Degenerate polynomial patches of degree 11 for almost GC^2 interpolation over triangles

Bas J. Hogervorst and Ruud van Damme

*Department of Applied Mathematics, University of Twente, P.O. Box 217,
7500 AE Enschede, The Netherlands*

We consider the problem of interpolating scattered data in $\mathbb{R}^3$ by an *almost* geometrically smooth GC^2 surface, where almost GC^2 means GC^2 except in a finite number of points (the vertices), where the surface is GC^1 . A local method is proposed, based on employing so-called degenerate triangular Bernstein–Bézier patches. We give an analysis of quintic patches for GC^1 and patches of degree eleven for almost GC^2 interpolation.

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1. Introduction

In this paper we consider the problem of finding a *geometrically smooth* surface interpolating *position* and *direction* given in a finite number of data points in three-dimensional Euclidean space. So, we are given a set of distinct vectors called *vertices*, say

$$V = \{v^i \in \mathbb{R}^3 | i = 1, \dots, v\},$$

and a corresponding set of *normals*, associated with the vertices. In case the normals are not given, they can be estimated (and the locality condition will necessarily be somewhat more loose). This is a problem of its own and we shall not give attention to how it can be solved. We assume that a *triangulation*, which defines the topology of the reconstructed surface, is given. For information on methods to construct a suitable triangulation we refer to the paper of Schumaker [13] and references therein. Every triangulation of the vertices defines a piecewise linear interpolating surface. Hence, the problem we consider can also be viewed as *smoothing* of this piecewise linear surface.

We propose a *parametric* patch method which has the following properties: (i) the data may be *scattered*, (ii) the interpolating surface is *geometrically twice differentiable almost everywhere* (AGC^2), and (iii) the scheme is *local*, by which we

mean that each patch only depends on data associated with its corresponding triangle from the triangulation.

First, in section 2, we give some prerequisites which will be used throughout the remainder of this paper. An analysis for the construction of a local GC^1 scheme can be found in section 3. By GC^1 we mean geometric smoothness of order one or *oriented tangent plane continuity*, in accordance with commonly accepted terminology [1,5]. The scheme combines the ideas of two papers, one by Cottin and Van Damme [2], the other by Pfluger and Neamtu [12]. Both papers propose a local GC^1 scheme. Cottin and Van Damme [2] use a triangular split on which they construct a piecewise cubic surface, whereas Pfluger and Neamtu [12] employ so-called *degenerate patches*. These are patches with coalescent control points in the vertices. (Then the surface contains points which are non-regular; hence the notion of GC^1 cannot be formulated by the existence of a regular parametrization.)

Then, having obtained a GC^1 scheme, it is natural to ask ourselves if we can use the same techniques to construct a GC^2 scheme. This turns out not to be the case. However, if we somewhat weaken the condition of GC^2 smoothness, the question can be answered in the affirmative; the analysis hereof can be found in section 4. Section 5 provides some concluding remarks.

2. Preliminaries

Let us introduce some notation and definitions which will be used throughout the sequel. For more details, proofs and background information the reader is referred to any of the standard papers such as [4] or [6]. We use boldface notation to indicate points or vectors in $\mathbb{R}^3$, so $\mathbf{x} \in \mathbb{R}^3$. Superscripts are used to number vectors $\mathbf{x}^i$ ($i = 0, \dots, m; m \geq 0$). Further, we use standard multi-index notation, i.e. for $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}_+^3$, $|\alpha| = \alpha_0 + \alpha_1 + \alpha_2$, $\alpha! = \alpha_0! \alpha_1! \alpha_2!$ and for $\tau = (\tau_0, \tau_1, \tau_2) \in \mathbb{R}^3$ we have $\tau^\alpha = \tau_0^{\alpha_0} \tau_1^{\alpha_1} \tau_2^{\alpha_2}$. The *standard 2-simplex* is denoted by S and will be thought of as the unit triangle in $\mathbb{R}^3$ with vertices $\mathbf{e}^0 = (0, 0, 1)$, $\mathbf{e}^1 = (1, 0, 0)$, $\mathbf{e}^2 = (0, 1, 0)$. Then, any polynomial on a triangular patch can be considered as a map from S . The *Bernstein–Bézier basis functions* are defined by

$$B_\alpha(\tau) := \frac{|\alpha|!}{\alpha!} \tau^\alpha, \quad \alpha \in \mathbb{Z}_+^3.$$

For representing a polynomial $p(\mathbf{x})$ of degree n , we express the location of a point $\tau = (\tau_0, \tau_1, \tau_2) \in S$, which are the usual *barycentric coordinates* of $\mathbf{x}$ with respect to S , as the value of a Bernstein–Bézier polynomial:

$$p(\mathbf{x}) = B_n[\mathbf{c}](\tau) := \sum_{|\alpha|=n} \mathbf{c}_\alpha B_\alpha(\tau), \quad (1)$$

where $\mathbf{c}_\alpha \in \mathbb{R}^3$ are *vector-valued Bézier coefficients* or *control points*. The set $\mathbf{c} := \{\mathbf{c}_\alpha\}_{|\alpha|=n}$ is called the *Bézier net* or *control polygon*. We shall need the well-

known formula which expresses the directional derivatives of a Bernstein polynomial in terms of other Bernstein polynomials of lower degree,

$$D_{v^i - v^j} B_n[c](x) = n \sum_{|\alpha|=n-1} (E_i - E_j) c_\alpha B_\alpha(x), \quad (2)$$

where $E_i c_\alpha := c_{\alpha + e^i}$, $i = 0, 1, 2$, denotes the i th shift operator. Likewise, higher order derivatives are conveniently expressed in terms of differences of the control points.

Note that the definition of the basis functions immediately implies

$$p(v^i) = B_n[c](e^i) = c_{ne^i}, \quad i = 0, 1, 2,$$

so that the polynomial interpolates the “extreme” coefficients c_{ne^i} . Thus, in particular,

$$D_{v^i - v^j} p(v^j) = n(c_{(n-1)e^j + e^i} - c_{ne^j}).$$

3. A quintic GC^1 scheme

A parametric piecewise polynomial surface is of GC^1 smoothness if it is continuous and the surface normals of abutting patches are uniquely defined and agree at every point of the boundary. We have the following algebraic characterization of the necessary and sufficient constraints for oriented tangent plane continuity [11].

Consider a patch p , parametrized by u and v , and its neighbour patch q , parametrized by u and w and the definitions

$$p_u := \frac{\partial}{\partial u} p(u, 0), \quad p_v := \frac{\partial}{\partial v} p(u, 0) \quad \text{and} \quad q_w := \frac{\partial}{\partial w} q(u, 0).$$

Then p and q are part of a GC^1 -surface if and only if there exist scalar-valued weight functions λ, μ and ν such that, at each point of the boundary,

$$\lambda p_u = \mu p_v + \nu q_w \quad [\text{common tangent plane}], \quad (3)$$

$$p_v \times p_u \neq 0 \quad [\text{regularity}], \quad (4)$$

$$\mu\nu > 0 \quad [\text{proper orientation}]. \quad (5)$$

The weight functions may be chosen as polynomials of degree at most the degree of p and q [11]. Once the degree of the interpolant is chosen, the degrees of λ , μ and ν characterize the transition (match) between the patches. Most developed schemes use a match $q_w = \alpha p_u + \beta p_v$, where α and β are restricted to be polynomials instead of rationals (cf. [10, table 1]). Much can be gained by allowing all three weight functions to be non-constant.

We remark that, in general, (3) forms a *nonlinear system of equality and inequality constraints* in the coefficients of p , q , λ , μ and ν . An important feature of any surface construction is the selection of (preferably geometrically meaningful) coefficients to be fixed a priori (input or derived from the data), so as to arrive at a sufficient and consistent sequence of local and linear constraints.

We employ the Bernstein–Bézier form to represent the polynomial triangular patches (cf. section 2, definition 1), because this form gives easy access to value and derivative information along the boundaries of a patch, as well as geometric meaning to its coefficients (cf. [6, 3]). To construct the interpolating surface we determine the control points. The equality constraint, applied to patches in Bernstein–Bézier form, involves interior control points both along the patch boundaries and around the data points, thus linking the corresponding system of equations globally. In order to reduce this system to local blocks, we fix some degrees of freedom. This is done in such a way that the constraints become *linear and symmetrical* in the remaining degrees of freedom along the boundaries. In disposing of the ambiguity we would like to meet at least three objectives: (i) the interpolant is *independent of the labeling* of the vertices; (ii) the *additional conditions are simple*; (iii) the interpolant is *visually pleasing*. The first two objectives are met by taking the weight functions in (3) in an appropriate way by noting that λ as defined in (9) belongs to the derivative along and μ, ν belong to the cross-boundary derivative of a common edge of the patches p and q . Of these conditions the third is hardest to quantify and to enforce. Attempts were made to dispose of the remaining degrees of freedom by requiring that the interpolant minimizes the norm of certain distances subject to interpolation.

Consider two patches of degree n ,

$$p = B_n[P] \text{ and } q = B_n[Q], \quad (6)$$

defined by the control points $P = \{P_\alpha\}_{|\alpha|=n}$, $Q = \{Q_\alpha\}_{|\alpha|=n}$, respectively. The patches p and q join continuously if (and only if) they are equal on the common edge, i.e. if $p(\tau) = q(\tau)$, $\tau_0 = 0$, $\tau_1 + \tau_2 = 1$. Therefore, we take

$$P_\alpha = Q_\alpha, \quad \alpha_0 = 0, \alpha_1 + \alpha_2 = n.$$

We shall employ directional derivatives in specific directions depending on the triangulation. These derivatives are those *along* the common edge of two adjacent patches and the *cross-boundary* derivative, denoted by $D_{\parallel}$ and $D_{\perp}$, respectively. So, using the directions $(v^2 - v^1)$ and $(v^0 - v^1) + (v^0 - v^2)$, we define

$$D_{\parallel} = D_{v^2-v^1}, \quad D_{\parallel} = D_{v^1-v^0} + D_{v^2-v^0}. \quad (7)$$

The degeneracy as proposed by Pfluger and Neamtu [12] means that on each patch at each vertex the control point in the vertex coincides with the nearest control point on both the two edges having that vertex in common, so we have three coalescent control points. If so, then GC^1 over the edges implies GC^1 in the vertices [12] and it is not difficult to construct a local scheme.

In this section we look at quintic patches ($n = 5$), as in this case we have enough flexibility to construct a GC^1 scheme which can handle not too restricted data. Employing a degenerate quintic patch amounts to setting

$$P_{4,1,0} = P_{5,0,0} = P_{4,0,1}, \quad P_{1,4,0} = P_{0,5,0} = P_{0,4,1}, \quad P_{1,0,4} = P_{0,0,5} = P_{0,1,4},$$

leaving two free control points on the boundary. In order to increase readability we make a change of variables (cf. figure at the end of this section):

$$P_{1,3,1} = r^1, P_{1,2,2} = r, P_{1,1,3} = r^2,$$

$$P_{0,5,0} = v^1, P_{0,3,2} = s^1, P_{0,2,3} = s^2, P_{0,0,5} = v^2,$$

$$Q_{0,5,0} = v^1, Q_{0,3,2} = s^1, Q_{0,2,3} = s^2, Q_{0,0,5} = v^2,$$

$$Q_{1,3,1} = t^1, Q_{1,2,2} = t, Q_{1,1,3} = t^2.$$

Unknowns are the control points $r^1, r, r^2, s^1, s^2, t^1, t, t^2$, whereas the values of v^1 and v^2 are assumed given.

For GC^1 , the first derivatives of patches p and q along and over the edges have to be coplanar, or, for some functions λ, μ and ν , eq. (3) has to be satisfied, or

$$\lambda D_{\parallel} p = \mu D_{\perp} p + \nu D_{\perp} q. \quad (8)$$

For constructing a quintic surface, it is not difficult to see that we can take linear λ, μ and ν .

We require the method to be independent of the labeling of the vertices. Remark that λ belongs to the “parallel” and μ, ν belong to the “perpendicular” derivative, that $D_{\perp}$ is symmetric and that $D_{\parallel}$ is anti-symmetric on the common edge. Then, relabeling the control points up-down, i.e. replacing (r^1, r, r^2) by (t^1, t, t^2) and vice versa while exchanging μ and ν , must yield the same solution; this must also be the case if the control points are relabeled left-right, i.e. exchanging the indices 1 and 2 of the control points while replacing $(\mu_1, \nu_1, \lambda_1)$ by $(\mu_2, \nu_2, \lambda_2)$ and vice versa, where μ_i, ν_i and λ_i denote the coefficients of the linear weight functions. We call the transformation up-down T_{UD} and the exchanging of the indices T_{LR} .

By defining λ, μ, ν as

$$\lambda = \lambda_1 \tau_1 - \lambda_2 \tau_2, \quad \mu = \mu_1 \tau_1 + \mu_2 \tau_2, \quad \nu = \nu_1 \tau_1 + \nu_2 \tau_2, \quad (9)$$

the independence of the labeling of the control points is automatically met. This means that the method will be invariant under the transformations T_{UD} ,

$$\begin{aligned} & (\mathbf{r}^1, \mathbf{t}^1, \mathbf{r}^2, \mathbf{t}^2, \mu_1, \mu_2, \nu_1, \nu_2) \\ & \rightarrow (\mathbf{t}^1, \mathbf{r}^1, \mathbf{t}^2, \mathbf{r}^2, \nu_1, \nu_2, \mu_1, \mu_2), \end{aligned} \quad (10)$$

and T_{LR} ,

$$\begin{aligned} & (v^1, s^1, s^2, v^2, \mathbf{r}^1, \mathbf{t}^1, \mathbf{r}^2, \mathbf{t}^2, \mu_1, \mu_2, \nu_1, \nu_2, \lambda_1, \lambda_2) \\ & \rightarrow (v^2, s^2, s^1, v^1, \mathbf{r}^2, \mathbf{t}^2, \mathbf{r}^1, \mathbf{t}^1, \mu_2, \mu_1, \nu_2, \nu_1, \lambda_2, \lambda_1). \end{aligned} \quad (11)$$

We substitute the expressions for two adjacent patches (6) and linear weight functions as defined above in (8). Then, taking into account the degeneracy, applying the change of variables and defining the functions $\mathcal{F}_j, j = 1, 2$, for which $T_{UD}\mathcal{F}_j = \mathcal{F}_j$, by

$$\begin{aligned} \mathcal{F}_1(\mathbf{P}, \mathbf{Q}, \lambda, \mu, \nu) &:= \lambda_1(s^1 - v^1) + \mu_1(s^1 - 2\mathbf{r}^1 + v^1) + \nu_1(s^1 - 2\mathbf{t}^1 + v^1), \\ \mathcal{F}_2(\mathbf{P}, \mathbf{Q}, \lambda, \mu, \nu) &:= -\lambda_2(s^1 - v^1) + \mu_2(s^1 - 2\mathbf{r}^1 + v^1) + \nu_2(s^1 - 2\mathbf{t}^1 + v^1), \\ &\quad + \xi_n(-\lambda_1(s^2 - s^1) + \mu_1(s^2 - 2\mathbf{r} + s^1) + \nu_1(s^2 - 2\mathbf{t} + s^1)), \end{aligned}$$

where $\xi_n = (n+1)/(n-1)$, gives the equations for the control points of the degenerate quintic patches p and q to join GC^1 :

$$T_{UD}\mathcal{F}_j = 0, \quad T_{LR}\mathcal{F}_j = 0, \quad j = 1, 2. \quad (12)$$

The equations for $j = 1$ can be solved for s^i , yielding

$$s^i = v^i + \tilde{\mu}_i(\mathbf{r}^i - v^i) + \tilde{\nu}_i(\mathbf{t}^i - v^i), \quad i = 1, 2, \quad (13)$$

where $\tilde{\mu}_i = 2\mu_i/(\lambda_i + \mu_i + \nu_i)$, $\tilde{\nu}_i = 2\nu_i/(\lambda_i + \mu_i + \nu_i)$, $i = 1, 2$.

The equations for $j = 2$ can be thought of as a linear system for $\mathbf{r}$ and $\mathbf{t}$ with determinant

$$\Delta = \mu_1\nu_2 - \mu_2\nu_1.$$

In case of a linearly independent system we can find $\mathbf{r}$ and $\mathbf{t}$, after inserting the expressions for $s^i, i = 1, 2$, in terms of $v^i, \mathbf{r}^i$ and $\mathbf{t}^i$. If $\Delta = 0$, however, a zero denominator appears, which we shall tackle giving a condition on λ_i, μ_i and ν_i ($i = 1, 2$) which guarantees the numerator to be zero too. Each coefficient of the control points $v^i, \mathbf{r}^i, \mathbf{t}^i$ must have a finite limit of unity for $(\mu_1\nu_2)/(\mu_2\nu_1)$. Introducing a new variable for μ_1/ν_1 , taking μ_2/ν_2 also equal to this variable in the numerator of $\mathbf{r}$ (and of $\mathbf{t}$) and looking at the coefficients of $v^i, \mathbf{r}^i, \mathbf{t}^i$, we find that there is only one sensible choice to make the numerator zero.

If $\lambda_1\nu_2 + \lambda_2\nu_1 = 0$, then the system (12) with $j = 2$ is solvable for $\mathbf{r}$ and $\mathbf{t}$.

We still have the requirement of independence of the labeling of the control points, which the weight functions have inherited. Therefore, we give a condition which is anti-symmetric in λ and symmetric in μ, ν if the system for $\mathbf{r}$ and $\mathbf{t}$ is linearly independent and that deteriorates to $\lambda_1\nu_2 + \lambda_2\nu_1 = 0$ in case $\Delta = 0$:

$$\lambda_1/(\mu_1 + \nu_1) + \lambda_2/(\mu_2 + \nu_2) = 0, \quad (14)$$

or, in terms of $\tilde{\mu}_i$ and $\tilde{\nu}_i$,

$$1/(\tilde{\mu}_1 + \tilde{\nu}_1) + 1/(\tilde{\mu}_2 + \tilde{\nu}_2) = 1. \quad (15)$$

As solution of the linear system under the constraint (14) we take $\mathbf{r}$ as

$$\mathbf{r} = \mathbf{s}^{12} + \frac{\nu_2}{\xi_n(\mu_1 + \nu_1)}(\mathbf{r}^1 - \mathbf{t}^1) + \frac{\nu_1}{\xi_n(\mu_2 + \nu_2)}(\mathbf{r}^2 - \mathbf{t}^2), \quad (16)$$

where $\mathbf{s}^{12} = \frac{1}{2}(\mathbf{s}^1 + \mathbf{s}^2) + (\rho/2)(\mathbf{s}^2 - \mathbf{s}^1)$ and $\rho \in \mathbb{R}$ is such that

$$\lambda_1 = \rho(\mu_1 + \nu_1), \quad \lambda_2 = -\rho(\mu_2 + \nu_2). \quad (17)$$

We remark that $\lambda_1 \lambda_2 < 0$, $\tilde{\mu} > 0$ and $\tilde{\nu} > 0$ imply $|\rho| < 1$. Hence (16) will be a very reasonable choice of $\mathbf{r}$, since $\mathbf{r}$ lies approximately “between” $\mathbf{r}^1$ and $\mathbf{r}^2$. Further,

$$T_{UD}\rho = \rho, \quad T_{LR}\rho = -\rho \quad (18)$$

and $\mathbf{t}$ is determined by the transformation T_{UD} . We come to the following method for the construction of a quintic GC^1 surface, given (scattered) position and normal vectors:

- The vertices $\mathbf{v}^i$ are given.
- Choose the control points $\mathbf{r}^i, \mathbf{t}^i$ ($i = 1, 2$) in the tangent plane, characterized by the normals in the vertices.
- Choose $\mathbf{s}^i$, $i = 1, 2$ such that (13) and (15) are satisfied.
- The control points $\mathbf{r}$ and $\mathbf{t}$ are determined by (16) and $\mathbf{t} = T_{UD}\mathbf{r}$.

4. A burptic AGC^2 scheme

Now that we have derived a local GC^1 scheme which uses degenerate patches, we try, in the same spirit, to construct a local GC^2 scheme. For a degenerate patch, GC^1 at the vertices and GC^2 over the edges does not imply GC^2 at the vertices. It can easily be shown that, in general, the curvature of a degenerate patch tends to infinity if we approach a vertex. Therefore, we shall restrict ourselves to a scheme which constructs a surface which will be GC^2 except in the vertices where it will be GC^1 . We call this an almost GC^2 (AGC^2) scheme and this is the “most” we can get using degenerate patches.

Consider two abutting patches p and q of the form (6) which meet in GC^1 with weights λ, μ, ν (see (8)). These patches meet in GC^2 if and only if there exist weight functions α, β, γ and $\delta \neq 0$ such that at each point of the common boundary [11]

$$\delta((v^2 q_{ww} - \lambda \nu q_{wu}) - (\mu^2 p_{vv} - \lambda \mu p_{vu})) = \alpha p_u + \beta p_v - \gamma q_w. \quad (19)$$

As in the previous section, we use derivatives along and across a common edge, i.e.

$$D_{\parallel}, D_{\perp}, D_{\parallel}^2 = D_{\parallel} D_{\parallel}, D_{\perp}^2 = D_{\perp} D_{\perp}, D_{\parallel\perp} = D_{\parallel} D_{\perp} = D_{\perp} D_{\parallel},$$

where $D_{\parallel}$ and $D_{\perp}$ are defined in (7).

Interpolation by a twice differentiable *function* that is polynomial on each triangle of the triangulation would require a polynomial of degree at least nine [14]. To be able to construct a *local* GC^2 scheme we need at least degree ten. Next to the requirement of locality we want to employ degenerate patches with three coalescent control points in the vertices. A corollary of this degeneracy and the GC^2 eq. (19) is that we need patches of at least degree eleven with fifteen (!) coalescent control points in each vertex. We call a patch of degree $n = 11$ a *burptic*¹ patch. So, in this section, we derive a condensed parametric AGC^2 scheme that is piecewise burptic on each triangle and that requires only position and normal data in the vertices. Although, in general, we do agree with Farin (and many others) who remark that “for practical purposes, degrees exceeding ten are prohibitive” [7], in this case polynomials of degree more than ten can be used, because the algorithm we arrive at is relatively simple to implement and, above all, gives good results.

We remark that it is actually possible to derive conditions on the control points so that we *can* construct a proper local GC^2 scheme employing degenerate patches. However, the requirement of locality then implies the use of patches of a considerably higher degree.

In order to increase readability we again make a change of variables:

$$\begin{aligned} P_{2,6,3} &= b^0, P_{2,5,4} = b^1, P_{2,4,5} = b^2, P_{2,3,6} = b^3, \\ P_{1,6,4} &= r^1, P_{1,5,5} = r, P_{1,4,6} = r^2, \\ P_{0,11,0} &= v^1, P_{0,6,5} = s^1, P_{0,5,6} = s^2, P_{0,0,11} = v^2, \\ Q_{0,11,0} &= v^1, P_{0,6,5} = s^1, P_{0,5,6} = s^2, P_{0,0,11} = v^2, \\ Q_{1,6,4} &= t^1, Q_{1,5,5} = t, Q_{1,4,6} = t^2, \\ Q_{2,6,3} &= u^0, Q_{2,5,4} = u^1, Q_{2,4,5} = u^2, Q_{2,3,6} = u^3. \end{aligned}$$

We shall construct a burptic surface by first fixing the control points on the common edge, viz. v^1, v^2, s^1, s^2 , and those on the “ GC^1 layer”, viz. r^1, r, r^2, t^1, t, t^2 . This is done analogously as in the previous section. With the introduced notation we

¹ This name is a result of our poor Latin education during high school. It was meant to serve as a temporary substitute. However, we began to get accustomed to it.

find that the GC^1 equations are given by (12) and hence the control points $s^i, i = 1, 2$, and r by (13), respectively (16) (then $t = T_{UD}r$). The remaining unknowns, i.e. b^i, u^i ($i = 0, 1, 2, 3$), are then determined by considering the GC^2 eq. (19).

Considering the degrees of the different polynomial parts in (19), taking into account the independence of the labeling of the vertices, we choose the weight functions as (cf. section 3)

$$\delta = 1, \quad \alpha = \alpha_1\tau_1 - \alpha_2\tau_2, \quad \beta = \beta_1\tau_1 + \beta_2\tau_2, \quad \gamma = \gamma_1\tau_1 + \gamma_2\tau_2.$$

Then, the method will be invariant under the transformations T_{UD} ,

$$(10) \quad \text{and} \\ (b^0, b^1, b^2, b^3, u^0, u^1, u^2, u^3, \beta_1, \beta_2, \gamma_1, \gamma_2) \\ \rightarrow (u^0, u^1, u^2, u^3, b^0, b^1, b^2, b^3, \gamma_1, \gamma_2, \beta_1, \beta_2),$$

and T_{LR} ,

$$(11) \quad \text{and} \\ (b^0, b^1, b^2, b^3, u^0, u^1, u^2, u^3, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2) \\ \rightarrow (b^3, b^2, b^1, b^0, u^3, u^2, u^1, u^0, \alpha_2, \alpha_1, \beta_2, \beta_1, \gamma_2, \gamma_1).$$

The analysis above defines (in the same way as for the GC^1 equations) a linear system of six vector equations for the remaining control points of the degenerate burptic patches p and q to join GC^2 ; two of these equations (belonging to the terms of lowest and highest degree of τ_2) are (cf. (12))

$$T_{UD}\mathcal{F}_3 = \mathcal{F}_3 = 0, \quad T_{LR}\mathcal{F}_3 = 0, \quad (20)$$

where

$$\mathcal{F}_3(P, Q, \lambda, \mu, \nu) := \lambda_1\nu_1(v^1 - 2t^1 + s^1) - \lambda_1\mu_1(v^1 - 2r^1 + s^1) \\ + \mu_1^2(v^1 - 4b^0 + 4r^1 - s^1) - \nu_1^2(v^1 - 4u^0 + 4t^1 - s^1).$$

We determine $\mathbf{u}^0, \mathbf{b}^0$ by minimizing the measure $|\mathbf{u}^0 - \mathbf{u}_*^0|^2 + |\mathbf{b}^0 - \mathbf{b}_*^0|^2$, where $\mathbf{u}_*^0, \mathbf{b}_*^0$ are some target points in the tangent plane through the nearest vertex, e.g., coming from a GC^1 interpolant. Writing the first equation of (20) as

$$\mathbf{u}^0 = \zeta \mathbf{b}^0 + \mathbf{z}^0,$$

where $\zeta = \mu_1^2/\nu_1^2$ and

$$\mathbf{z}^0 = \frac{\mu_1^2 - \nu_1^2}{4\nu_1^2} (\rho(\mathbf{v}^1 + \mathbf{s}^1) + (\mathbf{s}^1 - \mathbf{v}^1)) - \frac{\rho + 2}{2\nu_1^2} (\mu_1^2 \mathbf{r}^1 - \nu_1^2 \mathbf{t}^1) - \frac{\rho \mu_1}{2\nu_1} (\mathbf{r}^1 - \mathbf{t}^1)$$

is a point in the tangent plane through $\mathbf{v}^1$, the solution of the minimization problem is

$$\mathbf{u}^0 = (\zeta \mathbf{b}_*^0 + \zeta^2 \mathbf{u}_*^0 + \mathbf{z}^0)/(\zeta^2 + 1), \quad \mathbf{b}^0 = (\mathbf{b}_*^0 + \zeta \mathbf{u}_*^0 - \zeta \mathbf{z}^0)/(\zeta^2 + 1). \quad (21)$$

The control points $\mathbf{u}^3, \mathbf{b}^3$ are determined analogously.

If the remaining GC^2 equations are solved directly with respect to $\mathbf{u}^1, \mathbf{u}^2, \mathbf{b}^1, \mathbf{b}^2$ (i.e. with arbitrary $\alpha_i, \beta_i, \gamma_i$) then, as in the previous section, badly behaved denominators, viz. third powers of $\Delta = \mu_1 \nu_2 - \mu_2 \nu_1$, occur. We derive conditions for $\alpha_i, \beta_i, \gamma_i$ ($i = 1, 2$) in such a way that these cancel automatically.

Let (λ, μ, ν) be part of a solution of the GC^1 eq. (3) and let (α, β, γ) be the corresponding solution of the GC^2 equation. Now, $c(\lambda, \mu, \nu)$, where $c \in \mathbb{R}$ is an arbitrary constant, is also a solution of (3). Obviously, the corresponding solution of (19) must be of the form $c^2(\alpha, \beta, \gamma)$. Therefore, we assume that

$$\alpha, \beta, \gamma \text{ are homogeneous quadratic polynomials in } \lambda, \mu, \nu. \quad (22)$$

The system involving α, β, γ (satisfying assumption (22)) and the remaining control points $\mathbf{u}^1, \mathbf{u}^2, \mathbf{b}^1, \mathbf{b}^2$ was set up and solved using the symbolic manipulation language REDUCE [8]. Eliminating the free parameters yields the following constraints on the coefficients of the weight functions α, β and γ :

$$\begin{aligned} (\mu_1 - \nu_1)\beta_2 - (\mu_2 - \nu_2)\beta_1 &= \Delta(\lambda_1 + \lambda_2), \\ \alpha_1 = 0, \alpha_2 = 0, \gamma_1 &= \beta_1, \gamma_2 = \beta_2. \end{aligned} \quad (23)$$

Thus, with the assumption that α, β, γ are homogeneous quadratics in λ, μ, ν , we obtain a variety of solutions. We found that (23) is also a sufficient condition for a solution to exist. This means that the imposition of (23) enables assumption (22) to be dropped. We write the second condition in a more symmetric form, viz.

$$\begin{aligned} \beta_1 &= \kappa(\mu_1 - \nu_1) + (\mu_1 + \nu_1)(\lambda_1 + \lambda_2)/2, \\ \beta_2 &= \kappa(\mu_2 - \nu_2) + (\mu_2 + \nu_2)(\lambda_1 + \lambda_2)/2, \end{aligned}$$

where κ is a function in μ_i, ν_i, λ_i ($i = 1, 2$), for which $T_{UD}\kappa = -\kappa, T_{LR}\kappa = \kappa$ (cf. (18)). Finally, the remaining GC^2 equations are solved for $\mathbf{u}^1, \mathbf{u}^2, \mathbf{b}^1, \mathbf{b}^2$, which gives as solution

$$\begin{aligned} \mathbf{u}^1 = & \eta_1 \mathbf{v}^1 + \eta_2 \mathbf{s}^1 + \eta_3 \mathbf{s}^2 + \eta_4 \mathbf{v}^2 \\ & + \eta_5 (\mathbf{t}^1 - \mathbf{r}^1) + \eta_6 (\mathbf{t}^2 - \mathbf{r}^2) + \eta_7 (\mathbf{b}^0 + \mathbf{u}^0) + \eta_8 (\mathbf{b}^3 + \mathbf{u}^3), \end{aligned} \quad (24)$$

where the coefficients are given by the following complicated expressions

$$\begin{aligned} \eta_1 &= \frac{(1-\rho)^2}{4} - \frac{2(3+\rho)(1-\rho)\mu_1\mu_2}{3(\mu_1^2 + \nu_1^2)}, \\ \eta_2 &= \frac{1-\rho^2}{2} - \frac{2(1+\rho)^2\mu_1\mu_2}{3(\mu_1^2 + \nu_1^2)}, \\ \eta_3 &= \frac{(1+\rho)^2}{4} - \frac{(1-\rho)^2\mu_1^2}{3(\mu_2^2 + \nu_2^2)}, \\ \eta_4 &= -\frac{(1+\rho)(3-\rho)\mu_1^2}{3(\mu_2^2 + \nu_2^2)}, \\ \eta_5 &= \frac{24\mu_1 - \rho(\nu_2 - \nu_1 + 21\mu_1) - 2\kappa}{24(\mu_1 + \nu_1)} + \frac{\mu_2\nu_1(20 + 17\rho)}{24(\mu_1^2 + \nu_1^2)} \\ &\quad - \frac{\mu_1\mu_2(\mu_1 - \nu_1)(12 + 15\rho)}{24(\mu_1^2 + \nu_1^2)(\mu_1 + \nu_1)}, \\ \eta_6 &= \frac{5(1+\rho)\mu_1}{6(\mu_2 + \nu_2)} - \frac{2(1-\rho)(\mu_2 - \nu_2)\mu_1^2}{3(\mu_2^2 + \nu_2^2)(\mu_2 + \nu_2)}, \\ \eta_7 &= \frac{4\mu_1\mu_2}{3(\mu_1^2 + \nu_1^2)}, \\ \eta_8 &= \frac{2\mu_1^2}{3(\mu_2^2 + \nu_2^2)}. \end{aligned}$$

The expressions for the control points $\mathbf{u}^2$, $\mathbf{b}^1$ and $\mathbf{b}^2$ can be found by the symmetry of the method, using the transformations T_{LR} and T_{UD} .

The algorithm for the construction of a burptic AGC^2 surface, given (scattered) position and normal vectors, now reads

- The vertices $\mathbf{v}^i$ are given.
- Choose the control points $\mathbf{r}^i$, $\mathbf{t}^i$ ($i = 1, 2$) in the tangent plane, characterized by the normals in the vertices.
- Choose $\mathbf{s}^i$, $i = 1, 2$, such that (13) and (15) are satisfied.
- The control points $\mathbf{r}$ and $\mathbf{t}$ are determined by (16) and $\mathbf{t} = T_{UD}\mathbf{r}$.
- The control points $\mathbf{u}^0$, $\mathbf{u}^3$, $\mathbf{b}^0$, $\mathbf{b}^3$ are determined by choosing target points in the appropriate tangent planes and taking a linear combination according to (21).
- Finally, $\mathbf{u}^1$ is calculated from (24) and $\mathbf{u}^2$, $\mathbf{b}^1$ and $\mathbf{b}^2$ follow from the symmetry of the method, i.e. from the transformations T_{LR} and T_{UD} .

5. Discussion

In this paper we have designed interpolation methods based on quintic and burp-tic patches. The former method yields surfaces of GC^1 smoothness, whereas surfaces constructed with the latter are of almost GC^2 smoothness. The use of degenerate patches has enabled us to design schemes which have the advantage of being local in the sense that the control points are determined separately around vertices and along edges of adjacent patches, and, at the same time, to remove much of the overwhelming freedom. The numerical results are quite satisfactory (for some pleasing pictures, see [9]).

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