely one set of values for μ and δ so that there is a p/q orbit at that point. For $\varepsilon = 0$ these orbits are readily found: if one takes $\delta = 0$ the system is integrable and all periodic orbits are on invariant circles. For $x = 0$ p/q orbits occur if $\mu = 0$. At other x such orbits are found if one adapts μ , i.e. by shifting the frequency curve upward.

2.2. The bifurcation equations

To prove Theorem 2.1 the expressions (2.7) are substituted in (2.5) so that the r.h.s. of the latter is a function of the four variables x , γ , $\mathbf{b}$ and $\boldsymbol{\psi}$. This is indicated with a hat

$$\hat{\mathbf{G}}_\phi(x, \gamma; \mathbf{b}, \boldsymbol{\psi}) = \mathbf{G}_\phi(\mathbf{e} + x\mathbf{e} + S\mathbf{b}, \boldsymbol{\sigma} + \gamma\mathbf{e} + \boldsymbol{\psi}) \text{ etc.} \quad (2.10)$$

For our further analysis we also need

$$\mathbf{Q}(\mathbf{e} + x\mathbf{e} + S\mathbf{b}) = \mathbf{Q}(1+x)\mathbf{e} + \mathbf{Q}'(1+x)S\mathbf{b} + O(\mathbf{b}^2), \quad (2.11a)$$

$$\mathbf{M}'(\mathbf{e} + x\mathbf{e} + S\mathbf{b}) = \mathbf{M}'(1+x)\mathbf{e} + \mathbf{M}''(1+x)S\mathbf{b} + O(\mathbf{b}^2). \quad (2.11b)$$

Now observe that the nullspace and range of $1 - S$ are $\text{Span}\{\mathbf{e}\}$ and its orthogonal complement respectively. Thus the equations (2.5) can be split in two by projecting on these two subspaces

$$(1 - S)\mathbf{b} = \varepsilon P \hat{\mathbf{G}}_\phi(x, \gamma; \mathbf{b}, \boldsymbol{\psi}) + \delta \mathbf{b}, \quad (2.12)$$

$$(S - 1)\boldsymbol{\psi} = \varepsilon P \hat{\mathbf{G}}_a(x, \gamma; \mathbf{b}, \boldsymbol{\psi}) - \mathbf{Q}'(1+x)S\mathbf{b} + \mu \mathbf{M}''(1+x)S\mathbf{b} + O(\mathbf{b}^2),$$

where P denotes the orthogonal projection on the range of $1 - S$, and

$$0 = \delta(1+x) + \varepsilon(\mathbf{e}, \hat{\mathbf{G}}_\phi(x, \gamma; \mathbf{b}, \boldsymbol{\psi}))/q, \quad (2.13)$$

$$0 = -\mathbf{Q}(1+x) + \varepsilon(\mathbf{e}, \hat{\mathbf{G}}_a(x, \gamma; \mathbf{b}, \boldsymbol{\psi}))/q + \mu \mathbf{M}'(1+x) + O(\mathbf{b}^2).$$

Here $(\cdot, \cdot)$ is the standard inner product in R^q (so that $(\mathbf{e}, \mathbf{e}) = q$). The latter equations say that the component of the r.h.s. of (2.5) in the null space of $1 - S$ must be equal to zero.

We first consider (2.12).

Lemma 2.2. For small ε, μ, δ , and all allowed x and γ the equations (2.12) have unique solutions $\mathbf{b}^*$ and $\boldsymbol{\psi}^*$ near zero depending on $\varepsilon, \mu, \delta, x, \gamma$ which can be written as

$$[\mathbf{b}^*, \boldsymbol{\psi}^*](\varepsilon, \mu, \delta, x, \gamma) = \varepsilon \{[b_0^*, \psi_0^*](x, \gamma) + O(\varepsilon, \mu, \delta)\} \quad (2.14a)$$

where, the subscript r denoting the restriction to the range of $1 - S$,

$$\mathbf{b}_0^* = (1 - S)_r^{-1} P \hat{\mathbf{G}}_\phi(x, \gamma; \mathbf{0}, \mathbf{0}), \quad (2.14b)$$

$$\boldsymbol{\psi}_0^* = (S - 1)_r^{-1} [P \hat{\mathbf{G}}_a(x, \gamma; \mathbf{0}, \mathbf{0}) - \mathbf{Q}'(1+x)S\mathbf{b}_0^*].$$

Proof. Consider (2.12) as an equation on the space formed by the direct product $\text{Range}(1 - S) \otimes \text{Range}(1 - S)$. This equation has a solution $\mathbf{b} = \boldsymbol{\psi} = \mathbf{0}$ for $\varepsilon = \delta = 0$ and all x and γ . To be able to apply the implicit function theorem [11] the derivative with respect to $[\mathbf{b}, \boldsymbol{\psi}]$ must be invertible near $\varepsilon = \mu = \delta = 0$ and for some range of x and γ . At $\varepsilon = \mu = \delta = 0$ this

derivative is the restriction of

$$\begin{pmatrix} 1-S & 0 \\ Q'(1+x)S & S-1 \end{pmatrix} \quad (2.15)$$

to the direct product space mentioned above. This restriction is invertible for all x . Consequently it is invertible in a neighborhood of $\varepsilon = \mu = \delta = 0$ and for all x and γ so that the implicit function theorem proves the existence of the solution, depending regularly on the parameters. Moreover, if $\varepsilon = 0$, then $\mathbf{b} = \boldsymbol{\psi} = \mathbf{0}$ is a solution. Hence ε is a factor as in (2.14a). Substituting a series in ε for $\mathbf{b}^*$ and $\boldsymbol{\psi}^*$ in (2.12) and calculating the lowest order one obtains (2.14b). $\square$

Next consider (2.13) to obtain a relation between x , γ , ε , μ and δ . Substituting the expression (2.14) for $\mathbf{b}$ and $\boldsymbol{\psi}$ into (2.13) one obtains two equations for these five unknowns (after division by $M'(1+x)$ and $(1+x)$)

$$\begin{aligned} \delta &= -\varepsilon(\mathbf{e}, \hat{\mathbf{G}}_\phi(x, \gamma; 0, 0))/[q(1+x)] + O(\varepsilon^2), \\ \mu &= Q(1+x)/M'(1+x) - \varepsilon(\mathbf{e}, \hat{\mathbf{G}}_a(x, \gamma; 0, 0))/[qM'(1+x)] + O(\varepsilon^2). \end{aligned} \quad (2.16)$$

Observe that the r.h.s. depends on δ and μ via the ε^2 terms. Again one applies the implicit function theorem to prove

Lemma 2.3. For small ε and for all allowed x and γ the equations (2.16) have a unique solution δ^* and μ^* which have the form

$$[\delta^*, \mu^*](\varepsilon, x, \gamma) = [0, \mu_0^*(x)] + \varepsilon[\delta_1^*, \mu_1^*](x, \gamma) + O(\varepsilon^2) \quad (2.17a)$$

with

$$\begin{aligned} \mu_0^*(x) &= Q(1+x)/M'(1+x), \\ \delta_1^* &= -(\mathbf{e}, \hat{\mathbf{G}}_\phi(x, \gamma; 0, 0))/[q(1+x)], \\ \mu_1^* &= -(\mathbf{e}, \hat{\mathbf{G}}_a(x, \gamma; 0, 0))/[qM'(1+x)]. \end{aligned} \quad (2.17b)$$

Proof. We must solve $[\delta, \mu]$ as a function of the other variables. Observe that (2.16) has a solution $\delta = 0$ and $\mu = Q(1+x)/M'(1+x)$ for $\varepsilon = 0$. Considering the two equations as one relation in R^2 one readily applies the implicit function theorem. The given form of the solution readily follows directly from (2.16). $\square$

These equations (2.16) are the bifurcation equations. They are the basis for determining the bifurcation set in parameter space in Section 3.

2.3. Set up for differential equations

We now describe in main lines the bifurcation analysis for a perturbed differential equation that is similar to the mapping (2.1). We find that the results still hold, and in particular that the bifurcation equations are the same in both cases.

The differential equation we consider is

$$\dot{a} = -\lambda a - \varepsilon K_\phi(a, \phi, t), \quad \phi = N'(a) + \mu M'(a) + \varepsilon K_a(a, \phi, t). \quad (2.18)$$

Here a and ϕ have to be seen as action angle variables. N and M satisfy the condition (2.2b). The Hamiltonian perturbation K is periodic in t and ϕ with period one. This system and its resulting flow is indeed similar to the mapping (2.2). For details we refer to Section 4. Here we remark that for parameters λ, ε equal to zero this is an integrable Hamiltonian system, Hamiltonian $N(a) + \mu M(a)$. Solving his system and considering discrete time steps one finds the mapping (2.3) for ε, δ equal to zero. Furthermore the divergence of the vector field equals $-\lambda$ so that the flow reduces volume with a constant rate $\exp -\lambda t$, which is similar to the constant Jacobian property of the mapping.

We consider bifurcation of periodic orbits near $a(t) = 1$ and we write (cf. (2.7))

$$a(t) = 1 + x + b(t), \quad \phi(t) = p/q t + \gamma + \psi(t), \quad (2.19)$$

where b and ψ are periodic one C^1 functions with zero average. Substitution of these expressions in (2.18) yields the counterpart of (2.5)

$$\begin{aligned} \dot{b} &= -\lambda(1 + x + b) - \varepsilon K_\phi(1 + x + b, p/qt + \gamma + \psi, t), \\ \dot{\psi} &= -Q(1 + x + b) + \mu M'(1 + x + b) \\ &\quad + \varepsilon K_a(1 + x + b, p/qt + \gamma + \psi, t) + O(b^2). \end{aligned} \quad (2.20)$$

As in Section 2.2 observe that the null space and range of the operator ' d/dt ' are the set of constant functions and its orthogonal complement respectively. Let P denote the orthogonal projection on the latter and let the inner product be defined as usual, $(f, g) = \int fg dt$. Then (2.20) decomposes into

$$\begin{aligned} \dot{b} &= -\lambda b + \varepsilon P K_\phi(1 + x + b, p/qt + \gamma + \psi, t), \\ \dot{\psi} &= -Q'(1 + x)b + \mu M''(1 + x)b \\ &\quad + \varepsilon P K_a(1 + x + b, p/qt + \gamma + \psi, t) + O(b^2) \end{aligned} \quad (2.21)$$

and

$$\begin{aligned}
 0 &= \lambda(1+x) + \varepsilon \int dt K_\phi(1+x+b, p/qt + \gamma + \psi, t), \\
 0 &= -Q(1+x) + \mu M'(1+x) \\
 &\quad + \varepsilon \int dt K_a(1+x+b, p/qt + \gamma + \psi, t) + O(b^2).
 \end{aligned} \tag{2.22}$$

These two sets of equations have the same structure as (2.12) and (2.13) and are treated along similar lines: in particular (2.21) can be solved and after substitution of the solution in (2.22), one finds the bifurcation equations (cf. (2.16)), which are analyzed in the next Section.

3. Bifurcation of saddle node pairs near the maximum of the rotation number

Analyzing the bifurcation equations (2.16) and their solutions (2.17) we first obtain the set of phase space coordinates where bifurcations occur. Then we discuss the resulting bifurcation set in parameter space and finally we give a short description of the resulting phase portraits.

3.1. The bifurcation set in phase space

Central in the analysis are the bifurcation equations (2.16), which with the solutions (2.17) in Lemma 2.3 are written equivalently as

$$\delta = \delta^*(x, \gamma, \varepsilon), \quad \mu = \mu^*(x, \gamma, \varepsilon). \tag{3.1}$$

Now consider for a given ε a periodic orbit at certain x' and γ' . If (3.1) are invertible with respect to x and γ in a neighborhood of x' and γ' there are unique solutions x and γ as function of ε , δ and μ in the neighborhood. In other words the periodic orbit can be continued uniquely and there is no bifurcation for the corresponding parameter values. If (3.1) are not invertible we possibly have a bifurcation point.

To elaborate this condition, δ^* and μ^* are given a simpler form with the abbreviations

$$A(x) = Q(1+x)/M'(1+x) = x^2/2 + A_3x^3/3 + O(x^4), \tag{3.2a}$$

$$C(x, \gamma) = -(e, \hat{G}_\phi(x, \gamma; 0, 0))/[q(1+x)], \tag{3.2b}$$

$$D(x, \gamma) = -(e, \hat{G}_a(x, \gamma; 0, 0))/[qM'(1+x)].$$

so that δ^* and μ^* read

$$\delta^* = \varepsilon C(x, \gamma) + O(\varepsilon^2), \quad \mu^* = A(x) + \varepsilon D(x, \gamma) + O(\varepsilon^2). \tag{3.3}$$

Figure 3

Sketch of $C_0(\gamma)$ (a) and $D_0(\gamma)$ (b) in the simplest possible case, i.e. when the lowest relevant Fourier component in the generating function term G dominates (cf. Appendix A). In c) another possibility for $C_0(\gamma)$ is shown which leads to more complicated bifurcation diagrams (cf. [4]).

$C(x, \gamma)$ and $D(x, \gamma)$ are periodic in γ with period $1/q$ and their precise forms are determined by the periodic $1/q$ components (and harmonics) of the generating function G , cf. Appendix A. To determine our thoughts let $C(0, \gamma)$ and $D(0, \gamma)$ have the space as in Fig. 3.

Since we are interested in bifurcations near the top of the frequency curve, i.e. near $x = 0$, we restrict to small x , in particular $x = O(\varepsilon^{1/2})$ appears to be a good choice. Then with the scaled variable

$$y = x/\varepsilon^{1/2}, \quad \text{with } y = O(1) \quad (3.4)$$

and Taylor expansions

$$C(x, \gamma) = C_0(\gamma) + xC_1(\gamma) + O(x^2), \quad D(x, \gamma) = D_0(\gamma) + xD_1(\gamma) + O(x^2) \quad (3.5)$$

the r.h.s. in (3.3) read

$$\delta^* \varepsilon^{-1} = C_0(\gamma) + \varepsilon^{1/2} y C_1(\gamma) + O(\varepsilon), \quad (3.6a)$$

$$\mu^* \varepsilon^{-1} = y^2/2 + D_0(\gamma) + \varepsilon^{1/2} y \{D_1(\gamma) + A_3\} + O(\varepsilon). \quad (3.6b)$$

The bifurcation condition now is that the Jacobian of the r.h.s., seen as a function of γ and y , equals zero, i.e.

$$yC'_0 + \varepsilon^{1/2}\{y^2C'_1 + C'_0D_1 + C'_0A_3 - C_1D'_0\} + O(\varepsilon) = 0. \quad (3.7)$$

This is a scalar equation in 3 variables for which one can (try) to find solutions $y(\gamma, \varepsilon)$ or $\gamma(y, \varepsilon)$. These solutions represent the set in phase space where bifurcations may occur. We find these solutions in two steps. First taking into account only the zero order ε terms in the r.h.s. of (3.6, 7) one finds the 'skeleton' of the solutions. Subsequently this result is refined with the $\varepsilon^{1/2}$ terms. The $O(\varepsilon)$ terms appear not to alter the results essentially. A parametric representation of the bifurcation set in parameter space in terms of x and γ is found by substitution of one of these solutions in (3.1).

Figure 4

Analysis of bifurcation equations in lowest order in ϵ . a) Bifurcations occur for x and γ values as shown, according to (3.8); b) shows the resulting bifurcation set from (3.6) if one takes into account only the zero order terms in the rhs. The curves are parametrized with γ and y respectively. The closed curve corresponds to the solution (3.8i), x zero for all γ , whereas the horizontal lines correspond to (3.8ii).

Neglecting all ϵ terms the equation (3.7) reduces to $yC'_0(\gamma) = 0$ which has two solutions

$$\text{i) } y = 0 \quad \text{for all } \gamma, \quad \text{ii) } \gamma = \gamma_0 \text{ with } C'_0(\gamma_0) = 0, \quad \text{for all } |y| = O(1). \quad (3.8)$$

The result is depicted in Fig. 4a. The resulting bifurcation set is shown in Fig. 4b. This set is obtained after substitution of the two solutions (3.8) in (3.6), taking into account the ϵ^0 terms only. The closed curve is parametrized with γ , the horizontal line with y . They coincide at $\gamma = \gamma_0$, $y = 0$. Remark in particular that the horizontal curve is in fact twofold corresponding to positive and negative values of y respectively.

Next we investigate how these graphs change under the influence of the $\epsilon^{1/2}$ terms. Clearly the interesting point is the way in which the double line and its connection point to the closed curve change. Furthermore we concentrate on positive δ , i.e. actual damping in the system. The region of interest in the $\delta - \mu$ plane is indicated in Fig. 4b. First we look for a solution $\gamma(y, \epsilon)$ of (3.7) in the corresponding region in the $y - \gamma$ plane, cf. Fig. 4a, satisfying $\gamma(y, 0) = 0$. We restrict a priori to small $\gamma - \gamma_0$ and requiring

$$\gamma - \gamma_0 = O(\epsilon^{1/4}) \quad (3.9)$$

appears to be a good choice. To obtain $\gamma(y, \epsilon)$ we proceed as follows. Assuming for simplicity $\gamma_0 = 0$ the requirement on the shape of C and D as in Fig. 3a,b is made more specific if we write the Taylor expansion

$$\begin{aligned} C_0(\gamma) &= C_0(0) - a\gamma^2/2 - m\gamma^3/3 + O(\gamma^4), \\ D_0(\gamma) &= D_0(0) + d_1\gamma + d_2\gamma^2/2 + O(\gamma^3) \end{aligned} \quad (3.10a)$$

and require

$$a > 0, \quad d_1 > 0. \quad (3.10b)$$

Since $\gamma - \gamma_0$ is supposed to be small we consider the Taylor expansion of the l.h.s. of (3.7) around $\gamma = \gamma_0 = 0$. With the a priori assumption (3.9) and with the constants b and g given by

$$C'_1(0) = b, \quad C_1(0) = g, \quad (3.11)$$

(3.7) turns into

$$-y\gamma a(1 + m\gamma) + \varepsilon^{1/2}\{y^2b - d_1g\} + O(\varepsilon^{3/4}) = 0, \quad \text{for } \gamma = O(\varepsilon^{1/4}). \quad (3.12)$$

Dividing by $1 + m\gamma$ and again using $\gamma = O(\varepsilon^{1/4})$ one finds the expression for γ as a function of y

$$\gamma = \varepsilon^{1/2}[-d_1gy^{-1} + yb]a^{-1} + \gamma^{-1}O(\varepsilon^{3/4}). \quad (3.13a)$$

This expression is relevant as a solution to (3.7) if and only if $\gamma = O(\varepsilon^{1/4})$. Clearly this is guaranteed if we restrict to

$$y = O(\varepsilon^s), \quad \text{with } 0 \leq s \leq 1/4. \quad (3.13b)$$

To investigate the relative importance of the three terms in (3.13a) we look along lines in the y - ε^s plane given by $|y| = \varepsilon^s$, with s fixed. Doing so one readily concludes that the order term is always smaller in order than one of the two others. Furthermore these two are of equal importance only if $s = 0$. Otherwise the first term is dominant, showing that the relation is essentially $\gamma y = \varepsilon^{1/2}$ (cf. Fig. 6a).

The solution of (3.7) can be completed if we find the solution $y(\gamma, \varepsilon)$ for γ outside the restriction (3.9). Requiring a priori $y = O(\varepsilon^{1/4})$ relation (3.7) reduces to

$$yC'_0 + \varepsilon^{1/2}\{C'_0D_1 + C'_0A_3 - C_1D'_0\} + O(\varepsilon^{3/4}) = 0, \quad \text{for } y = O(\varepsilon^{1/4}). \quad (3.14a)$$

Since $C'_0(\gamma) = -a\gamma + \dots$ and assuming that $C'_0(\gamma)$ has no other zeros apart from the one at γ_0 , we obtain with γ_0 a suitable constant

$$y = -\varepsilon^{1/2}a^{-1}gd_1\gamma^{-1} + O(\varepsilon^{1/2}) \quad \text{for } \varepsilon^{1/4} \leq |\gamma| \leq \gamma_0. \quad (3.14b)$$

This expression coincides with (3.13a) in the region where both γ and y are of order $\varepsilon^{1/4}$. One of the resulting possibilities for the y - γ plane is sketched in Fig. 5.

Summarizingly, if one changes parameters δ and μ for nonzero (but small) ε , bifurcating orbits may be observed at x - γ values in the phase plane as shown in Fig. 4a, more accurately, as in Fig. 5.

Figure 5
Correction due to the $\varepsilon^{1/2}$ term in (3.7) of the solution branches in Fig. 4a that cross at γ_{01} . The point of crossing unfolds according to $\gamma y = -\varepsilon^{1/2} a^{-1} d_1 g$ (cf. (3.13a) and (3.14b)). The parameter g , here positive, must be nonzero.

3.2. The bifurcation set in the parameter plane

To obtain the bifurcation set that describes the unfolding in the boxed region in Fig. 4b we will use (3.13a) to eliminate γ from (3.1). First define for convenience

$$\delta_r^* = \delta^*/\varepsilon - C_0(0), \quad \mu_r^* = \mu^*/\varepsilon - D_0(0) \quad (3.15a)$$

and make a Taylor expansion of (3.6) around $\gamma = \gamma_{01} = 0$ to obtain, using again $\gamma = O(\varepsilon^{1/4})$,

$$\delta_r^* = -a\gamma^2/2 + \varepsilon^{1/2}y\gamma + O(\varepsilon^{3/4}), \quad (3.15b)$$

$$\mu_r^* = y^2/2 + d_1\gamma + d_2\gamma^2/2 + \varepsilon^{1/2}y[D_1(0) + A_3] + O(\varepsilon^{3/4}). \quad (3.15c)$$

From these expressions γ can be eliminated with (3.13). We obtain

$$\delta_r^* = -\varepsilon y^{-2} a^{-1} d_1^2 g^2/2 + \varepsilon^{1/2}y\gamma + O(\varepsilon^{3/4}), \quad (3.16a)$$

$$\begin{aligned} \mu_r^* &= y^2/2 + \varepsilon^{1/2}y[D_1(0) + A_3 + d_1 b a^{-1}] \\ &\quad - \varepsilon^{1/2}y^{-1} d_1^2 g/a + \varepsilon y^{-2} d_2 d_1^2 g^2 a^{-2} + O(\varepsilon^{3/4}). \end{aligned} \quad (3.16b)$$

We first investigate (3.16a). To this end again substitute $y = \pm \varepsilon^s$ and consider δ_r^* as a function of s . Equating the exponents (cf. Fig. 6b) one readily sees that the order term is always smaller than one of the two other terms. Furthermore, only if $s = 1/6$ the remaining terms are of equal importance: if $s < 1/6$ the second term is dominant, otherwise the first term. The result is shown in Fig. 7, where δ_r^* is shown as a function of s . The two branches correspond to positive and negative y respectively. We have chosen g to be positive. This is not a restriction since changing the sign of g corresponds to changing the sign of y , i.e. taking the other branch.

The second equation (3.16b) is investigated in the same way (cf. Fig. 6b). In fact only the first and the third term appear to be relevant so

Figure 6

Exponents $m_j(s)$ of ε for $|y| = \varepsilon^s$ of the different terms in (3.13a) (a) and (3.16b) (b). They are numbered according to their order of appearance. The three marked lines in (b) correspond to terms in (3.16a). The terms corresponding to the thickly drawn curves are the relevant ones (see text). The dashed line corresponds to the order term in each case.

Figure 7

Graphs of δ_r^* and μ_r^* (3.15, 16) for fixed ε as a function of $s, y = \pm \varepsilon^s$. The right and left graphs correspond to positive and negative y respectively.

that there remains

$$\mu_r^* = y^2/2 - \varepsilon^{1/2} y^{-1} d_1^2 g/a + O(\varepsilon^{1/2}). \quad (3.16c)$$

In this expression again $s = 1/6$ appears to be the turning point for dominance of the second and the first term respectively. The graph is sketched in Fig. 7 where we have chosen $g > 0$ as above.

Combining these results a parametric representation of the bifurcation set in the $\delta - \mu$ plane (Fig. 8) is easily obtained from $\delta/\varepsilon = C_0(0) + \delta_r^*$ and

Figure 8

a) Unfolding of the bifurcation set. The thick lines correspond to the expressions given in (3.18a, b) and (3.19a, b). The thinner one is from (3.18c, 19c), where $s = 1/6$.

$\mu/\varepsilon = D_0(0) + \mu_r^*$. The two branches correspond to positive and negative y respectively. Observe in particular that this behavior is found under the genericity conditions (cf. 3.10, 11))

$$g \neq 0, \quad d_1 \neq 0, \quad a \neq 0. \quad (3.17)$$

If (one of) these are not fulfilled, the second term in (3.16c) would either disappear or blow up. Note that the meaning of these conditions is easily interpreted in terms of properties of the perturbation G of the generating function, via the functions C_0 , D_0 and C_1 , cf. (3.2b), (3.5), (3.10), Fig. 3.

One readily verifies that choosing the signs of these parameters different from above, has no essential influence on the unfolding as in Fig. 8. For example if $g < 0$ the graphs of δ_r^* and μ_r^* in Fig. 7 are reflected with respect to the vertical axis. Thus the δ - μ graphs are the same as before, each corresponding to a differently signed range of y .

Explicit expressions for δ/ε as function of μ/ε can be found for $s \neq 1/6$ as follows. The argument above shows

$$\delta_r^* = \varepsilon^{1/2}yg, \quad 0 \leq s < 1/6, \quad (3.18a)$$

$$\delta_r^* = -\varepsilon y^{-2}a^{-1}d_1^2g^2/2, \quad 1/6 < s \leq 1/4. \quad (3.18b)$$

If $s = 1/6$ both terms are of equal importance and we obtain

$$\delta_r^* = \varepsilon^{2/3}[-z^{-2}a^{-1}g^2d_1^2/2 + zg] \quad \text{with} \quad z = ye^{-1/6}. \quad (3.18c)$$

For μ_r^* the dominant terms are

$$\mu_r^* = y^2/2, \quad 0 \leq s < 1/6, \quad (3.19a)$$

$$\mu_r^* = -\varepsilon^{1/2}y^{-1}d_1^2g/a, \quad 1/6 < s \leq 1/4, \quad (3.19b)$$

$$\mu_r^* = \varepsilon^{1/3}[z^2/2 - z^{-1}d_1^2g/a] \quad \text{with} \quad z = ye^{-1/6}. \quad (3.19c)$$

Explicit expressions for the relation between δ and μ are easily obtained after elimination of y from (3.17, 18a) and (3.17, 18b). In each case one finds two branches of a parabola.

Figure 9

Parameter space. Shown is the bifurcation set that results from the analysis of Section 3.2. Note that this is a 'refined' version of the one shown in Fig. 4b. Also shown are the routes as given in (3.22a). The phase portraits of Fig. 11 appear if one follows the route A–I.

Summarizingly, if one changes parameters δ and μ for nonzero (but small) ε , bifurcation of orbits may be observed at δ – μ values as shown in Fig. 4b, more accurately, as in Fig. 8.

3.3. Phase portraits

Finally we discuss qualitatively the phase portraits which are observed in a real or numerical experiment if one follows one of the two different routes in parameter plane

$$\text{i) } \delta/\varepsilon \text{ fixed, } \mu/\varepsilon \text{ increases, } \quad \text{ii) } \mu/\varepsilon \text{ fixed, } \delta/\varepsilon \text{ decreases.} \quad (3.22a)$$

These routes are shown schematically in Fig. 9, which also shows the 'refined' (Section 3.2) bifurcation set of Fig. 4b. In the latter the routes i) and ii) cross the closed curve and the horizontal line respectively. The ratio's δ/ε and μ/ε are chosen so that the paths stay away from the 'touching point' γ_{01} , i.e. we require

$$\begin{aligned} \text{i) } & \delta/\varepsilon < C_0(\gamma_{01}) - \alpha_1, \quad \text{and} \\ \text{ii) } & \mu/\varepsilon > D_0(\gamma_{01}) + \alpha_2, \quad \alpha_{1,2} \text{ positive constants.} \end{aligned} \quad (3.22b)$$

The bifurcating orbits can be found along the lines of the constructive procedure used in Section 2 to prove their existence. To obtain a general picture however, it is sufficient to determine the values of x and γ of these periodic orbits. This can in principle be done by solving x and γ as a function of the parameters from (3.1). To simplify the calculations as much as possible we calculate x and γ from (3.6) omitting the ε terms in the r.h.s., i.e. from

$$\delta/\varepsilon = C_0(\gamma), \quad \mu/\varepsilon = y^2/2 + D_0(\gamma), \quad (3.23)$$

which is sufficient to see some global aspects of the phase portraits.

In case i) of (3.22), the first equation of (3.23) yields for a $C_0(\gamma)$ as in Fig. 3 two solutions $\gamma_{1,2}(\delta/\varepsilon)$ corresponding to the γ values at the points where the route intersects the closed contour in Fig. 4b. Substitution of

these values in the second of (3.23) yields for each of the $\gamma_{1,2}$ two values of y

$$y = \pm \sqrt{2\{\mu/\varepsilon - D_0(\gamma_{1,2}(\delta/\varepsilon))\}}. \quad (3.24)$$

These are the radial positions of the saddle and the node that bifurcate for increasing μ/ε , one inside and the other outside the circle in the phase plane where the winding number has its maximum.

The expression (3.24) suggests that the bifurcation takes place at this circle, at $y = 0$. Recall however that (3.23), and the corresponding bifurcation set Fig. 4b, gives only a lowest order approximation. From (3.14) one sees that the bifurcation actually takes place at distance $y = O(\varepsilon^{1/2})$, i.e. $x = O(\varepsilon)$, to this circle. Figure 10a,b, taken from [4] shows some phase portraits for a particular mapping, for $\delta/\varepsilon = 0$.

Now consider the second case ii), i.e. descending along the vertical line in Fig. 9, or correspondingly crossing the horizontal line in Fig. 4b. Since $\gamma - \gamma_{01}$ is small here, cf. (3.13a) we use the Taylor expansion of C_0 to obtain from the first of (3.23) two values for γ as a function of δ/ε (cf. (3.10))

$$\gamma_{\pm} = \gamma_{01} \pm \sqrt{2/a\{C_0(\gamma_{01}) - \delta/\varepsilon\}}. \quad (3.25a)$$

The second of (3.23) then yields for y approximately

$$y_{\pm} = \pm \sqrt{2\{\mu/\varepsilon - D_0(\gamma_{01}) - d_1(\gamma_{\pm} - \gamma_{01})\}}. \quad (3.25b)$$

Figure 10
Phase portraits of the mapping (2.2) for increasing μ and $\delta = 0$ and $\varepsilon \neq 0$ fixed, showing subsequent bifurcation of $3/10$ orbits due to increase of μ (from [4]). The mapping is generated by (2.2) with $G = \cos(2\pi q\phi_n)a_{n+1}^2/(4\pi q)$, $M = 2\pi\mu a_{n+1}$ and $Q = 3\beta(a_{n+1} - 1)^2$. Parameter values were $\varepsilon = 0.0008$, $\delta = 0$ and μ takes values 0, 0.0003, 0.00065 and 0.001 respectively.

Recall that the first two terms under the root are fixed and taken together are positive and away from zero (cf. (3.22b)). The two expressions (3.25) describe two saddle node bifurcations, the saddle and the node separating axially according to (3.25a) at two different positions of y : one saddle node pair inside and the other one outside the circle in the phase plane where the rotation number is maximal.

In this approximation these two bifurcations take place at equal value of δ/ε , in agreement with the bifurcation diagram Fig. 4b. Recall however that including higher order terms splits the horizontal line in Fig. 4b as is shown in Fig. 9. Correspondingly the bifurcations take place at different values of δ/ε with interval depending on μ/ε , cf. Fig. 8.

Figure 11 from [4] shows bifurcation behavior of an actual mapping along a path as sketched in Fig. 9. Observe the subsequent appearing and disappearing of the periodic orbits. Note in particular the rearrangement of the homo- and heteroclinic connections between the points in D , E , F . A more detailed investigation of this reconnection process and its relation to the unfolding of the point of tangency in Fig. 4b at γ_{01} seems worthwhile.

The above analysis makes clear that if $C_0(\gamma)$ and $D_0(\gamma)$ have a shape as in Fig. 3, and if the genericity conditions on the properties of these functions near the extrema of $C_0(\gamma)$ hold true, there are exactly two saddle node bifurcations, resulting in 4 periodic orbits. The argument can be extended for functions $C_0(\gamma)$ which have an even number, say $2s$ of 'generic' extrema, $s = 1, 2, \dots$. Then there are $2s$ saddle node bifurcations, resulting in $4s$ periodic orbits. In [12] an alternate treatment, based on a Melnikov approach, is given to determine the number of periodic orbits in the special (conservative) case $\delta = \mu = 0$, $\varepsilon \neq 0$.

Figure 11

Observed saddle node pairs and their homo- and heteroclinic connections from the same system as in Fig. 10, if one follows the route as in Fig. 9.

4. The stroboscopic mapping of a dissipative Hamiltonian oscillator

In this Section we connect the foregoing analysis for mappings to a realistic physical situation: a driven and damped oscillator. An expression for the stroboscopic mapping in lowest order in the perturbation parameters is derived. In particular it is shown in which way the parameters ε , μ and δ in the mapping of Section 2 are related to the driving strength, the driving frequency and friction coefficient of the oscillator.

We investigate a perturbed oscillator in one degree of freedom, with Hamiltonian in Euclidean coordinates

$$H(p, q, t) = H_0(p, q) + \varepsilon K(p, q, \alpha t), \quad K = K_0(p, q) + K_1(p, q, \alpha t). \quad (4.1)$$

H_0 equals $1/2p^2 + V(q)$. K_1 is period one in αt , so that the parameter α is the driving frequency. If friction is added, the equations of motion are

$$\dot{p} = -\partial H/\partial q - \lambda p, \quad \dot{q} = \partial H/\partial p, \quad (4.2)$$

where λ is the friction coefficient. Typically the three parameters, ε , α and λ which represent driving strength, frequency of driving and friction are the ones that are available in real experiments. Being time independent, the unperturbed system H_0 is integrable, suppose with action angle variables a and ϕ and new Hamiltonian N

$$p = p(a, \phi), \quad q = q(a, \phi) \quad \text{with} \quad N(a) = H_0(p(a, \phi), q(a, \phi)). \quad (4.3)$$

Suppose that the frequency $N'(a)$ has a maximum, at $a = 1$ say, (as in Fig. 2b). Such a situation is easily realized (cf. Fig. 12 and Appendix B).

4.1. Equations of motion in action angle variables

The equations (4.2) will be formulated in terms of a set of action angle variables which is suitable to express the stroboscopic mapping with a

Figure 12

a) Potential $V(q) = q^2/2 + q^3/3 + cq^4/4$ for $c = c_0 = 1/4$. In that case a particle moving in this potential has an unstable equilibrium at $q = -2$ with $V = 1/3$. b) Sketch of frequency versus energy for a particle moving in this potential (no friction). If c is chosen slightly above c_0 the equilibrium at $1/3$ disappears and correspondingly (by continuity) the minimum in the ω - E curve is lifted up a little and turns into a smooth minimum.

generating function as used in Section 2. To do this let M denote the Jacobian matrix of the transformation (4.3), with its inverse M^{-1}

$$M = \begin{pmatrix} p_a & p_\phi \\ q_a & q_\phi \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} q_\phi & -p_\phi \\ -q_a & p_a \end{pmatrix}, \quad p_a q_\phi - p_\phi q_a = 1. \quad (4.4)$$

Then one finds straightforwardly that (4.2) is equivalent to

$$\dot{a} = -\partial G / \partial \phi - \lambda p q_\phi, \quad \dot{\phi} = \partial G / \partial a + \lambda p q_a \quad (4.5)$$

with the new Hamiltonian G given in the usual way, $G(a, \phi, t) = H(p(a, \phi), q(a, \phi), t)$. The dissipative terms can be rewritten as follows: There exists a function $F(a, \phi)$ such that

$$\partial F / \partial \phi = p q_\phi - a, \quad \partial F / \partial a = p q_a. \quad (4.6)$$

This follows from standard analysis: differentiating the right hand sides with respect to a and ϕ respectively one finds that their difference equals zero so that F can be solved. Here one uses the fact that the Poisson bracket $p_a q_\phi - p_\phi q_a = 1$, a consequence of the transformation (4.3) being symplectic. The result is that (4.6) can be rewritten as

$$\dot{a} = -\partial G / \partial \phi - \lambda \partial F / \partial \phi - \lambda a, \quad \dot{\phi} = \partial G / \partial a + \lambda \partial F / \partial a. \quad (4.7)$$

This equation has the same structure as (4.2): it is Hamiltonian, with Hamiltonian $G + \lambda F$, with a dissipation term linear in the momenta. In deriving the latter we used only the fact that (4.3) is symplectic. Consequently any such transformation transforms an equation as (4.2) to a similar one with a new Hamiltonian $H + \lambda F$, for some F . This F is intimately related to the generating function of the transformation and the conservation of structure is a direct consequence of the fact that solutions of (4.2) represent critical points of the action functional $\int e^{\lambda t} (p\dot{q} - H) dt$ (see below). Henceforth we will call an equation of the form (4.2), i.e. a Hamiltonian system with linear friction, a **dissipative Hamiltonian system** $[H, \lambda]$, where we always assume $\lambda > 0$.

To continue now observe that the Hamiltonian of (4.7) reads

$$N(a) + \lambda F(a, \phi) + \varepsilon K_0(q(a, \phi), p(a, \phi)) + \varepsilon K_1(q(a, \phi), p(a, \phi), \alpha t). \quad (4.8)$$

Since we are dealing with a one degree of freedom system, the time independent part itself is integrable and obviously we would prefer to have action angle variables such that the integrable part depends on the action only. This can be achieved easily. For small λ and ε we can find following standard methods [10] a near identity transformation to action angle variables $[b, \psi]$ which depends smoothly on the parameters and which satisfies

$$b = a + O(\varepsilon, \lambda), \quad \psi = \phi + O(\varepsilon, \lambda), \quad (4.9)$$

which transforms the time independent part of (4.8) to

$$N(b) + \lambda F_{av}(b) + \varepsilon K_{0av}(b) + O(\lambda^2, \varepsilon\lambda, \varepsilon^2) \quad (4.10)$$

where the subscript ‘*av*’ denotes averaging over the angle. With (4.9) and (4.6) one sees that the resulting F for this near identity transformation is of order one in the small parameters. Consequently its contribution to the equations of motion is of second order. Furthermore for Hamiltonians H_0 which are even in the Euclidean momenta $F_{av}(b)$ appears to be equal to zero. Thus we can formulate our results so far in

Theorem 4.1. For a dissipative Hamiltonian system (4.1, 2, 3) with Hamiltonian even in the momentum, there is a set of action angle variables $[b, \psi]$ that transforms the system to another dissipative Hamiltonian system $[H, \lambda]$ with H given by

$$H(b, \psi, t) = N(b) + \varepsilon K_{0av}(b) + \varepsilon K_1(q(b, \psi), p(b, \psi)), \alpha t) + O(\lambda^2, \varepsilon\lambda, \varepsilon^2). \quad (4.11)$$

Proof. It only remains to show that $F_{av}(b) = 0$. Consider a Hamiltonian which is even in p , e.g. $H_0 = p^2/2 + V(q)$. Then its level lines are symmetric with respect to reflection to the q axis. Choosing ϕ zero at $p = 0$, clearly p is an odd function of ϕ and q is even. Consequently pq_a is odd and its average is zero. So the average of $\partial F/\partial a$ is zero, cf. (4.6), which means that the average of F is a constant, which always can be chosen to be zero. $\square$

Remark. Since the Hamiltonian in (4.8) depends smoothly on ε and λ , the same holds for the transformation (4.9) and the resulting Hamiltonian (4.11). The resulting equations of motion then are, omitting square order terms in the parameters

$$\dot{b} = -\varepsilon \partial K_1/\partial \psi - \lambda b, \quad \dot{\psi} = N'(b) + \varepsilon K'_{0av}(b) + \varepsilon \partial K_1/\partial b. \quad (4.12)$$

4.2. The stroboscopic mapping

To prepare for the calculation of the stroboscopic mapping, we choose a time scale suitable for our purposes. Since we will investigate the bifurcation of p/q orbits near the maximum $v_{\max}$ of the frequency $N'(b)$ (at $b = 1$) the driving frequency will be varied around a v_c such that $N'(1)/v_c = p/q$. Equivalently we will assume that the unit of time is chosen in such a way that $N'(1) = p/q$ and the driving frequency will be varied near unity. The latter effect is easily brought into the equations by writing $\alpha = 1/(1 + \beta)$

with β small and introducing the scaled time τ

$$\tau = t/(1 + \beta). \quad (4.13)$$

Substitution in (4.12) gives trivially, with b and ψ considered as functions of τ and omitting again terms of square order in the small parameters,

$$\begin{aligned} \dot{b} &= -\varepsilon \partial K_1 / \partial \psi - \lambda b, \\ \dot{\psi} &= N'(b) + \beta N'(b) + \varepsilon K'_{0av}(b) + \varepsilon \partial K_1(b, \psi, \tau) / \partial b, \end{aligned} \quad (4.14a)$$

which is a dissipative Hamiltonian system $[H(b, \psi, \tau), \lambda]$ with Hamiltonian

$$H(b, \psi, \tau) = N(b) + \beta N(b) + \varepsilon K_{0av}(b) + \varepsilon K_1(b, \psi, \tau). \quad (4.14b)$$

The basis for the construction of the generating function of the stroboscopic mapping is provided by the following Lemma, which is an extension of well known properties of Hamiltonian systems [10].

Lemma 4.1. i) Trajectories of a dissipative Hamiltonian system (4.2) that connect $q_0 = q(t_0)$ with $q_1 = q(t_1)$ are critical points of the action functional

$$\int_{t_0}^{t_1} e^{\lambda t} (p \dot{q} - H) dt. \quad (4.15)$$

ii) Let $[q(t), p(t)]$ denote an actual trajectory. Suppose that q_1 can be written as a function of $p_1 (= p(t_1))$ and q_0 , i.e. $q_1 = \tilde{q}_1(p_1, q_0)$. Then the discrete flow mapping from t_0 to t_1 is generated by

$$S(p_1, q_0) = -e^{-\lambda} \int_{t_0}^{t_1} e^{\lambda t} (p \dot{q} - H) dt + p_1 \tilde{q}_1(p_1, q_0), \quad (4.16a)$$

where the integral is taken along the actual trajectory, and given by the rules

$$p_0 e^{-\lambda} = \partial S / \partial q_0, \quad q_1 = \partial S / \partial p_1. \quad (4.16b)$$

Proof. i) This follows in exactly the same way as modified Hamilton's principle for Hamiltonian flows with the variation conditions $\delta q(t_0) = \delta q(t_1) = 0$ and δp free [10].

ii) Consider two nearby trajectories $[q(t), p(t)]$ and $[q(t) + \delta q(t), p(t) + \delta p(t)]$. Then the difference in action S between the two equals in first order

$$\delta S = -e^{-\lambda} \int e^{\lambda t} (\dot{q} \delta p + p \delta \dot{q} - \delta H) dt + \delta p_1 \tilde{q}_1(p_1, q_0) + p_1 \delta \tilde{q}_1(p_1, q_0). \quad (4.17)$$

Inserting the identity $e^{\lambda t} p \delta \dot{q} = d/dt(e^{\lambda t} p \delta q) - \delta q(\lambda e^{\lambda t} p + e^{\lambda t} \dot{p})$ in the integrand and using the fact that $q(t)$ and $p(t)$ satisfy the equations of motion,

one obtains $\delta S = e^{-\lambda} p_0 \delta q_0 + q_1 \delta p_1$ from which the relations (4.16b) for the partial derivatives of S follow. $\square$

Obviously to calculate the generating function one must know the orbits so that this Lemma is of not much practical use in finding the stroboscopic mapping in general. But it justifies the search for a generating function, which we presently obtain from the equations of motion directly. We find only the lowest order terms, which serves our purpose to relate the parameters in the foregoing Sections to those of a driven oscillator.

Theorem 4.2. The generating function for the time one map of the dissipative Hamiltonian system (4.1) reads up to first order in the small parameter $\varepsilon, \lambda, \beta$

$$S(b_1, \psi_0) = b_1 \psi_0 + N(b_1) + \lambda[b_1 N'(b_1) - N(b_1)]/2 + \beta N(b_1) + \varepsilon K_{0av}(b_1) + \varepsilon G(b_1, \psi_0) \quad (4.18a)$$

with G given by

$$G(b_1, \psi_0) = \int_0^1 K_1(b_1, \psi_0 + N'(b_1)\tau, \tau) d\tau. \quad (4.18b)$$

Proof. From the equations (4.14a) we obtain by integration

$$b_1 = b(\tau) - \varepsilon \int_\tau^1 K_{1\psi}(b, \psi, t) dt - \lambda \int_\tau^1 b dt, \quad (4.19a)$$

$$\psi(\tau) = \psi_0 + (1 + \beta) \int_0^\tau N'(b(t)) dt + \varepsilon \int_0^\tau K_{1b}(b, \psi, t) dt. \quad (4.19b)$$

For compactness we use subscripts b and ψ to denote partial differentiation. From these expressions we obtain

$$b_1 = b(\tau) + O(\varepsilon, \lambda), \quad \psi(\tau) = \psi_0 + \tau N'(b_1) + O(\varepsilon, \lambda, \beta). \quad (4.20)$$

After substitution of $\tau = 0$ and the first of (4.20), (4.19a) yields

$$b_0 = b_1 + \varepsilon \int_0^1 K_{1\psi}(b_1, \psi_0 + \tau N'(b_1), \tau) dt + \lambda b_0 \quad (4.21a)$$

which we write as

$$(1 - \lambda)b_0 = \varepsilon \partial/\partial \psi_0 \int_0^1 K_1(b_1, \psi_0 + \tau N'(b_1), \tau) dt. \quad (4.21b)$$

This equation is the same as the first of (4.16b), taking into account that λ is small. Now turn to (4.19b) which for $\tau = 1$ up to square order terms

reads

$$\psi_1 = \psi_0 + \int_0^1 N'(b(t)) dt + \beta N'(b_1) + \varepsilon \int_0^1 K_{1b}(b_1, \psi_0 + N'(b_1)t, t) dt. \quad (4.22)$$

Take Taylor expansion of the second term with respect to the small parameters

$$\begin{aligned} \int_0^1 N'(b(t)) dt &= N'(b_1) + \lambda N''(b_1) \int_0^1 d\tau \int_\tau^1 b_1 dt \\ &\quad + \varepsilon N''(b_1) \int_0^1 d\tau \int_\tau^1 s(t) dt \end{aligned}$$

with

$$s(t) = K_{1\psi}(b_1, \psi_0 + N'(b_1)t, t). \quad (4.23)$$

The middle term is simplified to

$$\lambda N''(b_1) \int_0^1 d\tau \int_\tau^1 b_1 dt = \lambda/2 \partial/\partial b_1 [N'(b_1)b_1 - N(b_1)]. \quad (4.24)$$

The last term in (4.23) is simplified with the identity

$$\int_0^1 d\tau \int_\tau^1 s(t) dt = \int_0^1 \tau s(\tau) d\tau \quad (4.25)$$

which one verifies by partial integration of each side. The last term in (4.23) then reads

$$\varepsilon N''(b_1) \int_0^1 d\tau \tau K_{1\psi}(b_1, \psi_0 + N'(b_1)\tau, \tau). \quad (4.26)$$

This term combines with the last in (4.22) to

$$\varepsilon \partial/\partial b_1 \int_0^1 K_1(b_1, \psi_0 + \tau N'(b_1), \tau) dt \quad (4.27)$$

which shows, together with (4.24), that (4.22) has the form of the second of (4.16b). $\square$

Remark. Since the one critical point of the unperturbed system is nondegenerate, the time one flow of the perturbed system depends smoothly on the small parameters. This justifies the series expansion of the stroboscopic mapping with respect to these parameters. Note, however, that this generating function is derived from expressions that are valid at time scale $\tau = O(1)$ only. Consequently it can be used for investigation of periodic orbits, with a given period. It does not give a good approximation for infinite time behavior.

Conclusion

Using this result we are able to interpret the generating function (2.2a) in real physical terms. Clearly the terms G in each case correspond to one another, ε being the amplitude of the driving. Furthermore δ is related to the friction by $1 - \delta = \exp(-\lambda)$, i.e. $\delta \approx \lambda$. Finally the effect of the term $\mu M(a_{n+1})$ corresponds to three terms

$$\text{i) } \lambda[b_1 N'(b_1) - N(b_1)]/2 \quad \text{ii) } \beta N(b_1), \quad \text{iii) } \varepsilon K_{0av}(b_1). \quad (4.25)$$

Thus the shift of the frequency curve represented by the term μM in (2.2a) is caused by the combined effects of friction, by the actual shift of driving frequency in the experiment and by the integrable part of the perturbation.

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Appendix A

Relevant Fourier components

To investigate the relation between C and D in (3.2b) and the ε term G in the generating function F (2.2), write G as

$$G(a_{n+1}, \phi_n) = \sum_{m \neq 0} g_m(a_{n+1}) \exp(2\pi i m \phi_n). \quad (\text{A.1})$$

Then one readily sees, cf. (2.6, 7, 10)

$$q^{-1}(e, \hat{G}(x, \gamma; 0, 0)) = \sum_{m \neq 0} g_m(1+x) c_m \exp(2\pi i m \gamma), \quad (\text{A.2a})$$

with c_m defined by

$$c_m = q^{-1} \sum_{j=0}^{q-1} \exp(2\pi i m j p / q). \quad (\text{A.2b})$$

Now c_m equals zero unless mjp/q is an integer for $q = 0, 1, \dots, q^{-1}$, which is satisfied only if m contains a factor q . Thus one readily sees

$$c_m = 1 \quad \text{if } m = kq \text{ for some integer } k, \quad c_m = 0 \text{ otherwise.} \quad (\text{A.3})$$

In conclusion $(e, \hat{G}(x, \gamma; 0, 0))$ equals precisely the sum of the sum of the kq -Fourier components of G , and consequently its period is $1/q$.

To consider a typical case in more detail, suppose that the q component in $(e, \hat{G}(x, \gamma; 0, 0))$ and its derivative with respect to γ dominates. Let the phase be such that this dominating component is a cosine, then C and D have approximately the shape as shown in Fig. 3 (the detailed investigation in [Van der Weele 1990] uses this special case: $G(a_{n+1}, \phi_n) = a_{n+1} \cos(q2\pi\phi_n)$).

Appendix B

A potential well with nonmonotone twist

An example with nonmonotone twist is readily found. Consider a particle with unit mass in a potential well

$$V(q) = q^2/2 + aq^3/3 + bq^4/4 \quad (\text{B.1})$$

with a and b both positive. Via such a potential an ω - E relation with an extremum can easily be realized, as is demonstrated in Fig. 11. An analytical argument runs as follows.

A periodic orbit can be written as $q(\phi)$ with $\phi = \omega t$. A simple bifurcation argument tells that q and ω can be written as

$$\begin{aligned} q(\phi) &= R \cos(\phi) + R^2 g_2(\phi) + \cdots \quad \text{with} \quad g_i(\phi) \perp \cos(\phi), \\ \omega &= 1 + \omega_1 R + \omega_2 R^2 + \cdots \end{aligned} \quad (\text{B.2})$$

A uniqueness argument shows that the frequency must be even in R . The same holds for the energy expressed as a function of R . Furthermore, for small R , the energy necessarily increases with R . Substitution of these expressions in the Lagrange equations of motion and equating the terms with respect to powers of R one finds $\omega_1 = 0$ indeed and further

$$\omega_2 = -2a^2/3 + 3b/4, \quad g_1(\phi) = a(\cos 2\phi - 3)/6. \quad (\text{B.3})$$

This expression shows that the cubic term in the potential causes that ω decreases with R^2 , whereas the quartic term causes an increase, provided $b > 0$. Consequently if a is large enough ω will decrease for increasing small R . It shows the same behavior with respect to E , since the latter is an increasing function of R^2 . On the other hand, for larger energy values, the quartic term in the potential dominates causing an increasing ω as a function of E .

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Abstract

We investigate analytically the effect of perturbations on an integrable oscillator in one degree of freedom whose frequency shows a maximum as a function of the energy, i.e. a system with nonmonotone twist. The perturbation depends on three parameters: one parameter describes friction such that the Jacobian is constant and less than one. A second and a third describe the variation of the frequency and of the strength of the driving force respectively. The main result is the appearance of two chains of saddle node pairs in the phase portrait. This contrasts with the bifurcation of one chain of periodic orbits in the case of perturbations of monotone twist systems. This result is obtained for a mapping, but it is demonstrated that the same formalism and results apply for time continuous systems as well. In particular we derive an explicit expression for the stroboscopic mapping of a particle in a potential well, driven by a periodic force and under influence of friction, thus giving a clear physical interpretation to the bifurcation parameters in the mapping.

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