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Long cycles in graphs with prescribed toughness and minimum degree[☆]

Douglas Bauer[a,1], H.J. Broersma[b,*], J. van den Heuvel[b], H.J. Veldman[b]

[a] Department of Pure and Applied Mathematics, Stevens Institute of Technology, Hoboken, NJ 07030, USA
[b] Faculty of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, Netherlands

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Abstract

A cycle C of a graph G is a D_λ-cycle if every component of $G-V(C)$ has order less than λ. Using the notion of D_λ-cycles, a number of results are established concerning long cycles in graphs with prescribed toughness and minimum degree. Let G be a t-tough graph on $n\geqslant 3$ vertices. If $\delta>n/(t+\lambda)+\lambda-2$ for some $\lambda\leqslant t+1$, then G contains a D_λ-cycle. In particular, if $\delta>n/(t+1)-1$, then G is hamiltonian, improving a classical result of Dirac for $t>1$. If G is nonhamiltonian and $\delta>n/(t+\lambda)+\lambda-2$ for some $\lambda\leqslant t+1$, then G contains a cycle of length at least $(t+1)(\delta-\lambda+2)+t$, partially improving another classical result of Dirac for $t>1$.

Keywords: Hamiltonian graph; (D_λ-)cycle; Toughness; (Minimum) degree; Circumference

1. Introduction

We use [7] for terminology and notation not defined here and consider simple graphs only.

Let G be a graph and λ a positive integer. Following [16], a cycle C of G is called a *D_λ-cycle* if all components of $G-V(C)$ have order less than λ. A D_1-cycle is a *Hamilton cycle*, a D_2-cycle is also called a *dominating cycle*. A graph is *hamiltonian* if it contains a Hamilton cycle. We denote by $\omega_\lambda(G)$ the number of components of G of order at least λ, and we use ω instead of ω_1. As introduced in [8], G is *t-tough* ($t\in\mathbb{R}$, $t\geqslant 0$) if $|S|\geqslant t\cdot\omega(G-S)$ for any subset S of $V(G)$ with $\omega(G-S)>1$. The *toughness* of G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough ($\tau(K_n)=\infty$ for all $n\geqslant 1$). Two subgraphs H_1 and H_2 of G are *remote* if $V(H_1)\cap V(H_2)=\emptyset$ and there is no edge of G joining a vertex of H_1 and a vertex of H_2. We denote by $\alpha_\lambda(G)$ the maximum

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number of pairwise remote connected subgraphs of order λ of G. Thus α_1 coincides with the independence number α. The set of *neighbors of a subgraph* H of G, denoted $N(H)$, is the set of vertices in $V(G)-V(H)$ adjacent to at least one vertex of H; $d(H)=|N(H)|$ is the *degree of a subgraph* H of G. By $\delta_\lambda(G)$ we denote the minimum degree of a connected subgraph of order λ in G, so that δ_1 coincides with the minimum vertex degree δ. We use $c(G)$ to denote the *circumference* of G, i.e., the length of a longest cycle of G. A cycle C of G is called *nonextendable* if G contains no cycle C' with $V(C)\subsetneq V(C')$.

Our work was motivated by two classical results of Dirac.

Theorem 1 (Dirac [10]). *Let G be a graph of order $n\geqslant 3$. If $\delta\geqslant\frac{1}{2}n$, then G is hamiltonian.*

Theorem 2 (Dirac [10]). *Let G be a 2-connected nonhamiltonian graph. Then $c(G)\geqslant 2\delta$.*

We show that the lower bounds on δ in Theorem 1 and $c(G)$ in Theorem 2 can be improved if G is assumed to have toughness $\tau>1$. This idea is also reflected by a conjecture of Chvátal.

Conjecture 3 (*Chvátal* [8]). There exists a constant t_0 such that every t_0-tough graph is hamiltonian.

In [4] it was observed that a result in [1] has the following consequence.

Theorem 4 (Bauer et al. [4]). *Let G be a t-tough graph on $n\geqslant 3$ vertices, where $1\leqslant t\leqslant 2$. If $\delta>n/(t+1)-1$ then G is hamiltonian.*

In Sections 2 and 3 we obtain analogues of Theorems 1 and 2, respectively. The results in these sections involve some assumption on the toughness of a graph and the notion of a D_λ-cycle. In particular, we show that the requirement $t\leqslant 2$ can be removed in Theorem 4.

2. Analogues of Theorem 1

We start with our main result.

Theorem 5. *Let G be a t-tough 2-connected graph on n vertices with*

$$\delta_\lambda>\begin{cases}\dfrac{n}{t+\lambda}-1 & \text{if } \lambda\leqslant t+1,\\[2ex] \dfrac{n}{2t+1}+t-\lambda & \text{if } \lambda\geqslant t+1.\end{cases} \tag{1}$$

Then G contains a D_λ-cycle.

Before proving Theorem 5, we give the following lemma which is implicit in the proof of [16, Theorem 2].

Lemma 6. *Let G be a 2-connected graph containing a $D_{\lambda+1}$-cycle but no D_λ-cycle. Then* $\alpha_\lambda \geqslant \delta_\lambda + 1$.

Corollary 7. *Let G be a t-tough 2-connected graph of order n.*

(a) *If G contains a $D_{\lambda+1}$-cycle, but no D_λ-cycle, then $n \geqslant (\delta_\lambda + 1)(t+\lambda)$.*

(b) *If G contains no D_λ-cycle, then $n \geqslant (2t+1)(t+\lambda)$.*

Proof. Let G satisfy the hypothesis of the corollary.

(a) Suppose G contains a $D_{\lambda+1}$-cycle, but no D_λ-cycle. Clearly $t \leqslant (n-\lambda\alpha_\lambda)/\alpha_\lambda$ whenever $\alpha_\lambda \geqslant 2$. Hence from Lemma 6 we obtain $t \leqslant (n-\lambda(\delta_\lambda+1))/(\delta_\lambda+1)$, or equivalently, $n \geqslant (\delta_\lambda+1)(t+\lambda)$.

(b) Suppose G contains no D_λ-cycle. Setting $l+1 = \min\{s \mid G \text{ contains a } D_s\text{-cycle}\}$, we have $l \geqslant \lambda$. Since $\delta_l \geqslant \kappa(G) \geqslant 2t$, using (a) we obtain $n \geqslant (\delta_l+1)(t+l) \geqslant (2t+1)(t+\lambda)$. □

Proof of Theorem 5. Let G satisfy the hypothesis of the theorem and suppose G contains no D_λ-cycle. Setting $l+1 = \min\{s \mid G \text{ contains a } D_s\text{-cycle}\}$, we have

$$l \geqslant \lambda. \tag{2}$$

By Corollary 7(b),

$$n \geqslant (2t+1)(t+\lambda), \tag{3}$$

implying that

$$\max\left\{\frac{n}{t+\lambda}-1, \frac{n}{2t+1}+t-\lambda\right\} = \begin{cases} \dfrac{n}{t+\lambda}-1 & \text{if } \lambda \leqslant t+1, \\ \dfrac{n}{2t+1}+t-\lambda & \text{if } \lambda \geqslant t+1. \end{cases} \tag{4}$$

By (1) and (4),

$$\delta_\lambda > \max\left\{\frac{n}{t+\lambda}-1, \frac{n}{2t+1}+t-\lambda\right\}. \tag{5}$$

Let $f(x) = x^2 - x(\delta_\lambda + \lambda - t + 1) + n - t\delta_\lambda - \lambda t - t$. By Corollary 7(a), $n \geqslant (\delta_l+1)(t+l) \geqslant (\delta_\lambda - (l-\lambda)+1)(t+l)$, implying that

$$f(l) \geqslant 0. \tag{6}$$

By (5),

$$f(\lambda) < 0 \quad \text{and} \quad f\left(\frac{n}{2t+1}-t\right) < 0. \tag{7}$$

Noting that $\lambda \leqslant n/(2t+1)-t$ by (3), we conclude from (2), (6) and (7) that $l > n/(2t+1)-t$, or equivalently, $n<(2t+1)(t+l)$. By Corollary 7(b), G contains a D_l-cycle, a contradiction. □

By Corollary 7(b), Theorem 5 is of interest only for $n \geqslant (2t+1)(t+\lambda)$. For these values of n the following result is more general than Theorem 5.

Theorem 8. *Let G be a t-tough 2-connected graph on n vertices with* $\alpha_\lambda \leqslant \delta_\lambda$ *and*

$$\delta_{\lambda+1} > \begin{cases} \dfrac{n}{t+\lambda+1}-1 & \text{if } \lambda \leqslant t, \\ \dfrac{n}{2t+1}+t-\lambda-1 & \text{if } \lambda \geqslant t. \end{cases} \tag{8}$$

Then G contains a D_λ*-cycle.*

Theorem 8 is a direct consequence of Theorem 5 and Lemma 6.

Using $\delta_{\lambda+1} \geqslant \delta_\lambda - 1$, it is easy to show that (1) implies (8) if $n \geqslant (2t+1)(t+\lambda)$. To show that Theorem 8 is more general than Theorem 5 for these values of n, it remains to show that (1) implies $\alpha_\lambda \leqslant \delta_\lambda$. If $\alpha_\lambda \geqslant 2$, then $t \leqslant (n-\lambda\alpha_\lambda)/\alpha_\lambda$, or equivalently, $\alpha_\lambda \leqslant n/(t+\lambda)$. As in the proof of Theorem 5, $n \geqslant (2t+1)(t+\lambda)$ implies (5), hence $\delta_\lambda > n/(t+\lambda)-1$, implying $\alpha_\lambda \leqslant \delta_\lambda$.

The situation is similar to the fact that the following result of Nash-Williams is more general than Theorem 1.

Theorem 9 (Nash-Williams [14]). *Let G be a 2-connected graph on n vertices with* $\delta \geqslant \max\{\alpha, \frac{1}{3}(n+2)\}$. *Then G is hamiltonian.*

Using $\delta_\lambda \geqslant \delta - \lambda + 1$, we obtain the following consequences of Theorem 5 in terms of the minimum vertex degree.

Corollary 10. *Let G be a t-tough graph on* $n \geqslant 3$ *vertices with* $\delta > n/(t+\lambda)+\lambda-2$ *for some* $\lambda \leqslant t+1$. *Then G contains a* D_λ*-cycle.*

Corollary 11. *Let G be a 2-connected t-tough graph on n vertices with* $\delta > n/(2t+1)+t-1$. *Then G contains a* $D_{\lceil t \rceil+1}$*-cycle.*

Note that any graph satisfying the hypothesis of Corollary 10 is 2-connected.

The following examples show that the condition $\lambda \leqslant t+1$ cannot be relaxed in Corollary 10 for $t=1$. Let H_n denote the graph on $n=3k \geqslant 9$ vertices consisting of three disjoint complete graphs on k vertices and let G_n denote the graph obtained from H_n by adding the edges of two triangles between two disjoint triples of vertices, each containing one vertex of each component of H_n. Then G_n contains no D_{k-2}-cycle, while $\tau(G_n)=1$ and $\delta(G_n)=\frac{1}{3}n-1>n/(1+\lambda)+\lambda-2$ for $\lambda\geqslant 3$ and n large enough. For t with $1<t<2$, at least some upper bound on λ in terms of t is needed in Corollary 10. The verification of this claim is postponed to Section 4.

For $\lambda=1$ and $\lambda=2$, respectively, we obtain the following explicit forms of Corollary 10.

Corollary 12. *Let G be a t-tough graph on $n\geqslant 3$ vertices with $\delta>n/(t+1)-1$. Then G is hamiltonian.*

Corollary 13. *Let G be a t-tough graph ($t\geqslant 1$) on $n\geqslant 3$ vertices with $\delta>n/(t+2)$. Then G contains a dominating cycle.*

Corollary 12 has a number of consequences, one of which is that a t-tough $n/(t+1)$-regular graph is hamiltonian. We now know, however, that such graphs need not have a triangle. In particular, there exist t-tough $n/(t+1)$-regular graphs with no triangles for all t of the form $2-1/k$ where k is an integer and $k\geqslant 2$, and for all t of the form $3-4/(k+1)$ where k is an integer and $k\geqslant 3$ [5]). It is conjectured in [5] that for suitable arbitrarily large t, there exist t-tough $n/(t+1)$-regular graphs with no triangle. Therefore, graphs satisfying the hypothesis of Corollary 12 are not pancyclic in general. However, it is also conjectured in [5] that t-tough graphs on $n\geqslant 3$ vertices with $\delta>n/(t+1)$ are pancyclic.

In spite of the crude upper bound on the toughness used in the proof of Theorem 5 (via the application of Corollary 7), the following best possible results show that Corollaries 12 and 13 are surprisingly close to best possible for $t=1$. The first result is the minimum degree analogue of a result from [13], the second is a weak version of a result from [6]. We do not know how good the lower bounds on δ in Corollaries 12 and 13 are for $t>1$.

Theorem 14 (Jung [13]). *Let G be a 1-tough graph on $n\geqslant 11$ vertices with $\delta\geqslant\frac{1}{2}(n-4)$. Then G is hamiltonian.*

Theorem 15 (Bigalke and Jung [6]). *Let G be a 1-tough graph on $n\geqslant 3$ vertices with $\delta\geqslant\frac{1}{3}n$. Then G contains a dominating cycle.*

In fact, in [6] it is shown that in a graph satisfying the hypothesis of Theorem 15, every longest cycle is a dominating cycle.

Other related (and more general) results for the cases $t=1$, $\lambda=2$ and $t=2$, $\lambda=1$ can be found in [1]. We mention the following two only.

Theorem 16 (Bauer et al. [1]). *Let G be a 1-tough graph on n vertices such that $d(x)+d(y)+d(z)\geqslant n$ for all independent sets of vertices x, y, z. Then every longest cycle in G is a dominating cycle.*

Theorem 17 (Bauer et al. [1]). *Let G be a 2-tough graph on $n \geqslant 3$ vertices such that $d(x)+d(y)+d(z) \geqslant n$ for all independent sets of vertices x, y, z. Then G is hamiltonian.*

Another consequence of Corollary 12 is that Conjecture 3 is true within the class of graphs with minimum degree at least a constant times the number of vertices. Also, as was first observed by Jackson [12], Corollary 12 implies the following.

Corollary 18. *Let G be a t-tough graph on $n \geqslant 3$ vertices. If $t > -\frac{3}{4}+\sqrt{\frac{1}{2}n+\frac{1}{16}}$, then G is hamiltonian.*

Proof. If $G \neq K_n$ is a t-tough graph with $t > -\frac{3}{4}+\sqrt{\frac{1}{2}n+\frac{1}{16}}$, then $\delta \geqslant \kappa \geqslant 2t > n/(t+1)-1$. The result follows from Corollary 12. □

Using the fact that in a noncomplete graph G on n vertices $\alpha \leqslant n/(t+1)$ and $\kappa \geqslant 2t$, a result slightly weaker than Corollary 18 (with $t \geqslant \frac{1}{2}+\sqrt{\frac{1}{2}n+\frac{1}{4}}$) can be obtained from the result in [9] that any graph of order $n \geqslant 3$ for which $\alpha \leqslant \kappa$, is hamiltonian. In a similar way, a slightly stronger result (with $t \geqslant -\frac{3}{4}+\sqrt{\frac{1}{2}n+\frac{1}{16}}$) can be obtained from the following result in [6].

Theorem 19 (Bigalke and Jung [6]). *Let G be a 3-connected 1-tough graph with $\alpha \leqslant \kappa+1$. Then G is hamiltonian or G is the Petersen graph.*

The following result follows by letting $\lambda = 1$ in Theorem 8 (using $\delta_2 \geqslant \delta - 1$).

Corollary 20. *Let G be a t-tough graph ($t \geqslant 1$) on $n \geqslant 3$ vertices with $\alpha \leqslant \delta$ and $\delta > n/(t+2)$. Then G is hamiltonian.*

A result of the same type appears in [11]. It is a generalization of Theorem 9 above.

Theorem 21 (Fraisse [11]). *Let G be a k-connected graph ($k \geqslant 2$) on n vertices with*

$$\delta \geqslant \max\left\{\alpha+k-2, \frac{n+k(k-1)}{k+1}\right\}.$$

Then G is hamiltonian.

Since $\kappa \geqslant 2t$ in a noncomplete t-tough graph, we compare the following consequence of Theorem 21 with Corollary 20.

Corollary 22. *Let G be a 2t-connected graph ($t \geqslant 1$) on n vertices with $\alpha \leqslant \delta - 2t + 2$ and $\delta \geqslant (n+2t(2t-1))/(2t+1)$. Then G is hamiltonian.*

What we observe is that if we impose a stronger condition on α, the lower bound on δ can be decreased (and we can work with a “simple” connectivity condition instead of a “difficult” toughness condition).

3. Analogues of Theorem 2

Theorem 23. *Let C be a nonextendable cycle in a t-tough graph G and let H be a component of $G-V(C)$. Then $c(G)\geqslant|V(C)|\geqslant(t+1)d(H)+t$.*

Proof. Let C be a nonextendable cycle in a t-tough graph G and let H be a component of $G-V(C)$. By standard arguments in hamiltonian graph theory, the nonextendability of C implies that the immediate successors of the neighbors of H on C (in a specified orientation of C) form an independent set S with $|S|=d(H)$, and no vertex of H is adjacent to a vertex in S. Hence

$$t\leqslant\frac{|V(C)-S|}{|S|+1}=\frac{|V(C)|-d(H)}{d(H)+1},$$

so that $c(G)\geqslant|V(C)|\geqslant(t+1)d(H)+t$. $\square$

If C is a D_λ-cycle in a graph G, and H is a component of $G-V(C)$, then $d(H)\geqslant\delta-\lambda+2$. Using this, and considering a nonextendable D_λ-cycle, we obtain the following consequence of Theorem 23.

Corollary 24. *Let G be a t-tough nonhamiltonian graph and suppose G contains a D_λ-cycle. Then $c(G)\geqslant(t+1)(\delta-\lambda+2)+t$.*

Combining Corollaries 10 and 24 we obtain a partial improvement of Theorem 2.

Corollary 25. *Let G be a t-tough nonhamiltonian graph on $n\geqslant3$ vertices with $\delta>n/(t+\lambda)+\lambda-2$ for some $\lambda\leqslant t+1$. Then $c(G)\geqslant(t+1)(\delta-\lambda+2)+t$.*

Corollary 25 is not best possible for $t=1$ and $\lambda=2$, as shown by the following results, the first of which is a consequence of a result in [1]. The second is a consequence of a result in [2, 15].

Theorem 26 (Bauer et al. [1]). *Let G be a 1-tough nonhamiltonian graph on $n\geqslant11$ vertices with $\delta\geqslant\frac{1}{3}n$. Then $c(G)\geqslant n+\delta-\alpha\geqslant\frac{1}{2}n+\delta>2\delta+2$.*

The last inequality in Theorem 26 follows from Theorem 14.

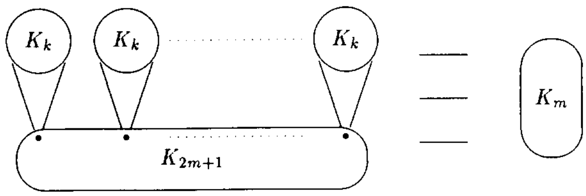

Fig. 1. The graph $G(k,m)$. The horizontal lines indicate a complete bipartite join.

Theorem 27 (Bauer and Schmeichel [2], Tian and Zhao [15]). *Let G be a 1-tough nonhamiltonian graph on $n \geqslant 3$ vertices. Then $c(G) \geqslant 2\delta + 2$.*

4. Examples related to Corollary 10

In Section 2 we claimed that for t with $1<t<2$, at least some upper bound on λ in terms of t is needed in Corollary 10. Here we verify this claim by showing that for every t with $1<t<2$ there exists an integer $\lambda(t)$ and infinitely many t-tough graphs satisfying $\delta > n/(t+\lambda(t)) + \lambda(t) - 2$ and containing no $D_{\lambda(t)}$-cycle.

Case 1. $1<t<\frac{3}{2}$: For $k,m \geqslant 1$ we construct the graphs $G(k,m)$ as follows. Let $H_1, \ldots, H_{2m+1}$ be $2m+1$ disjoint copies of K_k and let T be a copy of K_{2m+1}, disjoint from $H_1, \ldots, H_{2m+1}$, with $V(T) = \{u_1, \ldots, u_{2m+1}\}$. Form $H(k,m)$ by joining the vertex u_i to all vertices of H_i for $i=1,\ldots,2m+1$. Now $G(k,m)$ is the join of a copy H of K_m and $H(k,m)$. See Fig. 1 for a sketch of $G(k,m)$.

It is easy to verify that $|V(G(k,m))| = m+(2m+1)(k+1)$, $\delta(G(k,m)) = k+m$ and $G(k,m)$ contains no D_k-cycle. Furthermore, we have

$$\tau(G(k,m)) = \frac{3m}{2m+1},$$

since clearly the vertex cuts $S \subseteq V(G(k,m))$ that satisfy

$$\tau(G(k,m)) = \frac{|S|}{\omega(G(k,m)-S)}$$

are the sets containing all vertices of H and all but one of the vertices in T. The graphs $G(1,m)$ were constructed in [3] as examples of graphs containing no 2-factor (and hence no Hamilton cycle). The graphs $G(k,1)$ are well-known examples of 1-tough nonhamiltonian graphs.

Now suppose t with $1<t<\frac{3}{2}$ is given. Choose m such that $3m/(2m+1) \geqslant t$ and set $\lambda = \lambda(t) = 2m$. Now for all k with $k \geqslant \lambda$ and

$$k > \frac{m+\lambda+1+(\lambda-m-2)(t+\lambda)}{t-1},$$

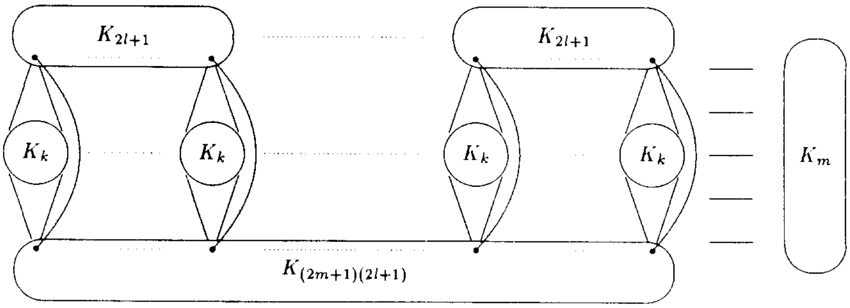

Fig. 2. The graph $G(k,m,l)$. The horizontal lines indicate a complete bipartite join.

the graph $G(k,m)$ is t-tough, contains no D_k-cycle, hence certainly no D_λ-cycle, and satisfies

$$\delta(G(k,m))=k+m>\frac{m+(\lambda+1)(k+1)}{t+\lambda}+\lambda-2=\frac{|V(G(k,m))|}{t+\lambda}+\lambda-2.$$

Case 2. $\frac{3}{2}\leqslant t<2$: For $k,l\geqslant 1$ and $m\geqslant 2$ we construct the graphs $G(k,m,l)$ as follows. Let $\{H_{i,j}\mid 1\leqslant i\leqslant 2m+1,\ 1\leqslant j\leqslant 2l+1\}$ be a collection of $(2m+1)(2l+1)$ disjoint copies of K_k and let $T_1,\dots,T_{2m+1}$ be $2m+1$ disjoint copies of K_{2l+1} that are also disjoint from all $H_{i,j}$. Let $V(T_i)=\{v_{i,1},\dots,v_{i,2l+1}\}$. Finally, let T_0 be a copy of $K_{(2m+1)(2l+1)}$, disjoint from all $H_{i,j}$ and all T_i, with $V(T_0)=\{u_{i,j}\mid 1\leqslant i\leqslant 2m+1, 1\leqslant j\leqslant 2l+1\}$. Form $H(k,m,l)$ by joining the vertices in $H_{i,j}$ to $v_{i,j}$ and to $u_{i,j}$ and adding the edge $u_{i,j}v_{i,j}$, for $i=1,\dots,2m+1$, $j=1,\dots,2l+1$. Now $G(k,m,l)$ is the join of a copy H of K_m and $H(k,m,l)$. The graph $G(k,m,l)$ is sketched in Fig. 2.

We have $|V(G(k,m,l))|=m+(2m+1)(2l+1)(k+2)$, $\delta(G(k,m,l))=k+m+1$ and $G(k,m,l)$ contains no D_k-cycle. The toughness of $G(k,m,l)$ is given by

$$\tau(G(k,m,l))=\frac{m+(2m+1)(2l+1)-1+(2m+1)2l}{(2m+1)(2l+1)}=2-\frac{m+2}{(2m+1)(2l+1)}$$

The vertex cuts $S\subseteq V(G(k,m,l))$ that satisfy

$$\tau(G(k,m,l))=\frac{|S|}{\omega(G(k,m,l)-S)}$$

are the sets containing all vertices of H, all but one of the vertices in T_0 and, for $i=1,\dots,2m+1$, all but one of the vertices in T_i. The graphs $G(1,m,l)$ without the edges $u_{i,j}v_{i,j}$ appear in [3] as examples of graphs containing no 2-factor.

Suppose t with $\frac{3}{2}\leqslant t<2$ is given. Choose m and l such that

$$2-\frac{m+2}{(2m+1)(2l+1)}\geqslant t$$

and set $\lambda=\lambda(t)=(2m+1)(2l+1)-1$. Now for all k with $k\geqslant\lambda$ and

$$k>\frac{m+2(\lambda+1)+(\lambda-m-3)(t+\lambda)}{t-1},$$

the graph $G(k,m,l)$ is t-tough, contains no D_k-cycle, hence certainly no D_λ-cycle, and satisfies

$$\delta(G(k,m,l))=k+m+1>\frac{m+(\lambda+1)(k+2)}{t+\lambda}+\lambda-2=\frac{|V(G(k,m,l))|}{t+\lambda}+\lambda-2.$$

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