Tidal-induced large-scale regular bed form patterns in a three-dimensional shallow water model

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Abstract. The three-dimensional model presented in this paper is used to study how tidal currents form wave-like bottom patterns. Inclusion of vertical flow structure turns out to be necessary to describe the formation, or absence, of all known large-scale regular bottom features. The tide and topography are treated as a coupled system, and the potential free instabilities in this system are investigated. A linear stability analysis is used to study the dynamics of bed forms. This yields information on the role of physical mechanisms; such information cannot be obtained by direct numerical nonlinear simulation. The basic state represents a horizontal elliptical flow, veering in vertical direction, over a flat bottom. It is shown that in such a system, modes can grow which have characteristics of both tidal sand banks and sand waves. The model predictions are compared with observations and show good agreement.

1. Introduction

The sandy sea bottom of many coastal seas shows a variety of regular patterns. These patterns have different spatial dimensions and are formed over different timescales which suggest that they are caused by different physical processes. Usually, they do not occur as isolated features, but they occur in patches, elongated in one horizontal spatial direction on a scale which is much longer than the distances between the crests of the sand bars. The largest-scale phenomena are tidal sand banks, sand waves, shoreface-connected ridges, and storm-generated ridges. Shoreface-connected ridges appear in seas with and without large tidal influence [see *Van de Meene*, 1994], whereas storm-generated ridges are present only in seas where the tidal currents are weak [see *Boczar-Karakiewicz and Bona*, 1986]. Tidal sand banks and sand waves, which are the subject of this paper, are only observed in seas with strong tidal currents.

Tidal sand banks typically have wavelengths (distance between two crests) of the order of 5–10 km and can be up to 30 m high, which is more than half of local water depth. They hardly move and their crests are rotated 10°–30° anticlockwise (in the northern hemisphere) with respect to the principal tidal current direction. The dimensions of sand waves are smaller than those of sand banks, a typical wavelength ranging from 100 to 800 m. The maximum height of sand waves is of the order of 5 m; they can be dynamically active and can show displacement rates of up to 0.5 m per year. Sand wave crests are perpendicular to the mean tidal current. Tidal sand banks and sand wave fields can sometimes overlap, even with heights of the same order [see *Huntley et al.*, 1993].

By considering the sandy sea bottom and the seawater as a dynamically coupled system, one can study whether regular

features are associated with free instabilities of this system. *Huthnance* [1982] was the first to do a study of this kind to describe tidal sand banks. He described the seawater motion using the depth-integrated shallow water equations, supplemented by conservation of sediment and a bed load transport parametrization. He found that the fastest growing mode had a wavelength and orientation comparable to those of observed tidal sand banks. This model was studied and extended in several ways by *De Vriend* [1990], *Hulscher et al.* [1993], and *De Swart and Hulscher* [1995]. The main shortcoming of the model based on the depth-integrated shallow water equations is that it only predicts a mode corresponding to sand banks; it does not predict sand waves.

Two models have been postulated to study the behavior of sand waves in a tidal environment. A model for describing the properties of already existing sand waves is discussed by *Fredsoe and Deigaard* [1993], which is a synthesis of earlier work by these authors. The study is based on their dune model in a river environment. The fluid motion is a steady current which may incorporate the effects of wind waves. The dimensions and dynamical properties of sand waves predicted by this model agree well with observations. A model for the formation of sand wave by oscillating tidal currents is proposed by *Hulscher et al.* [1993]. They consider a depth-averaged model with a modified bottom stress description which accounts parametrically for circulations in the vertical plane. This approach has recently been modified and extended by *De Swart and Hulscher* [1995].

In the present paper a three-dimensional model is described which can be used to calculate the bottom shear stress explicitly and to study vertical flow effects. It is shown that both sand banks and sand waves are generated by tidal currents. This connection between tidal currents and sand waves, strongly suggested by *Belderson* [1986], has not been explicitly demonstrated before [*Fredsoe and Deigaard*, 1993]. Furthermore, this three-dimensional model is able to distinguish between local conditions that lead to the formation of tidal sand banks, sand waves, or a combination of both or that determine a flat bottom. Such a model is needed for understanding of nonlinear amplitude behavior of tidal sand banks and sand waves, since

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Figure 1. Situation sketch (side view).

observations suggest that the two patterns can interact significantly.

The incorporation of vertical effects adds other mechanisms leading to formation of regular bottom patterns, besides the known depth-averaged flow mechanism leading to the growth of tidal sand banks. Frictionally induced shear stresses near an undulating bottom will cause residual circulations in the vertical plane, similar to residual circulations caused by headland eddies as described by Zimmerman [1981]. Tilting of vortex tubes might result in additional physical phenomena, as suggested by Zimmerman [1986]. These considerations have led to the development of a model in which the water motions are described in three spatial dimensions.

The paper is organized as follows. Section 2 introduces a model based on the three-dimensional shallow water equations because in a coastal sea the tidal wave is long compared to the depth. The changes in bottom location follow from conservation of sediment; interaction between seawater and topography is empirically modeled by a bed load transport formula. Scaling the equations yields two timescales, and averaging over the tidal period is used to derive the topographical changes on their own timescale. In section 3 the linear stability concept is presented and possible basic states with a flat bottom are described. These basic velocities show a veering of the tide due to earth rotational effects, known as Ekman spiraling. The dynamics of small perturbations evolving on the flat bottom is analyzed in section 4. This yields a set of differential equations with variable coefficients in the vertical coordinate. The main results of the linear stability problem are discussed in section 5. Initial bottom responses are found to vary according to the Stokes number of the tidal motion and resistance characteristics imposed by the bottom sediment. The basic configurations are a flat bottom, tidal sand banks, sand waves, and a mode that has crests nearly parallel to the principal tidal axis. Subsequently, in section 6 the physical mechanisms that supply a positive feedback to the different modes are identified and discussed. The validity of the results is examined in the final section, and conclusions of the work in the paper are presented.

2. Model

The here considered model consists of two coupled parts, a flow model and a part describing the sediment dynamics. Water motions driven by tides can be described with the three-dimensional shallow water equations, which are

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - f v = -g \frac{\partial \zeta}{\partial x} + \frac{\partial}{\partial z} \left(A_v \frac{\partial u}{\partial z} \right), \quad (1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + f u = -g \frac{\partial \zeta}{\partial y} + \frac{\partial}{\partial z} \left(A_v \frac{\partial v}{\partial z} \right), \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (3)$$

Here u , v , and w are the velocity components in the x , y , and z directions; $z = \zeta$ is the free surface elevation; and h is the bottom level with respect to the undisturbed water depth H ; see the situation sketch in Figure 1. Furthermore, f and g are the Coriolis parameter and the acceleration of gravity, respectively. The vertical viscosity is denoted by A_v ; it measures how the fluid resists to changes in horizontal velocity in vertical direction. A derivation of this parametrization is given by Pedlosky [1987]. The parameter A_v is a function of time and all spatial directions, but these dependencies are not explicitly known. The simplest possible approximation is adopted, see, e.g., Pedlosky [1987], namely A_v , a constant. This choice has the advantage that it provides the simplest possible mathematical description which enables to study the transfer of horizontal momentum in the vertical direction. Although the constant A_v may be a very crude assumption, Ekman dynamics has proved to be able to describe satisfactory the dynamics of the major part of the water column [Soulsby, 1990].

The boundaries in the horizontal directions are assumed to be infinitely far away; the boundary conditions in the vertical direction are as follows. In the situation studied the wind stress at the surface equals zero, so at the sea surface the conditions are

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0, \quad w = \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y}. \quad (4)$$

At the bottom [$z = -H + h(x, y, t)$] a partial slip condition is used to describe the horizontal flow components. The vertical velocity component at the bottom is given by the kinematic condition

$$A_v \frac{\partial u}{\partial z} = S u, \quad A_v \frac{\partial v}{\partial z} = S v, \quad w = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y}. \quad (5)$$

The x axis is chosen in the direction of the depth-averaged current. With the help of the partial slip condition, one can model the shear stress at the bottom without including explicitly the complicated processes in the thin bottom boundary layer. In this layer the shear stress is approximately constant, and the flow adjusts to the no-slip condition at the roughness height above the seabed. To describe these processes near the bed, the constant eddy viscosity is a too simple approximation; however, the resistance parameter S can be used still to describe the bottom shear stress at the seabed properly. So beneath the model, presented in this paper, a bottom boundary layer is present and the parameter S fixes the relation between the horizontal velocity and the shear stress at the top of this bottom boundary layer. As the shear stress is approximately constant in the layer beneath, this means that the shear stress given by the partial slip condition is indeed the shear stress at the sea bottom. From now on it is assumed that the bottom (at $z = -H + h$), which is in fact the top of this bottom boundary layer, can be interpreted as the physical sea bottom. Finally, it should be noted that the partial slip boundary condition as chosen

here does not consider the generation of residual components by nonlinear flow interactions in the bottom boundary layer.

In theory a partial slip boundary condition can be used to study all situations between perfect slip, no resistance ($S = 0$), and no slip, infinite resistance ($S \rightarrow \infty$) at the bottom [see *Maas and van Haren*, 1987; *Visser et al.*, 1994]. In reality its possible values will vary in a smaller range, and the resistance parameter is related to for instance the bed roughness and it is sometimes identified with the friction velocity used in depth-averaged shallow water models.

The sediment transport model includes only bed load transport, which is assumed to be dominant in tidal offshore regimes. Since the flow velocities in the vertical direction are being calculated explicitly, the bed load transport can be described directly as a function of the bottom shear stress. This transport formulation differs from the transport modeling in depth-averaged flow models in which the transport is parametrized as a function of the depth-averaged velocities as, for instance, given by *Huthnance* [1982] and *Hulscher et al.* [1993]. A general bed load transport formula is chosen:

$$\mathbf{S}_b = \alpha |\boldsymbol{\tau}|^b \left[\frac{\boldsymbol{\tau}}{|\boldsymbol{\tau}|} - \lambda \nabla h \right], \quad (6)$$

in which $\boldsymbol{\tau}$ denotes the bottom shear stress

$$\boldsymbol{\tau} = A_v \left(\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z} \right) \Big|_{z=\text{bottom}}. \quad (7)$$

Formally, $\boldsymbol{\tau}$ should be multiplied with the density of the seawater to obtain the physical bottom shear stress, but in this text the phrase “bottom shear stress” will be used to denote $\boldsymbol{\tau}$. Furthermore, $\mathbf{S}_b$ is the volumetric sediment transport vector and α a bed load transport proportionality parameter. The exponent b expresses the nonlinearity of transport in relation to the bed shear stress. The sediment transport vector contains a bed-slope correction term which is weighted by λ . In this formula a critical shear stress is not incorporated explicitly. This choice is based on results of *De Swart and Hulscher* [1995], who showed that inclusion of it changes results only quantitatively, as it is common in coastal seas that during the major part of the tidal period the bed shear stress exceeds the critical shear stress. Many transport parametrizations are of the type (7); see the review by *Van Rijn* [1993]. The sediment balance (without suspended sediment)

$$\frac{\partial h}{\partial t} + \nabla \cdot \mathbf{S}_b = 0, \quad (8)$$

couple the flow model and the sediment-transport model.

The equations of motion will be made dimensionless with relevant scales in order to enable study on the behavior of large-scale bed forms, as described by this shallow water model. The tide is assumed to be the main forcing mechanism of these morphological features; this gives a timescale σ^{-1} for the flow variations, in which σ is the tidal frequency. The bottom is assumed to vary on a much larger timescale which is denoted as T_{long} , a morphologically relevant timescale. There are also two length scales involved in this model; one (L) is the length over which the tidal wave varies; the other (l_m) is the scale on which the bottom varies. The horizontal velocities are of the order of the depth-averaged velocities, which are scaled by the tidal current amplitude U . The vertical velocity is much smaller, and an appropriate scaling is derived from the continuity equation (3). Observations show that l_m is of order of the tidal excursion ($U\sigma^{-1}$) or smaller. The previous considerations motivate the introduction of the following nondimensional variables (denoted by $\check{}$):

$$(u, v, w) = \left(U\check{u}, U\check{v}, U\frac{H}{l_m}\check{w} \right), \quad (9)$$

$$(x, y, z) = (l_m\check{x}, l_m\check{y}, H\check{z})$$

$$\boldsymbol{\tau} = UH\sigma\check{\boldsymbol{\tau}}, \quad t = \frac{\check{t}}{\sigma}, \quad h = H\check{h}, \quad \zeta = \frac{UL\sigma}{g}\check{\zeta}. \quad (10)$$

Furthermore, the dispersion relation for a shallow water wave

$$L\sigma = 2\pi\sqrt{gH}, \quad (11)$$

is being used. The nondimensional model is as follows (for convenience $\check{}$ is dropped):

$$\delta \left\{ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - \hat{f}v - \frac{E_v}{2} \frac{\partial^2 u}{\partial z^2} \right\} = - \frac{\partial \zeta}{\partial x}, \quad (12)$$

$$\delta \left\{ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \hat{f}u - \frac{E_v}{2} \frac{\partial^2 v}{\partial z^2} \right\} = - \frac{\partial \zeta}{\partial y}, \quad (13)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (14)$$

$$\frac{\partial h}{\partial t} = -\hat{\alpha} \nabla \cdot \left[|\boldsymbol{\tau}|^b \left(\frac{\boldsymbol{\tau}}{|\boldsymbol{\tau}|} - \hat{\lambda} \nabla h \right) \right], \quad (15)$$

$$\boldsymbol{\tau} = \frac{E_v}{2} \left(\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z} \right) \Big|_{z=-1+h}. \quad (16)$$

The boundary conditions at the free surface ($z = \delta\zeta$) become

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0, \quad w = \delta \left\{ \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} \right\}, \quad (17)$$

and at the bottom ($z = -1 + h$) the nondimensional conditions are

$$\frac{E_v}{2} \frac{\partial u}{\partial z} = \frac{\hat{S}}{2} u, \quad \frac{E_v}{2} \frac{\partial v}{\partial z} = \frac{\hat{S}}{2} v, \quad w = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y}. \quad (18)$$

In these equations and boundary conditions the following parameters are observed:

$$\delta = \frac{U}{\sigma L} = \frac{l_m}{L}, \quad \hat{f} = \frac{f}{\sigma}, \quad E_v = \frac{2A_v}{\sigma H^2}, \quad \hat{S} = \frac{2S}{\sigma H}, \quad \hat{\lambda} = \lambda \frac{H}{l_m}, \quad (19)$$

$$\hat{\alpha} = \frac{\alpha(UH\sigma)^b}{Hl_m\sigma} \equiv \frac{1}{\sigma T_{\text{long}}}, \quad (20)$$

in which δ gives the ratio between the two length scales and $\hat{f}$ is the Coriolis parameter. The square root of half the Stokes number $\sqrt{E_v/2}$ quantifies the part of the water column (the Stokes depth) in which viscous effects influence the tidal motion significantly. The resistance parameter $\hat{S}$ gives the amount of resistance at the bottom in dimensionless units. This definition of the resistance parameter differs from the resistance parameter s given by *Maas and Van Haren* [1987]; the relation is given by $s = \hat{S}/E_v$. The definition in this paper is chosen so that viscous and resistance effects can be treated separately. The parameter $\hat{\lambda}$ accounts for slope correction on morphologically relevant spatial scales, and T_{long} follows from a balance of the two terms in the bottom evolution equation (8). T_{long}

has to be interpreted as the timescale on which morphological changes take place.

Next, typical scales for a representative North Sea location are presented. They can be used to determine the relevant physical processes in the model. The maximum tidal velocity is of the order of 1 m s^{-1} . The frequency of the M_2 tide in the North Sea is $\sigma = 1.4 \times 10^{-4} \text{ s}^{-1}$. This means that the morphological length scale is of the order of 7 km. Taking a Coriolis parameter at 52°N one has $f \approx 1.15 \times 10^{-4} \text{ s}^{-1}$, which gives $\hat{f} \approx 0.82$. In the North Sea the tidal wavelength is of the order of hundreds of kilometers, which means that the parameter δ is small, $\delta \approx 0.01$. For the offshore North Sea location a reference depth of $H = 30 \text{ m}$ is chosen. The vertical viscosity coefficient A_v has a wide range of values [see *Pedlosky*, 1987]; values from 0.01 to $0.5 \text{ m}^2 \text{ s}^{-1}$ are usually chosen, which yield a Stokes number between 0.2 and 8. For these values the Stokes depth is of the same order as, or is even larger than, the local water depth, which is characteristic for most coastal seas. Observations are needed to determine the resistance parameter $\hat{S}$. It can for instance be calculated from s and the Stokes number, where s has to be determined experimentally and $\hat{S} = s \times E_v$. *Maas and Van Haren* [1987] found that s is between 0.1 (weak resistance) and 10 (strong resistance) in the south of the North Sea. The exponent in the bed load transport formula, b , follows from measurements in combination with physical considerations; typical values are $b \sim 1-3$. Here $b = 1.5$ is chosen, a value which is based on a balance between the work done by transporting the grain's weight and the available part of the energy coming from the bottom shear stress; see, for example, *Dyer* [1986]. Characteristic values for the parameter λ can be estimated theoretically as indicated by *Hulscher et al.* [1993], and they are 1–2, which give $\hat{\lambda} \approx 0.01$. The bed load parameter α , for which a value can be estimated from *Van Rijn* [1993], is here ($b = 1.5$) given by $\alpha = 0.3 \text{ s m}^{-2}$, which yields $\hat{\alpha} = \mathcal{O}(3 \times 10^{-6})$, and so the morphological timescale becomes

$$T_{\text{long}} \approx 70 \text{ years.} \quad (21)$$

The bottom variable h will hardly vary on the tidal timescale; thus $\partial h / \partial t = 0$ can be used in the flow model. Furthermore, the slow evolution of the bed is determined by the tidally averaged sediment fluxes. Thus the bottom equation (15) can be replaced by

$$\frac{\partial h}{\partial T_m} = -\nabla \cdot \left\langle \left[|\tau|^b \left[\frac{\tau}{|\tau|} - \hat{\lambda} \nabla h \right] \right] \right\rangle, \quad (22)$$

where $T_m \equiv \hat{\alpha} t$, a slow time coordinate. The brackets denote a tidal average. More details on this approximation are given by *Hulscher et al.* [1993]. In order to compute the net sediment fluxes, one has to solve the flow equations at the fast timescale, with $\partial h / \partial t = 0$. So the equations of motion to be considered from now on are (12), (13), and (14) with $\partial h / \partial t = 0$, supplemented with (22); the boundary conditions are given in (17) and (18).

3. Linear Stability Theory and the Basic State

3.1. Linear Stability Concept

The main purpose of this study is to investigate whether the presence of tidal sand banks and sand waves can be associated with free instabilities in this morphological system. One can

test this hypothesis by first defining a basic state which describes a tidal current over a flat bottom. The basic velocities will have a vertical structure due to the frictional terms in the momentum equations. Moreover, Coriolis effects will cause veering of the currents (Ekman spiraling).

The next step is to introduce bed perturbations with arbitrary wave numbers in the horizontal directions. Subsequently, the initial interaction between bed forms and currents is studied in order to determine whether the perturbations are amplified or reduced. A basic state is said to be stable if arbitrary bed form disturbances with nonzero wave numbers decay exponentially. This implies that all perturbations have negative growth rates. Conversely, if there is at least one perturbation with a positive growth rate, the basic state is unstable. From a mathematical point of view this procedure may be formulated as follows. The solution of the problem is written symbolically as the vector

$$\psi = (u, v, w, \delta\zeta, h). \quad (23)$$

Now the response of the system is studied when a small wave-like perturbation in the bottom is present. This is a way in which the stability of the basic state can be investigated. The amplitude of the perturbation with respect to the undisturbed water depth is denoted by γ where $\gamma \ll 1$.

Hulscher et al. [1993] used a double expansion in the two small parameters γ and δ to solve an analogous problem. Subsequently, the study [*Hulscher et al.*, 1993] restricts to leading order effects in δ , which means that $\mathcal{O}(\delta)$ corrections are neglected (note that $\delta \approx 0.01$ according to the scales chosen in the previous section). Physically, this corresponds to a rigid lid approximation; that is, spatial variations of the tidal wave due to its propagation on the scale L are neglected for the study of the generation of bed forms. In order to make the procedure here as uncomplicated as possible, the double expansion will not be repeated here. Accepting at forehand that results will be accurate till order $\mathcal{O}(\delta)$ allows making the rigid lid approximation already at this stage. This is done by setting $\delta = 0$ in (17), which implies that the free surface does not have dynamics of its own. Furthermore, one has to assume that $\delta^{-1} \nabla \zeta = \mathcal{O}(1)$ in (12) and (13), which physically means that the local tidal flow is generated by free surface gradients. The solution can now be expanded as follows:

$$\psi = \psi_0 + \gamma \psi_1 + \gamma^2 \psi_2 + \dots, \quad (24)$$

where $h_0 = 0$. The matching boundary conditions at the bottom are found by transferring the boundary conditions at $z = -1 + h$, (18), to the fixed position $z = -1$. This can be done by making a Taylor expansion in the small parameter γ , which yields

$$\frac{\partial u}{\partial z} = \frac{\hat{S}}{E_v} \left\{ u + \gamma h \frac{\partial u}{\partial z} \right\} - \gamma h \frac{\partial^2 u}{\partial z^2} + \mathcal{O}(\gamma^2), \quad (25)$$

$$\frac{\partial v}{\partial z} = \frac{\hat{S}}{E_v} \left\{ v + \gamma h \frac{\partial v}{\partial z} \right\} - \gamma h \frac{\partial^2 v}{\partial z^2} + \mathcal{O}(\gamma^2), \quad (26)$$

$$w = \gamma \left\{ u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} \right\} - \gamma h \frac{\partial w}{\partial z} + \mathcal{O}(\gamma^2). \quad (27)$$

Here the basic solution is ψ_0 and the perturbation is given by ψ_1 . The equations which determine the basic state flow are

$$\frac{\partial u_0}{\partial t} + w_0 \frac{\partial u_0}{\partial z} - \hat{f} v_0 - \frac{E_v}{2} \frac{\partial^2 u_0}{\partial z^2} = - \frac{\partial \zeta_0}{\partial x}, \quad (28)$$

Figure 2. The vertical structure of the tidal components of the basic state as a function of the dimensionless depth: (a) $u_s(z)$ and (b) $u_c(z)$ (solid line), $v_s(z)$ (long-dashed line), and $v_c(z)$ (short-dashed line). The physical parameters are chosen as follows: $\hat{f} = 0.82$, $E_v = 1.0$, and $\hat{S} = 2.7$.

$$\frac{\partial v_0}{\partial t} + w_0 \frac{\partial v_0}{\partial z} + \hat{f} u_0 - \frac{E_v}{2} \frac{\partial^2 v_0}{\partial z^2} = -\frac{\partial \zeta_0}{\partial y}, \quad (29)$$

$$\frac{\partial w_0}{\partial z} = 0. \quad (30)$$

The corresponding boundary conditions follow from $\delta = 0$ in (17) at $z = 0$:

$$\frac{\partial u_0}{\partial z} = \frac{\partial v_0}{\partial z} = 0 \quad w_0 = 0, \quad (31)$$

and at the bottom $z = -1$ they become following (25)–(27)

$$E_v \frac{\partial u_0}{\partial z} = \hat{S} u_0 \quad E_v \frac{\partial v_0}{\partial z} = \hat{S} v_0 \quad w_0 = 0. \quad (32)$$

From (30) together with boundary conditions (31) (last one) and (32) it is determined that the vertical velocity in the basic state is

$$w_0(z) = 0. \quad (33)$$

In this paper the function $\nabla \zeta_0$, horizontal variations of the tidal wave, is chosen to be only a function of time. This function determines the depth-averaged velocities U_0 and V_0 or vice versa. Here the reverse option will be chosen so that results are comparable as close as possible with the results from the depth-averaged model as presented by Hulscher *et al.* [1993]. Therefore the depth-averaged velocities are prescribed

$$[U_0(t), V_0(t)] \equiv \int_{-1}^0 [u_0(z), v_0(z)] dz = (\sin t, \beta \cos t). \quad (34)$$

Here $|\beta| \leq 1$ is assumed; hence the x axis is chosen to coincide with the principal direction of the depth-averaged tidal current. Thus $\beta = 0$ describes an oscillating unidirectional tide; $0 < |\beta| < 1$ describes elliptical tides. Setting $|\beta| = 1$, one gets a circular tide; due to the loss of a preferred direction the choice of the x axis has no meaning in this case. Similar models have been considered earlier by, e.g., Prandle [1982], Maas and Van Haren [1987], and Visser *et al.* [1994]. As the problem here is slightly different it is solved in Appendix A. Expressions for the basic velocities u_0 and v_0 , which are functions of time and the vertical coordinate, are derived. The basic solution can be presented either in tidal components or in terms of elliptical properties; both will be discussed in the next subsection.

3.2. Horizontal Basic Velocities

Two ways of expressing the basic velocities will be presented. The first one, expressed in tidal components, will be used to derive the linear stability problem to be discussed in the next section. As can be seen from (28) to (34) the solution of the horizontal velocities can be written as

$$u_0(z, t) = u_s(z) \sin t + u_c(z) \cos t, \quad (35)$$

$$v_0(z, t) = v_s(z) \sin t + v_c(z) \cos t. \quad (36)$$

The values of $u_s(z)$, etc., can be computed following (A11) in Appendix A. In Figure 2 the vertical structure of the tidal components is shown for a unidirectional tide ($\beta = 0$) for characteristic North Sea conditions, the Coriolis parameter $\hat{f} = 0.82$, the Stokes number $E_v = 1.0$, and a resistance parameter $\hat{S} = 2.7$. Although the depth-averaged values of $u_c(z)$, $v_s(z)$, and $v_c(z)$ are zero here, internal friction and Coriolis effects cause these tidal velocity components to have a vertical

Figure 3. The vertical structure of elliptical properties of the basic state as a function of the dimensionless depth. The physical parameters are chosen as in Figure 2. In succession are shown (a) length of major axis $A_{\text{maj}}(z)$ of the tidal ellipse, (b) ellipticity (z), ratio of the length of minor and length of major axis, (c) $\Theta(z)$, the angle between the tidal ellipse major axis and the depth-averaged velocity in radians, and (d) $\Psi(z)$, the elliptical phase in radians. For more explanation, see text.

structure. Due to the presence of slip at the bottom, the velocity at the bottom is nonzero. Since a Fourier expansion in time will be used for the linear perturbation problem (next section), the above-mentioned parametrization is the one used in this paper.

The description in elliptical properties is helpful for a physical interpretation and links in with existing literature [Prandle, 1982; Visser *et al.*, 1994]. In Appendix A the physical quantities used for this description are derived and discussed. In Figure 3 the same physical parameters are used as in Figure 2.

In Figure 3a it is shown how the maximum value of the tidal velocity, $A_{\text{maj}}(z)$, increases from $u_0 \sim 0.55$ to $u_0 \sim 1.2$ from the sea bottom to the sea surface. Figure 3b shows that the ellipticity is cyclonic near the bottom and anticyclonic in the top of the water column. Figure 3c demonstrates that near the bottom the major axis is rotated cyclonically with respect to the depth-averaged flow direction; near the surface the rotation is anticyclonic. In this case the difference between both axes is 4° . Due to internal friction the flow velocity reaches its maximum at a later stage at the bottom than at the surface, as can be concluded from Figure 3d. In this case the time difference is 10.5 min.

The basic bottom shear stress is completely dominated by the sine component in the x direction. The amplitude of this component can be calculated as a function of the Stokes number and the stress parameter. In Figure 4 its (nondimensional) value is shown as a function of the stress parameter for three values of the Stokes number.

4. The Perturbed State

As explained in previous sections, the linear stability problem is derived from the equations describing the $\mathcal{O}(\gamma)$ part of the general solution presented in (24). The corresponding equations of motion and boundary conditions can be found by substituting the expansion (24) into (12)–(14) and (22), by setting $\delta = 0$ in (17), and by adding the transferred boundary conditions at the bottom (25)–(27). This gives the following equations:

$$\frac{\partial u_1}{\partial t} + u_0 \frac{\partial u_1}{\partial x} + v_0 \frac{\partial u_1}{\partial y} + w_1 \frac{\partial u_0}{\partial z} - \hat{f}v_1 - \frac{E_v}{2} \frac{\partial^2 u_1}{\partial z^2} = -\frac{\partial \zeta_1}{\partial x} \quad (37)$$

$$\frac{\partial v_1}{\partial t} + u_0 \frac{\partial v_1}{\partial x} + v_0 \frac{\partial v_1}{\partial y} + w_1 \frac{\partial v_0}{\partial z} + \hat{f}u_1 - \frac{E_v}{2} \frac{\partial^2 v_1}{\partial z^2} = -\frac{\partial \zeta_1}{\partial y} \quad (38)$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0, \quad (39)$$

$$\frac{\partial h_1}{\partial T_m} = -\nabla \cdot \langle |\tau_0|^{b-1} \tau_1 + (b-1)|\tau_0|^{b-3} \tau_0 (\tau_0 \cdot \tau_1) - |\tau_0|^b \hat{\nabla} h_1 \rangle \quad (40)$$

$$\tau = (\tau_x, \tau_y) = \frac{E_v}{2} \left(\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z} \right) \Big|_{z=-1}. \quad (41)$$

The boundary conditions at the free surface, $z = 0$, become

$$\frac{\partial u_1}{\partial z} = \frac{\partial v_1}{\partial z} = w_1 = 0. \quad (42)$$

At the bottom, $z = -1$, the boundary conditions are

$$\frac{\partial u_1}{\partial z} = \frac{\hat{S}}{E_v} u_1 + h_1 \left(-\frac{\partial^2 u_0}{\partial z^2} + \frac{\hat{S}}{E_v} \frac{\partial u_0}{\partial z} \right), \quad (43)$$

$$\frac{\partial v_1}{\partial z} = \frac{\hat{S}}{E_v} v_1 + h_1 \left(-\frac{\partial^2 v_0}{\partial z^2} + \frac{\hat{S}}{E_v} \frac{\partial v_0}{\partial z} \right), \quad (44)$$

and

$$w_1 = \left\{ u_0 \frac{\partial h_1}{\partial x} + v_0 \frac{\partial h_1}{\partial y} \right\}. \quad (45)$$

The bottom perturbation enters this perturbed system at the

Figure 4. The maximum value of the (dimensionless) bottom shear stress as a function of the stress parameter $\hat{S}$, for $E_v = 5.0$, $E_v = 1.0$, and $E_v = 0.5$ (in order of magnitude); $\hat{f} = 0.82$.

kinematic bottom boundary condition. Interactions between the basic solution and the perturbations take place through the oscillatory advection terms in the equations of motion. In case of a finite resistance parameter $\hat{S}$, vertical velocity at the bottom is generated because at this position the normal velocity component must vanish. The generation of perturbed horizontal velocities at the bottom is determined by the interaction between the bottom perturbations and the basic horizontal velocities. In the case where $\hat{S} \rightarrow \infty$, only the latter mechanism is active because in this limit the basic horizontal velocities become zero at the bottom.

The solution technique as adopted is the one described by Hulscher *et al.* [1993]. The problem structure for a wavy bottom perturbation allows for a Fourier transformation in horizontal coordinates. This means that the solutions have the following structure:

$$\psi_1(\mathbf{x}, z, t) = \iint \tilde{\psi}(\mathbf{k}, z, t) e^{-i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k} + c.c., \quad (46)$$

in which *c.c.* stands for the complex conjugate expression and $\psi_1 = (u_1, v_1, w_1, \zeta_1, h_1)$. The $\tilde{\psi} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\zeta}, \tilde{h})$ as defined in (46) depends on the wave vector $\mathbf{k}$ of the bottom pattern, the vertical coordinate z , and the time t . The flow part of the basic state is an M_2 tide, and due to the structure of the linear problem its solution is represented as a Fourier series in tidal components, $M_0, M_2, M_4, \dots$. A harmonic truncation in time is performed, which means that the perturbation is restricted to a finite number of tidal components; for details, see Hulscher *et al.* [1993]. They demonstrated that an $N = 3$ truncation (including M_0, M_2, M_4, M_6) is the most appropriate one for elliptical tides, although an $N = 1$ truncation (including M_0, M_2) will contain the dominant physics related to unidirectional tides. Since the depth-averaged basic state in all the examples in this paper is a depth-averaged unidirectional tide, an $N = 1$ truncation will include the dominant physical processes. In case the transformed bottom variable $\tilde{h}$ is chosen to be real, the structure of the equations will determine that the amplitudes of the various tidal components in the solution vector $\tilde{\psi}$ are either real or imaginary. The harmonic truncated solution can then be written as

$$\tilde{u}_{\text{trunc}}(z, t) = \tilde{h}[ia_0(z) + a_s(z) \sin t + a_c(z) \cos t], \quad (47)$$

$$\tilde{v}_{\text{trunc}}(z, t) = \tilde{h}[ib_0(z) + b_s(z) \sin t + b_c(z) \cos t], \quad (48)$$

$$\tilde{w}_{\text{trunc}}(z, t) = \tilde{h}[c_0(z) + ic_s(z) \sin t + ic_c(z) \cos t], \quad (49)$$

$$\tilde{\xi}_{\text{trunc}}(t) = \tilde{h}[d_0 + id_s \sin t + id_c \cos t], \quad (50)$$

in which $(a - d)_{0,s,c}$ are real functions. Since all the time dependencies are now known, the tidal averaging in the bottom evolution equation (40) can be carried out. The result is

$$\begin{aligned} \omega_{gr} \equiv \frac{1}{\tilde{h}} \frac{\partial \tilde{h}}{\partial T_m} = \frac{E_v}{2} |\tau_0|^{b-1} \{ -ka'_0(-1) - lb'_0(-1) \} \\ - \hat{\lambda} |\tau_0|^b (k^2 + l^2) - \frac{(b-1)E_v^3}{32} |\tau_0|^{b-3} \{ (k[u'_s(-1)]^2 \\ + [u'_c(-1)]^2)a'_0(-1) + l[v'_s(-1)]^2 + [v'_c(-1)]^2)b'_0(-1) \\ + [ka'_0(-1) + lb'_0(-1)][u'_s(-1)v'_s(-1) \\ + u'_c(-1)v'_c(-1)] \}, \end{aligned} \quad (51)$$

in which a prime denotes the vertical derivative of that particular function. The bottom evolution equation shows that the growth of bed forms is determined by the bottom shear stresses induced by the basic velocity field, which are already known (see section 2) and by the bottom shear stresses induced by the residual horizontal components of the velocities at order ψ_1 of the solution. To obtain the latter, one has to solve the three-dimensional flow field. The equations and boundary conditions for the real coefficients $(a - d)_{0,s,c}$ can be derived by substituting (47)–(50) into (46) and the Fourier transformation of (37)–(39) and boundary conditions (42)–(45). The resulting set of differential equations and boundary conditions, which determines the vertical flow structure induced by the perturbations, is given in Appendix B. The set was solved numerically. Some results of the flow problem are given at the end of the next section, but the main aim is to compute the growth rates of the bed form perturbations in order to find out which patterns the system selects.

5. Results

5.1. Generation of Rhythmic Bottom Patterns

In this section the results concerning the growth rate of regular bed forms are presented and discussed as function of the two parameters $\hat{S}$ and E_v . The model yields much more information than is needed to determine the growth rate of this wavy bottom perturbation.

5.1.1. Growth rates of bottom perturbations. The perturbed flow field is needed to compute the net sediment transport and the growth rate of the bed form as given by (51). In this model the vertical structure of the flow field and the amplitude of the free surface elevation strongly depend on two parameters: the resistance parameter $\hat{S}$ and the vertical Stokes number E_v . First, some typical results for several values of the Stokes number and resistance parameter are shown. The dimensional values for the eddy viscosity A_v and resistance parameter S which correspond to the chosen parameters E_v and $\hat{S}$ are given in Table 1.

If a moderate resistance situation (small $\hat{S}$) and a large Stokes number are chosen, the vertical effects are expected to be rather small, since in this case the Stokes boundary layer is much larger than the water depth and the vertical shear in the horizontal velocities is weak. These choices simulate the flow

Table 1. Dimensional values for the eddy viscosity A_v and resistance parameter S and the nondimensional model parameters E_v and $\hat{S}$

A_v , $\text{m}^2 \text{s}^{-1}$	S , m s^{-1}	E_v	$\hat{S}$	Figure
2.2×10^{-1}	2.9×10^{-3}	3.5	1.4	5a
2.2×10^{-1}	4.8×10^{-3}	3.5	2.3	5b
2.2×10^{-1}	5.7×10^{-3}	3.5	2.7	6a
1.26×10^{-1}	5.7×10^{-3}	2.0	2.7	6b
6.3×10^{-2}	5.7×10^{-3}	1.0	2.7	7a
3.2×10^{-2}	1.1×10^{-3}	0.5	0.5	7b

The reference value for the depth $H = 30$ m is taken. The values in this table correspond to the parameters for which results are shown in section 5.1.

behavior in a depth-averaged model; the generation of vertical velocities at the bottom, see (45), is strong in this case. The growth rates for such a situation ($\hat{S} = 1.4$, $E_v = 3.5$) in the North Sea ($\hat{f} = 0.82$) are shown in Figure 5a for all sinusoidal perturbations with wave vector (k, l) . Here k is the (dimensionless) number of bottom waves which fit into 42 km in x direction; l is the analogous number for the waves in y direction. The dimensional wave numbers $(\tilde{k}, \tilde{l})$ are given by $(\tilde{k}, \tilde{l}) = (k, l)/l_m = (k, l)/(7 \text{ km})$, from which the physical characteristics of the bottom pattern, in two horizontal dimensions, can be calculated.

The exponent in the bed load transport formula is chosen as $b = 1.5$; the parameter which models the downslope correction in the sediment transport is taken as $\hat{\lambda} = 0.0085$. Only the positive growth rate contours are shown, as one is interested in situations in which regular features develop; this corresponds to an unstable flat bottom. The maximum growth rate is reached at a wave vector of $(5.0, -3.0)$ which corresponds to a regular bottom feature of a wavelength of 7.2 km and crests which are directed 60° anticlockwise with respect to the depth-averaged current. The growth rate of this perturbation is 0.37, which means that the initial amplitude has increased by a factor of 1.4 on the long timescale. These structures resemble the tidal ridges that are found in several coastal seas. They also agree with results produced by depth-averaged models given by Huthnance [1982], de Vriend [1990], and Hulscher *et al.* [1993]. These authors found that the most unstable modes were those having wavelengths of the order of the tidal excursion and crests rotated anticlockwise due to Coriolis effects.

By increasing the resistance parameter, which implies less slip (and more stress) at the bottom ($\hat{S} = 2.3$), and keeping the other parameters fixed, one arrives at the contour plot of positive growth rates shown in Figure 5b. The amount of unstable wave vectors has increased, mainly in the k direction. A maximum growth rate is observed for the wave vector $(6.5, -3.5)$ and the growth rate is 1.1. Compared to the previous result, obtained with the smaller resistance number, the wave vector of the fastest growing mode has changed little, but the growth rate of this perturbation and the area of unstable modes have increased significantly.

If the resistance parameter is increased further to $\hat{S} = 2.7$ (and other parameters are kept fixed) the contour plot of the positive growth rates will be as shown in Figure 6a. The number of unstable modes has increased even more in the direction of larger k modes. The fastest growing mode is $(k, l) = (31, -2.8)$, its growth rate being 5.3. This mode has a wavelength of 1.3 km and its crests are directed 85° anticlockwise with respect to the depth-averaged tidal current, almost perpendic-

Figure 5. Contour plot of positive growth rates for bed form perturbations characterized by (k, l) . The dimensionless k is the number of bottom waves which fit into 42 km in x direction; l is the number in the y direction. The dimensional wave numbers $(\tilde{k}, \tilde{l})$ are given by $(\tilde{k}, \tilde{l}) = (k, l)/(7 \text{ km})$. (a) $E_v = 3.5$ and $\hat{S} = 1.4$; (b) $E_v = 3.5$ and $\hat{S} = 2.3$. Please, note the different scaling in Figures 5a and 5b along the axes and in contour spacing. For other parameters, see text.

Figure 6. As in Figure 5 but for (a) $\hat{S} = 2.7$ and $E_v = 3.5$ and (b) $\hat{S} = 2.7$ and $E_v = 2.0$. Please note the different scaling between the figures in axes and contour spacing.

Figure 7. As in Figure 5 but for (a) $\hat{S} = 2.7$ and $E_v = 1.0$ and (b) $\hat{S} = 0.5$ and $E_v = 0.5$. Please note the different scaling between the figures in axes and contour spacing.

ular to the principal tidal current. The unstable area shows a large asymmetry between the wave numbers l and $-l$; this asymmetry is greatest for the long waves. These unstable modes, which are not the fastest growing ones, have the characteristics of tidal sand banks. So for these parameters, both sand waves and tidal sand banks are initially unstable. In a nonlinear stability analysis these modes will interact. This interaction can lead to a finite amplitude behavior of a pattern in which both sand waves and tidal sand banks exist simultaneously. Such patterns are often observed; see, for example, Huntley [1993].

A decrease in the Stokes number to $E_v = 2.0$, which means an increase in $\hat{S}/E_v$, yields the result shown in Figure 6b. Physically, this parameter change corresponds to increased veering of horizontal currents in the water column in relation to the previous parameter choice. The fastest growing mode turns out to be $(k, l) = (43.9, -3.8)$ with a growth rate of 16. This mode has a wavelength of 950 m and crests directed 85° with respect to the x axis. The growth rate indicates that the timescale of this growing process is of the order of years, much faster than that of the fastest growing modes in Figure 5. If we decrease the Stokes number to $E_v = 1.0$, the fastest growing mode has a wavelength (see Figure 7a) of 640 m and crests directed 82° with respect to the x axis. The growth rate indicates that the timescale of this growing process is also of the order of years.

By taking a small Stokes number $E_v = 0.5$ and a small resistance number $\hat{S} = 0.5$, one obtains a fastest growing mode which has crests nearly parallel to the principal tidal current (an anticlockwise 10° angle between the crests and basic current) and a wavelength of 660 m; see Figure 7b. There is another unstable area that has a relative maximum corresponding to tidal sand banks. This area disappears if even

smaller Stokes numbers are chosen. The unstable area surrounding the fastest growing mode does not extend to the zero mode (extra long waves). This area decreases when a small amount of negative ellipticity ε is added; beyond a certain value it vanishes completely. The behavior of the smaller wavelength mode might provide a starting point for a weakly nonlinear stability analysis, which requires the existence of a critical mode, being the first growing mode beyond critical conditions. This technique will yield equations that describe the finite amplitude behavior of the bed forms. This technique was used, in a different model context, by Vittori and Blondeaux [1992], who studied the nonlinear amplitude topography of sand ripples caused by sea waves.

5.1.2. Fastest growing modes. The previous examples show that the results of the linear stability analysis depend strongly on the choice of the two parameters $\hat{S}$ and E_v . We can determine the fastest growing mode and its growth rate for any combination of the Stokes number and the resistance parameter. The results are summarized in the three contour plots of Figure 8. The other parameters are chosen as mentioned in the foregoing part of this section. In the following, discussion will be discriminated between four regimes of parameter settings. Regime I has a very small resistance parameter and a small to moderate Stokes number. Regime II is characterized by a small resistance parameter and a small to moderate Stokes number. The parameters in regime III are typically a moderate resistance number and a large Stokes number. The last regime, regime IV, has a large resistance parameter such that $\hat{S}/E_v$ is large. Figure 8a shows a contour plot of the dimensionless length of the wave vector at which the maximum growth rate is reached. The dimensional wavelength of the fastest growing mode is $42 \text{ km}/|k|$ for the present parameter setting. A sharp transition is observed between regime III (modes with $|k|$

smaller than 10) and regime IV (modes with $|k|$ larger than 10). Note that the contours are drawn with different spacing in these regimes. A $|k|$ between 4 and 10 corresponds to dimensional wavelengths between 4.2 km and 10.1 km; the features with $|k|$ larger than 10 have dimensional wavelengths smaller than 4.2 km. This plot suggests that there are four regimes in which the wavelength behaves differently. Regime I (no contour lines in the plot) denotes bottom structures having a wavelength larger than the tidal excursion which are hardly noticeable on the tidal excursion scale. In regime II, two types of unstable modes exist: both tidal sand banks and a much shorter mode, as in Figure 7b. The unstable area surrounding the shorter mode does not extend to the zero mode. Two other regimes are shown: one for tidal sand banks (regime III) and one for sand wave modes (regime IV).

The orientation of a bottom wave vector is expressed in the angle (in radians) to the depth averaged principal direction. Figure 8b shows how that angle for the wave vector belonging to the fastest growing mode depends on E_v and $\hat{S}$. In regime I the direction of the modes has no physical meaning since there are no growing modes here. Modes having an angle of -0.6 ± 0.1 (crests are rotated up to $60 \pm 6^\circ$ anticlockwise with respect to the mean tidal current) are the fastest growing ones in regime III. They have the same preferred direction as tidal sand banks. With increasing $\hat{S}$ (regime III is entered from regime I) the angle increases rapidly from -1.5 to -0.3 radians. This transitional behavior is also observed at the boundary zone between regime III and IV in which a rapid increase from -0.5 to -0.2 radians occurs. The modes in regime IV to the left of the contour line of -0.1 radians have crests almost perpendicular to the mean tidal current, just like the sand waves. For a small Stokes number we note a sharp transition from sand wave modes to a wavy bottom pattern in which wave crests are nearly parallel to the principal tidal current. Figure 8b also indicates that a classification into four regimes can be made as in these regimes, the direction of the fastest growing mode is significantly different.

Figure 8c shows how the growth rate of the fastest growing mode depends on $\hat{S}$ and E_v . This plot (spacing in contour lines is not equal) also shows four areas. In regime I the growth rate is lower than 0.2, which corresponds to slowly growing bed forms. It takes a very long period for these perturbations to increase in amplitude, if they are growing anyway, since timescales are of order of millennia and longer. In regime III, in which the Stokes number is approximately 2.0 (and larger), an increase in the resistance parameter causes an increase in growth from 0.2 to 2.0, and timescales are now of the order of decades to centuries. In regime IV the growth rate increases very rapidly when the slip parameter is increased further; this points to a morphologic timescale of order years. Regime II for small E_v shows small growth rates again.

The results of the linear stability analysis demonstrate that one can distinguish between four different types of behavior of the (initial) perturbed bottom, as is indicated in Figure 9. Qualitatively identical results are obtained for other values of $\hat{f}$, $\hat{\lambda}$, and b .

Typical top views of the three regular bottom patterns are shown in Figure 10. The remaining type corresponds to a situation in which the flat bottom is stable. Thus in this case, no bed forms occur. This situation arises for small values of $\hat{S}$ and moderate Stokes numbers, regime I. The other combinations of $\hat{S}$ and E_v lead to an initial unstable flat bottom which can be divided into the three types of patterns.

Figure 8. Contour plots of (a) the wave number of the mode with the maximum growth rate, (b) the direction (in radians) of the mode with the maximum growth rate, and (c) the maximum growth rate, as a function of $\hat{S}$ and E_v . The other parameters are as before.

One of these patterns has spatial and time dimensions resembling those of tidal sand banks, as described by, e.g., Off [1963]. These tidal sand banks have a wavelength of between 2 and 12 km, their crests are oriented 30° anticlockwise with respect to the mean tidal current, and their timescales are of the order of centuries. The banks generated by this model can be oriented more anticlockwise, up to 60° ; this matter will be discussed in section 5 on the physical interpretation of the results. The values for E_v and $\hat{S}$ which give this pattern (regime

Figure 9. Characteristic bed forms predicted by the three-dimensional shallow water model as a function of $\hat{S}$ and E_v . Quantification depends on values of $\hat{f}$, $\hat{\lambda}$, and b .

III) describe a flow having little vertical shear; hence the generation of sand banks can be simulated in a depth-averaged model. Such a depth-averaged model in which the generation of tidal sand banks is described was first presented by *Huthnance* [1982], followed by *De Vriend* [1990] and *Hulscher et al.* [1993].

The model has a region of parameters for which initially regular patterns develop which have characteristics similar to those of sand waves, as described by, e.g., *Belderson* [1986]. This regime (IV) is characterized by large values of $\hat{S}$ and moderate values of E_v . These patterns have a wavelength of 100–800 m, their crests are more or less perpendicular to the tidal mean flow, and their timescale is in the order of years. These regular features have not been found before in previous depth-averaged models, although they can appear in such models if the necessary three-dimensional effects are included parametrically, as shown by *Hulscher et al.* [1993] and *De Swart and Hulscher* [1995].

Finally, regime II is discussed. This regime is characterized by small values of E_v and yields a fastest growing mode whose crests are rotated anticlockwise (from perpendicular to almost parallel) with respect to the principal tidal current. Initially, two patterns coexist; one has wavelengths of hundreds of meters and grows faster than the pattern with a wavelength of the order of kilometers. The growth rates of both are relatively low. This shorter wavelength pattern will be referred to as tide-parallel ridges; the pattern with longer wavelengths has characteristics of tidal sand banks.

One can gain a better understanding as to why these different patterns appear by studying the tide-topography interactions in detail. Such an analysis is carried out in subsection 4.2 in order to gain information on the formation mechanism of sand waves. The physical processes that supply the positive feedback to a specific regular pattern in this model are discussed in section 5 in relation to the vertical effects.

5.2. Flow Response to a Sand Wave Perturbation

In order to study the processes that lead to the formation of sand waves, one needs to show the residual sea level elevation and vertical structure of the residual flow components explicitly. The sand wave selected in this study is the fastest growing mode for $\hat{S} = 2.7$ and E_v , as is shown in Figure 6b. The other parameters are the same as those chosen in subsection 5.1. Figure 11a shows the bottom perturbation with wave vector (65.2, -9.1), using the definitions (47)–(50). The horizontal coordinates were rotated so that the coordinate axes are parallel and perpendicular to the crests of the bottom perturbation. The bed forms are sand waves with a wavelength of 680 m, the crests of which have an anticlockwise rotation of 80° with respect to the principal tidal current. The amplitude of the waves is chosen to be $\gamma = 0.2$, which corresponds to 6 m in dimensional values. By means of the linear perturbation theory presented in section 3, one can calculate the initial response of the tidal components to such a perturbation. The response of the residual sea level ζ_1 to this perturbation is also shown in Figure 1a. The residual sea level is 180° out of phase with the bottom perturbation. One unit of sea level corresponds to 0.17 m (if the wavelength is taken to be 770 km, which is the

Figure 10. Rhythmic bottom patterns; the arrow indicates the principal tidal current. From left to right are shown tidal sand banks, sand waves, and tide-parallel ridges.

Figure 11. Cross-bank residual flow and bottom structure due to the bottom perturbation of a wave vector $(k, l) = (65.2, -9.1)$. In this situation, 1 unit in x direction corresponds to 108 m. The physical parameters are chosen as follows: $E_v = 1.0$, $\hat{S} = 2.7$. (a) Bottom perturbation of amplitude $\gamma = 0.2$ and the residual sea level response. The contour plots show contour lines of equal velocity in dimensionless units. (b) Cross-bank directed residual velocity. (c) Along-bank directed residual velocity. (d) Vertical residual velocity. For more information, see text.

Figure 12. Strong near-bed circulation which supports the growth of the bottom perturbation. The backward circulation in the upper flow part uses a larger part of the water column and is weaker.

order of magnitude for the southern North Sea), so the amplitude of the residual elevation is 0.015 m.

Figure 11 depicts the cross structure of the residual flow components caused by this sand wave. Dimensional values for the velocities are obtained by multiplying the numbers given in the contour plots by the dimensionless amplitude (here γ) of the sand wave and the appropriate flow scale (see section 2). In case of sand waves with an amplitude of 6.0 m, one unit for the horizontal flow components corresponds to 0.2 m s^{-1} ; one unit of vertical velocity corresponds to $8.5 \times 10^{-4} \text{ m s}^{-1}$. The cross-structure horizontal residual flow reaches a maximum value at approximately 0.1 of the total depth above the sides of the sand wave. Close to the bed the flows are directed from the troughs to the crests. The residual component of the vertical velocity is 1 order of magnitude less than the horizontal residual velocity and is directed upward above the crests and downward into the troughs. These residual flow components cause a residual circulation which favors the growth of the bed form, as shown in Figure 12. During both ebb and flood period the tidal flow moving upslope is larger than the flow moving downslope. In this model the sediment transport increases with increasing bed shear stress, as can be seen from (51). This implies that, averaging over a tidal cycle, there will be a convergence of sediment transport at the crests. Hence this sand wave will grow. It should be noted that circulations as shown in Figure 12 can also be caused by different processes than studied here, e.g., interactions between internal tides and topography [Maas and Zimmerman, 1989].

6. Physical Mechanisms for Generation of Regular Patterns

This subsection deals with the mechanisms which lead to the growth of rhythmic features. These mechanisms can be described in terms of vorticity dynamics, as given by Zimmerman [1986], as well as in terms of momentum.

6.1. Tidal Sand Banks

The generation of tidal sand banks can be explained in terms of the effect that a perturbation of the bottom topography has on the depth-averaged velocity field [Zimmerman, 1981]. Zimmerman showed that if the crests of the bed forms are rotated anticlockwise (in the northern hemisphere) with respect to the

depth-averaged tidal currents, Coriolis and bottom frictional forces cause the formation of clockwise horizontal residual circulation around the crests. Consequently, flow velocities on the upstream side of the crest will be slightly higher than those on the downstream side. Since sediment transport increases with increasing velocity, this means that sediment will accumulate at the crest.

The bottom friction torque has the largest impact on perturbations with a wave vector rotated both $+$ or $-$ (45° to 90°) with respect to the principal tidal current direction; the Coriolis torque leads to the strongest growing mode with a wave vector rotated $-$ (30° to 90°) with respect to the principal tidal current direction, as can be derived from the growth rate formula (as function of the wave vector) given by Hulscher *et al.* [1993]. The combination of bottom friction and Coriolis torque leads to the generation of a wavy bottom pattern in which the crests are rotated at a maximum of 60° with respect to the principal tidal current direction. If the Coriolis force is dominant, this angle will be large; if friction dominates, the angle will be smaller. The wavelength of the fastest growing mode is found from the balance of the growth caused by the two torques and the gravity effect which hinders the development of steeper slopes (shorter wavelengths).

This three-dimensional model can be used to simulate the depth-averaged mechanism if small values are selected for $\hat{S}$ and large values are selected for E_v . The first choice guarantees that mechanism, as described by the boundary condition (45), is very effective. This is the mechanism that is responsible for the planetary vortex stretching (or Coriolis torques) in the Zimmerman [1981] model. Of course, $\hat{S}$ cannot be chosen too small since no bed forms will occur if the bed shear stresses are very small. A large Stokes number guarantees that the phase lag between bed shear stress and depth-averaged current is small. This corresponds to regime III, as discussed in section 5.2. Since in this regime, Coriolis effects dominate over bottom friction effects, the crest orientation of the sand banks is close to 60° .

To give an indication of the relevance of the mechanism that underlies perturbations in the depth-averaged flow, the last term in the right-hand side of (43) and (44) can be rewritten as $(\hat{S}/E_v)^2 \mathbf{u}_0$. The stabilizing term $\partial^2 \mathbf{u}_0 / E_v \partial z^2$ can be rewritten into the term $(1/E_v) \mathbf{u}_0$ and a term proportional with the pressure gradients. So for large $(\hat{S}/E_v)$ the new three-dimensional growing term is expected to be dominant. This mechanism and the corresponding bed pattern will be discussed in the next subsection. For smaller $(\hat{S}/E_v)$ the stabilizing term will become more important: it has the largest impact for small slip parameter $\hat{S}$ values. The Coriolis-forcing mechanism dominates for small $\hat{S}$, as it is relatively more effective for large Stokes numbers than for smaller Stokes numbers where the friction is important. Therefore the mechanism underlying perturbations in the depth-averaged flow is expected to be important only for small values of the resistance number.

6.2. Tidal Sand Waves and Tide-Parallel Ridges

The results of the model show that the generation of sand waves develops primarily when resistance parameters are large. The growth of the features is determined by a growing term based on residual flow stresses and a term which hinders the formation of steep slopes. The latter has no preferred direction and it causes the damping of large wave numbers. The growing term, which is related to the residual bottom shear stresses, will now be discussed in more detail.

The mechanism of sand waves can be explained as follows. First, the large resistance, no-slip case $\hat{S} \rightarrow \infty$ is considered. The boundary conditions (43) and (44) then show that perturbed horizontal velocities are generated at the bed due to tide-topography interaction. These velocities oscillate with the principal tidal frequency. The direction of this velocity field coincides with the basic shear stresses in the troughs of the bed forms. Above the crests, its direction is opposite to that of the basic shear stress. Since the boundary condition (42) requires the perturbed shear stress $\partial \mathbf{u}_1 / \partial z$ to vanish at the free surface, this guarantees that there will also be a perturbed horizontal velocity field at some distance from the bed. This also holds for the vertical velocity field which oscillates with the tidal frequency. This is because mass conservation (39) causes the generation of vertical gradients in the flow field, whereas from (45) it can be seen that $\partial w / \partial z$ is nonzero at the bed. In this region the advection terms in the momentum equations lead to the generation of both residual currents and a residual pressure gradient. These phenomena cause residual circulations close to the bed, as already sketched in Figure 12. Hence there will be a net convergence of sediment transport at the crests of the sand waves, as was already explained in section 4.2.

If one allows for slip at the bottom, the situation becomes more complex. The basic currents and the bottom-induced perturbed currents are no longer weak near the bottom, and advective interactions take place throughout the water column. As the bed shear stresses become smaller with decreasing values of the resistance parameter, the growth rates will decrease. Since the bed-slope correction term in the sediment transport becomes relatively more effective, the wave number of the most preferred mode shifts to smaller wave numbers (larger wavelengths). Consequently, Earth rotation effects will become more important. These then cause the residual currents, forced by the pressure gradient, to veer too before they reach the sea bottom. For small Stokes numbers the veering will be substantial and it reaches a maximum of -45° [Pedlosky, 1987], which causes the fastest growing mode to have crests almost parallel to the principal current.

7. Discussion/Conclusions

This paper has focused on the regular patterns in the seabed which appear as free instabilities due to the tide-topography interaction. The model presented is a three-dimensional extension of the depth-integrated model developed by Huthnance [1982]. Vertical effects have been taken into account by incorporation of a constant viscosity and a partial slip condition at the bottom, following Prandle [1982] and Maas and Van Haren [1987]. Depending on the values of the resistance parameter and the Stokes number a flat bottom turns out to be either stable for all possible regular perturbations or unstable. In the latter case a wavy bottom pattern develops. Three different types of the initially most unstable modes can be identified: tidal sand banks, sand waves, and tide-parallel ridges.

The model gives the mode (wavelength and crest orientation with respect to the principal tidal current) which will grow fastest and the growth rate of this mode, indicating the morphological timescale. The analysis in this paper shows that these values depend strongly on the values for the Stokes number and the resistance parameter.

One might remark that the Stokes number and the resistance parameter could be dependent parameters. As when the turbulent eddy viscosity profile $\nu_t(z)$ and the roughness height

z_0 are known, the velocity profile is determined and the parameter set $(E_v, \hat{S})$ is completely fixed. So if $\nu_t(z)$ is a function of the friction velocity u_* and depth H only, this would cause a serious problem here. However, $\nu_t(z)$ also depends on local turbulence characteristics in the upper part of the water column, e.g., stratification, presence of swell, and action of wind waves. This leads, e.g., to a more realistic parabolic function $\nu_t(z) = \kappa u_* z [1 - \varepsilon(z/H)]$ in which κ is the Von Karmann constant and the parameter ε expresses the local variations in the turbulence characteristics near the water surface. On the basis of such a profile, one can derive expressions for the set $(E_v, \hat{S})$ depending on the two free parameters ε and z_0 . This simple example shows that E_v and $\hat{S}$ are not necessarily dependent parameters.

Of course, physically realistic values for ε , z_0 give limits on the realistic values for $\hat{S}$, E_v . Therefore not all combinations $\hat{S}$, E_v in Figures 8 and 9 may be achieved in reality. However, no general procedure is available yet to determine the realistic region(s) with any accuracy. So in order to present the model results as general as possible in this paper, no combinations of $\hat{S}$, E_v have been excluded yet.

A comparison between locally observed bed forms and the bed form predicted by the model will demonstrate the model's predictive capacity. Thus comparison is discussed now in connection with two studies in which observations of current profiles are fitted to parameters related to E_v and $\hat{S}$.

First, the results of Maas and Van Haren [1987] are considered. They use observed current profiles at a location in the south of the North Sea which give $E_v \approx 0.01$ and $\hat{S} \approx 0.1$. At this location, there are no regular bed forms. The local water depth at this location is $H = 48$ m and the tidal current amplitude is $U = 0.25$ m s $^{-1}$. These values differ from the reference values ($H = 30$ m and $U = 1$ m s $^{-1}$) used in this paper. However, only quantitative differences will be obtained for different values, and a comparison with the results presented in this paper gives an indication of the model's predictive capacity. Figure 8 shows that for such small resistance parameters the model predicts a stable flat model, so at this location, predicted behavior and observed bottom behavior do not contradict each other.

Observations near the Dutch coast reported by Visser *et al.* [1994] give $E_v \approx 1.5$; the resistance parameter increases from $\hat{S} \approx 1$ near the coast to $\hat{S} \approx 10$ offshore (A. W. Visser, personal communication, 1995). The model (Figure 8) shows a stable flat bottom for small resistance parameters and a transition to sand waves for larger values of the resistance parameter. The growth rate increases rapidly for increasing resistance parameters. The morphological study by Van Alphen and Damoiseaux [1989] demonstrates that it indeed appears that no regular large-scale bed patterns exist near the coast. However, offshore there are sand waves which increase in amplitude in offshore direction. Since an initially faster growing mode might, but does not necessarily, correspond to larger finite amplitudes, it is concluded here that these comparisons support the model results. One should realize, however, that these two comparisons cannot give more than an indication. Due to the scarcity of experimental results concerning the detailed vertical structure of tidal movements (from which the vertical viscosity and resistance parameter can be estimated), further comparisons should be performed in the future.

The physical mechanisms which cause the different types of wavy patterns are discussed in terms of generated residual flow near the bottom. The mechanism underlying the formation of

tidal sand banks operates due to positive feedback from the residual horizontal torques generated by friction and Coriolis force to the tidal sand banks, as shown by *Huthnance* [1982]. This mechanism can be simulated in this three-dimensional model by choosing parameters such that vertical effects are only weakly present. Sand waves are generated as follows. Residual pressure gradients and residual currents are generated due to the interaction of the basic current and the perturbed tidal components in the water column. Dissipation is concentrated in a layer near the bottom, which determines the uphill residual shear stress. For nonslip at the bottom the crests are perpendicular with respect to the principal tidal current. Less flow resistance at the bottom leads to the rotation of the crests due to the veering of the currents in the vertical direction. At low resistance the crests become nearly parallel to the main flow direction. The formation of sand waves is due to three-dimensional effects and thus cannot be described by a depth-averaged model. However, if the three-dimensional mechanisms responsible for sand waves are known, they can be included parametrically in a depth-averaged model as shown by *Hulscher et al.* [1993] and extended by *De Swart and Hulscher* [1995].

It has been shown explicitly that sand waves are generated by tidal movements. This was suggested by *Belderson* [1986], but so far it has not been demonstrated explicitly [*Fredsoe and Deigaard*, 1993]. In this model the migration of sand waves is absent, due to the choice of a symmetric basic tide which never generates migration of bed forms. *Fredsoe and Deigaard* [1993] presented a model for sand wave migration due to a steady current, without tidal influence. This implies that in order to model sand waves accurately, both tidal and residual currents should be incorporated.

This study is restricted to a linear stability analysis, which means that the height of the bed forms has to be small and formally infinitesimal and their shapes are sinusoidal. As such a topography corresponds to small-slope features, flow separation does not occur and therefore there is no need to incorporate it in this study. This in contradistinction with the finite amplitude sand waves for which the possibility of flow separation has to be investigated in further detail. If it turns out that flow separation can be neglected, one can study the sand wave behavior by performing a nonlinear stability analysis of the sand wave perturbations. A nonlinear stability analysis can allow for the interaction of unstable modes and yields information on the finite height and shape of the bed forms. The linear analysis in this paper shows that for small Stokes numbers in combination with slip at the bottom, a critical mode can be found. This might be a starting point for a weakly nonlinear stability analysis. Since the height of the sand waves in relation to the local water depth is usually of order 0.3 or less and the fastest growing mode has characteristics of sand waves, it should be possible to describe the shape of these phenomena by weakly nonlinear theory. This method has been used to describe the topography of sand ripples [see *Blondeaux*, 1990; *Vittori and Blondeaux*, 1990, 1992], and the results agree reasonably well with experiments and numerical flow simulations. *Komarova and Newell* [1995] suggested that such an analysis concerning sand waves should include a description of the interactions between unstable bed forms with $\mathbf{k} \neq 0$ and the neutrally stable infinite long wave $\mathbf{k} = 0$.

The three-dimensional model shows that vertical effects have to be included to explain the generation of sand waves due to tide-topography interactions. The parameters express-

ing the vertical profile structure yield regimes for discriminating between types of bottom behavior. This might help us to understand why there are locations where only tidal sand banks occur or where only sand waves occur or locations where both or neither of these bed forms are observed.

Appendix A: Solution of the Basic State

The basic state is now computed. The method is similar to that used by *Prandle* [1982] and *Visser et al.* [1994]. The starting point is the equations that describe the dynamics of the basic solution, (28)–(30), with the boundary conditions (31) and (32). The solution will be presented in tidal components (used in the paper) as well as in elliptical properties, the latter being more appropriate for physical interpretation.

The first step is to solve the vertical structure of the basic tide which is driven by pressure gradients. In section 3, equation (33), it was argued that at order of the basic state the vertical velocity equals zero. The set of equations thus becomes

$$\frac{\partial u_0}{\partial t} - \hat{f}v_0 = -\frac{\partial \zeta_0}{\partial x} + \frac{E_v}{2} \frac{\partial^2 u_0}{\partial z^2}, \quad (\text{A1})$$

$$\frac{\partial v_0}{\partial t} + \hat{f}u_0 = -\frac{\partial \zeta_0}{\partial y} + \frac{E_v}{2} \frac{\partial^2 v_0}{\partial z^2}. \quad (\text{A2})$$

The boundary conditions at $z = 0$ are

$$\frac{\partial u_0}{\partial z} = \frac{\partial v_0}{\partial z} = 0, \quad (\text{A3})$$

and at the bottom $z = -1$

$$E_v \frac{\partial u_0}{\partial z} = \hat{S}u_0 \quad E_v \frac{\partial v_0}{\partial z} = \hat{S}v_0. \quad (\text{A4})$$

The time behavior of the basic solution is prescribed as an M_2 tide, and subsequently, the following complex variables are introduced:

$$(u_0, v_0) = \frac{1}{2} \{ [\hat{u}_+(z) + \hat{u}_-(z)], i[\hat{u}_-(z) - \hat{u}_+(z)] \} \exp it + c.c., \quad (\text{A5})$$

$$\nabla \zeta_0 = -\frac{1}{2} \{ [\hat{\zeta}_+(z) + \hat{\zeta}_-(z)], i[\hat{\zeta}_-(z) - \hat{\zeta}_+(z)] \} \exp it + c.c.,$$

in which *c.c.* means the complex conjugate expression. The problem in complex variables becomes

$$i(1 \pm \hat{f})\hat{u}_\pm = \hat{\zeta}_\pm + E_v \frac{d^2 \hat{u}_\pm}{dz^2}, \quad (\text{A6})$$

with boundary conditions at $z = 0$

$$\frac{d\hat{u}_\pm}{dz} = 0 \quad (\text{A7})$$

and at $z = -1$

$$E_v \frac{d\hat{u}_\pm}{dz} = \hat{S}\hat{u}_\pm. \quad (\text{A8})$$

It is preferable to express the vertical structure of the horizontal currents in terms of depth-averaged flow quantities than to express it in pressure gradients ($\hat{\zeta}_\pm$). If the depth-averaged velocity equals $[U_0(t), V_0(t)] = (\sin t, \beta \cos t)$, the solution becomes

$$\hat{u}_{\pm}(z) = \frac{i}{2}(-1 \pm \beta) \cdot \alpha_{\pm}^{-2} \left(\frac{\cosh(\alpha_{\pm} z) - \cosh(\alpha_{\pm}) - \frac{\alpha_{\pm} E_v}{\hat{S}} \sinh \alpha_{\pm}}{\left(\frac{1}{\alpha_{\pm}} - \frac{\alpha_{\pm} E_v}{\hat{S}} \right) \sinh \alpha_{\pm} - \cosh \alpha_{\pm}} \right), \quad (\text{A9})$$

in which $\alpha_{\pm}$ is defined as

$$\alpha_{\pm} = (1 + i) \sqrt{\frac{1 \pm \hat{f}}{E_v}}. \quad (\text{A10})$$

This expression gives $\hat{u}_{\pm}$, from which the basic flow field can be described in either tidal components or in elliptical properties.

A.1. Tidal Components

The basic solution will here be expressed in M_2 tidal components, which is the notation used in this paper. From definitions (35), (36), and (A5) one gets

$$\begin{aligned} u_s(z) &= -\frac{1}{2} \text{Im}[\hat{u}_+(z) + \hat{u}_-(z)], \\ u_c(z) &= \frac{1}{2} \text{Re}[\hat{u}_+(z) + \hat{u}_-(z)], \\ v_s(z) &= \frac{1}{2} \text{Re}[\hat{u}_+(z) - \hat{u}_-(z)], \\ v_c(z) &= \frac{1}{2} \text{Im}[\hat{u}_+(z) - \hat{u}_-(z)]. \end{aligned} \quad (\text{A11})$$

A.2. Elliptical Properties

The horizontal flow field can be described in elliptical properties, instead of in tidal components. This paper uses definitions and derivation given by *Visser et al.* [1994]. Elliptical quantities are physically more suitable for describing the flow field. Following (A5), the elliptical tide is decomposed into two circular phasors, one rotating cyclonically, the other anticyclonically.

When both phasors are rectified, the semimajor axis of the ellipse is reached. The length of the semi-major axis is

$$A_{\text{maj}}(z) = |\hat{u}_+(z)| + |\hat{u}_-(z)|. \quad (\text{A12})$$

This gives the maximum value of the tidal velocity vector at a certain depth. If the depth-averaged tide is given by $(U, V) = (\sin t, \beta \cos t)$, the depth-averaged value of $A_{\text{maj}}(z)$ equals 1.

When both phasors are opposite, the semiminor axis of the ellipse is reached. The length of the semiminor axis is given by

$$A_{\text{min}}(z) = |\hat{u}_+(z)| - |\hat{u}_-(z)|. \quad (\text{A13})$$

This gives the minimum value of the tidal velocity vector at a certain depth. The ellipticity ε is defined as the quotient of both elliptical axes

$$\varepsilon(z) = \frac{A_{\text{min}}(z)}{A_{\text{maj}}(z)} = \frac{|\hat{u}_+(z)| - |\hat{u}_-(z)|}{|\hat{u}_+(z)| + |\hat{u}_-(z)|}. \quad (\text{A14})$$

If the tide is oscillating unidirectionally, $\varepsilon = 0$, and both phasors have equal length. If $\varepsilon > 0$, the cyclonic phasor has a larger amplitude than the anticyclonic phasor, and the tide is cyclonic. A negative ellipticity corresponds to an anticyclonic tide. *Hulscher et al.* [1993] describe the ellipticity as β . The relation between these parameters is $\beta = -\varepsilon$. For a depth-

averaged tide with ellipticity β , the depth-averaged value of $\varepsilon(z)$ equals $-\beta$.

The angle with respect to the x axis at which the maximum of the velocity vector is reached is given by $\Theta(z)$ and is defined uniquely if one assumes $\Theta(z) \in < -\frac{1}{2}\pi, \frac{1}{2}\pi$. This angle can be computed using

$$\Theta(z) = \frac{1}{2} \{ \arg[\hat{u}_+(z)] + \arg[\hat{u}_-(z)] \}. \quad (\text{A15})$$

If a depth-averaged tide is assumed to be given by $(U, V) = (\sin t, \beta \cos t)$, the depth-averaged absolute value of $\Theta(z)$ must be 0.

The time at which the maximum of this velocity vector is reached is given by $\Psi(z)$, which has a value between 0 and 2π , the duration of the tidal period. This can be calculated as follows

$$\Psi(z) = \frac{1}{2} \{ \arg[\hat{u}_+(z)] - \arg[\hat{u}_-(z)] \}. \quad (\text{A16})$$

The definition is unique if $\Psi(z) \in < -\frac{1}{2}\pi, \frac{1}{2}\pi$. If a depth-averaged tide is given by $(U, V) = (\sin t, \beta \cos t)$, the depth-averaged absolute value of $\Psi(z)$ must be $\frac{1}{2}\pi$.

Appendix B: The Vertical Structure of the Linear Problem

The resulting set of equations of the linear stability problem derived in section 4 describes the dependence of the flow field on the vertical coordinate. The equations for the perturbed tidal flow components (as defined in (47)–(50)) are

$$\begin{aligned} \frac{E_v}{2} a''_0 &= -\frac{1}{2} (ku_s + lv_s) a_s - \frac{1}{2} (ku_c + lv_c) a_c \\ &\quad - \hat{f} b_0 + \frac{1}{2} u'_s c_s + \frac{1}{2} u'_c c_c - kd_0, \end{aligned} \quad (\text{B1})$$

$$\frac{E_v}{2} a''_s = (ku_s + lv_s) a_0 - a_c - \hat{f} b_s + u'_s c_0 + kd_s, \quad (\text{B2})$$

$$\frac{E_v}{2} a''_c = (ku_c + lv_c) a_0 + a_s - \hat{f} b_c + u'_c c_0 + kd_c, \quad (\text{B3})$$

$$\begin{aligned} \frac{E_v}{2} b''_0 &= \hat{f} a_0 - \frac{1}{2} (ku_s + lv_s) b_s - \frac{1}{2} (ku_c + lv_c) b_c + \frac{1}{2} v'_s c_s \\ &\quad + \frac{1}{2} v'_c c_c - ld_0, \end{aligned} \quad (\text{B4})$$

$$\frac{E_v}{2} b''_s = \hat{f} a_s + (ku_s + lv_s) b_0 - b_c + v'_s c_0 + ld_s, \quad (\text{B5})$$

$$\frac{E_v}{2} b''_c = \hat{f} a_c + (ku_c + lv_c) b_0 + b_s + v'_c c_0 + ld_c, \quad (\text{B6})$$

$$c'_0 = -ka_0 - lb_0, \quad (\text{B7})$$

$$c'_s = ka_s + lb_s, \quad (\text{B8})$$

$$c'_c = ka_c + lb_c, \quad (\text{B9})$$

in which d_0 , d_s , and d_c are unknown constants. The boundary conditions at $z = 0$ are

$$\frac{\partial a_0}{\partial z} = \frac{\partial a_s}{\partial z} = \frac{\partial a_c}{\partial z} = \frac{\partial b_0}{\partial z} = \frac{\partial b_s}{\partial z} = \frac{\partial b_c}{\partial z} = c_0 = c_s = c_c = 0, \quad (\text{B10})$$

and at the bottom $z = -1$ they are

$$\begin{aligned} a'_s &= \frac{\hat{S}}{E_v} (a_s + u'_s) - u''_{ss}, & a'_c &= \frac{\hat{S}}{E_v} (a_c + u'_c) - u''_{cs}, \\ a_0 &= b_0 = c_0 = 0, \\ b'_s &= \frac{\hat{S}}{E_v} (b_s + v'_s) - v''_{ss}, & b'_c &= \frac{\hat{S}}{E_v} (b_c + v'_c) - v''_{cs}, \\ c_s &= -(ku_s + lv_s) & c_c &= -(ku_c + lv_c). \end{aligned} \quad (\text{B11})$$

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References

- Belderson, R. H., Offshore tidal and non-tidal sand ridges and sheets: Differences in morphology and hydrodynamic setting, in *Shelf Sands and Sandstones*, edited by R. J. Knight and J. R. McLean, *Mem. Can. Soc. Pet. Geol.*, 2, 293–301, 1986.
- Blondeaux, P., Sand ripples under sea waves, 1, Ripple formation, *J. Fluid Mech.*, 218, 1–17, 1990.
- Boczar-Karakiewicz, B., and J. L. Bona, Wave-dominated shelves: A model of sand ridge formation by progressive infragravity waves, in *Shelf Sands and Sandstones*, edited by R. J. Knight and J. R. McLean, *Mem. Can. Soc. Pet. Geol.*, 2, 163–179, 1986.
- De Swart, H. E., and S. J. M. H. Hulscher, Dynamics of large-scale bed forms in coastal seas, in *Nonlinear Dynamics and Pattern Formation in the Natural Environment*, edited by A. Doelman and A. van Harten, pp. 315–331, Longman, White Plains, N. Y., 1995.
- De Vriend, H. J., Morphological processes in shallow tidal seas, in *Residual Currents and Long-Term Transport, Coastal Estuarine Stud.*, vol. 38, edited by R. T. Cheng, pp. 276–301, AGU, Washington, D. C., 1990.
- Dyer, K. R., *Coastal and Estuarine Sediment Dynamics*, pp. 180–181, John Wiley, New York, 1986.
- Fredsoe, J., and R. Deigaard, *Mechanics of Coastal Sediment Transport*, *Adv. Ser. Ocean Eng.*, vol. 3, pp. 260–289, World Sci., River Edge, N. J., 1993.
- Hulscher, S. J. M. H., H. E. De Swart, and H. J. De Vriend, The generation of offshore tidal sand banks and sand waves, *Cont. Shelf Res.*, 13(11), 1183–1204, 1993.
- Huntley, D. A., J. M. Huthnance, M. B. Collins, C.-L. Liu, R. J. Nicholls, and C. Hewitson, Hydrodynamics and sediment dynamics of North Sea sand waves and sand banks, *Philos. Trans. R. Soc. London A*, 343, 461–474, 1993.
- Huthnance, J., On one mechanism forming linear sand banks, *Estuarine Coastal Shelf Sci.*, 4, 79–99, 1982.
- Komarova, N. L., and A. C. Newell, The mean flows driven by sandbar instabilities, in *Nonlinear Dynamics and Pattern Formation in the Natural Environment*, edited by A. Doelman and A. van Harten, pp. 147–167, Longman, White Plains, N. Y., 1995.
- Maas, L. R. M., and J. J. M. van Haren, Observations on the vertical structure of tidal and inertial currents in the central North Sea, *J. Mar. Res.*, 45, 293–318, 1987.
- Maas, L. R. M., and J. T. F. Zimmerman, Tide-topography interactions in a stratified shelf sea I, Basic equations for quasi-nonlinear internal tides, *Geophys. Astrophys. Fluid Dyn.*, 45, 1–35, 1989.
- Off, T., Rhythmic linear sand bodies caused by tidal currents, *Am. Assoc. Pet. Geol. Bull.*, 47, 324–341, 1963.
- Pedlosky, J., *Geophysical Fluid Dynamics*, 2nd ed., chap. 4, Springer-Verlag, New York, 1987.
- Prandle, D., The vertical structure of tidal currents, *Geophys. Astrophys. Fluid Dyn.*, 22, 29–48, 1982.
- Soulsby, R. L., Tidal-current boundary layers, in *The Sea*, vol. 9, part A, *Ocean Engineering Science*, edited by B. Le Mehaute and D. M. Hanes, pp. 523–566, John Wiley, New York, 1990.
- Van Alphen, J. S. L. J., and M. A. Damoiseaux, A geomorphological map of the Dutch shoreface and adjacent part of the continental shelf, *Geol. Mijnbouw*, 68, 433–444, 1989.
- Van de Meene, J., The shoreface connected ridges along the central Dutch coast, 256 pp., *Neth. Geogr. Stud.* 174, Kon. Ned. Aardrijks. Genoot., Utrecht, 1994.
- Van Rijn, L. C., *Handbook of Sediment Transport by Currents and Waves*, Aqua, Amsterdam, Netherlands, 1993.
- Visser, A. W., A. J. Souza, K. Hessner, and J. H. Simpson, The effect of stratification on tidal current profiles in a region of freshwater influence, *Oceanol. Acta*, 17(4), 369–381, 1994.
- Vittori, G., and P. Blondeaux, Sand ripples under sea waves, 2, Finite amplitude development, *J. Fluid Mech.*, 218, 19–39, 1990.
- Vittori, G., and P. Blondeaux, Sand ripples under sea waves, 3, Brick pattern ripple formation, *J. Fluid Mech.*, 239, 23–45, 1992.
- Zimmerman, J. T. F., Dynamics, diffusion and geomorphological significance of tidal residual eddies, *Nature*, 290, 549–555, 1981.
- Zimmerman, J. T. F., Principal differences between 2D- and vertically averaged 3D-models of topographic tidal rectification, in *Physics of Shallow Estuaries and Bays, Coastal Estuarine Stud.*, vol. 16, edited by J. van de Kreeke, pp. 120–129, AGU, Washington, D. C., 1986.

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