On Some Approximately Balanced Combinatorial Cooperative Games

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Abstract: A model of taxation for cooperative n -person games is introduced where proper coalitions are taxed proportionally to their value. Games with non-empty core under taxation at rate ε are ε -balanced. Sharp bounds on ε in matching games on (not necessarily bipartite) graphs are established. Upper and lower bounds on the smallest ε in bin packing games are derived and euclidean random TSP games are seen to be, with high probability, ε -balanced for $\varepsilon \approx 0.06$.

Key Words: cooperative game, balanced, core, tax, matching, bin packing, traveling salesman

1 Introduction

For our purposes, a *cooperative game* is just a function v , the *characteristic function* of the game, that assigns to each *coalition* $S \subseteq N$ of a finite set N of *players* a real number $v(S)$. We consider games where $v(S)$ arises as the optimal value of a combinatorial optimization problem defined by S . Typically, “optimization” will mean “maximization” (e.g., matching, bin packing). In case of “minimization”, we speak of a *cost game* (e.g., minimum spanning tree, traveling salesman). The discussion of minimization games is completely analogous to maximization games.

A *solution concept* tries to distribute the value $v(N)$ of the grand coalition N among the individual players in an acceptable manner. Doubtless, the intuitively most attractive solution concept is the *core* of a game, which consists of those vectors $x \in \mathbb{R}^N$ with component sum $x(N) = v(N)$ and S -restricted component sum $x(S) \geq v(S)$ for all $S \subset N$. Games with a non-empty core are *balanced*.

Unfortunately, many interesting games are not balanced. This means that one either has to introduce a completely different solution concept (e.g., Shapley value) or one has to modify the notion of “core”. A model for the latter goes back to Shapley and Shubik [1966]. In that model, a proper coalition $S \subset N$ is *taxed* by a constant ε or with rate ε proportionally to its size so that the value is reduced to $v(S) - \varepsilon$ or $v(S) - \varepsilon|S|$ resp.. Tijs and Driessen [1986] propose a tax proportional to the difference $v(S) - \sum_{i \in S} v(i)$. One now seeks a core vector in the taxed game, thereby trying to keep the tax(rate) ε “small”.

In the present note, we suggest a model of taxation where the tax imposed on a proper coalition $S \subset N$ is proportional to its value $v(S)$. This is the usual idea behind a sales tax and, therefore, appears to be quite realistic. We thus reduce that value $v(S)$ to $(1 - \varepsilon)v(S)$. A game with non-empty core after taxation with rate ε is ε -balanced. In this sense, ε -taxation provides an ε -approximation to balancedness.

In Section 2, we relate ε -balancedness to the usual definition of balancedness via the analogue of Bondareva's [1963] and Shapley's [1967] theorem. The optimal ε for matching games on arbitrary (possibly non-bipartite) graphs is determined in Section 3. Section 4 introduces bin packing games and derives (not sharp) bounds on the best ε . Finally, we discuss traveling salesman games and argue that large euclidean game instances are, with high probability, ε -balanced for $\varepsilon \approx 0.06$.

2 ε -Balancedness and ε -Core Allocations

In this section, we introduce the notion of ε -balanced games and the ε -core of a game. Arguing along the same lines as in Bondareva [1963] and Shapley [1967], we conclude that ε -balanced games are characterized by the existence of non-empty ε -cores (Theorem 1 below).

Let the cooperative game v with the set N of players be given by its characteristic function

$$v: 2^N \rightarrow \mathbb{R},$$

where $v(\emptyset) = 0$. Denoting by 1_S the indicator function of an arbitrary coalition $S \subseteq N$, we say that v is ε -balanced for $\varepsilon \geq 0$ if for all $S_1, S_2, \dots, S_k \subset N$ and parameters $\lambda(S_1), \dots, \lambda(S_k) \in \mathbb{R}_+$,

$$\sum_{j=1}^k \lambda(S_j) 1_{S_j} = 1_N \quad \text{implies} \quad (1 - \varepsilon) \sum_{j=1}^k \lambda(S_j) v(S_j) \leq v(N)$$

The ε -core $K_\varepsilon(v)$ of v consists of all allocation vectors $x \in \mathbb{R}^N$ such that

- i) $x(N) = v(N)$
- ii) $x(S) \geq (1 - \varepsilon)v(S)$ for all $S \subset N$,

where, as usual, $x(S) = \sum_{i \in S} x(i)$.

Our first result seemingly extends the theorem of Bondareva and Shapley (the case $\varepsilon = 0$ in our notation). Actually, it is easily seen to be equivalent to that theorem by considering the game $v_\varepsilon: 2^N \rightarrow \mathbb{R}$, where

$$v_\varepsilon(S) = \begin{cases} (1 - \varepsilon)v(S) & \text{if } S \neq N \\ v(S) & \text{if } S = N \end{cases}.$$

We therefore only sketch the simple direct proof.

Theorem 1: The game $v: 2^N \rightarrow \mathbb{R}$ is ε -balanced if and only if $K_\varepsilon(v) \neq \emptyset$.

Proof: Consider the linear program (LP):

$$\max \lambda_N v(N) + (1 - \varepsilon) \sum_{S \subset N} \lambda_S v(S)$$

$$\text{such that } \lambda_N + \sum_{S \ni i} \lambda_S \leq 0$$

$$\lambda_N \text{ free, } \lambda_S \geq 0 \text{ (} S \subset N \text{)}.$$

By linear programming duality, $K_\varepsilon(v) \neq \emptyset$ is tantamount to saying that each feasible solution λ for (LP) yields objective function value ≤ 0 , i.e., any choice of numbers $\lambda'_S \geq 0$ ($S \subset N$) and λ'_N with the property

$$\sum_{S \ni i} \lambda'_S = \lambda'_N \quad \text{for all } i \in N,$$

necessarily satisfies $(1 - \varepsilon) \sum_{S \subset N} \lambda'_S v(S) \leq \lambda'_N v(N)$. □

Rather than verifying the condition for ε -balancedness directly, Theorem 1 allows us to exhibit allocation vectors in the ε -core instead. We will now turn our attention to the study of ε -balancedness of some combinatorial games that are known to be generally not O -balanced.

Before doing so, we want to relate our ε -core with the ε -tax core of Tijs and Driessen [1986] that consists of those vectors $x \in \mathbb{R}^N$ with the properties

- i) $x(N) = v(N)$
- ii) $x(S) \geq (1 - \varepsilon)v(S) + \varepsilon \sum_{i \in S} v(i) \quad \text{if } S \subset N.$

It is not hard to find games with empty ε -tax core but non-empty ε -core. Indeed, for non-negative games, the ε -core is a more general solution concept. On the other hand, both concepts coincide for games v with $v(i) = 0$ for each $i \in N$. Examples of the latter are the matching games in Section 3.

A vector $x \in \mathbb{R}^N$ is an *imputation* for the game v if x distributes $v(N)$ and is *individually rational*, i.e., if

- i) $x(N) = v(N)$
- ii') $x(i) \geq v(i)$ for all $i \in N$.

Note that any vector in the ε -tax core is an imputation while the ε -core may contain also vectors that are not individually rational. In the games we consider below, however, we will always exhibit imputations in the ε -core.

3 Matching Games

Let $G = (N, E)$ be a graph on the vertex set N with the set E of edges. We assume that G is (edge-) weighted via

$$w: E \rightarrow \mathbb{R}.$$

Thinking of N as a set of players, define the value $v(S)$ of the coalition $S \subseteq N$ in the *matching game* v relative to w by

$$v(S) = \max \{w(M) \mid M \subseteq E(S) \text{ matching}\},$$

where $E(S)$ is the set of edges joining two vertices in S . (Recall that a **matching** in G is a set of pairwise disjoint edges).

It is well-known that the matching game is O -balanced if the graph G is *bipartite*, i.e., if G contains no circuit of odd length ≥ 3 (cf. Shapley and Shubik [1972]). On the other hand, there are matching games with empty O -core (take G to be triangle with $w \equiv 1$).

Theorem 2: Assume that $G = (N, E)$ contains no circuit of odd length smaller than k . Then the associated matching game v is $\frac{1}{k}$ -balanced.

Proof: For any coalition $S \subseteq N$, $v(S)$ is the optimal value of the integer linear program $ILP(S)$:

$$\max \sum_{e \in E} w(e)x_e$$

$$\text{such that } \sum_{e \ni i} x_e \leq 1 \quad \text{if } i \in S$$

$$\sum_{e \ni i} x_e \leq 0 \quad \text{if } i \notin S$$

$$x_e \in \{0, 1\} \quad .$$

Relaxing the integrality constraints $x_e \in \{0, 1\}$ to $0 \leq x_e \leq 1$, we obtain the associated linear program $LP(S)$, whose optimal value we denote by $\bar{v}(S)$. Clearly $\bar{v}(S) \geq v(S)$.

The game $\bar{v}$ is a linear production game in the sense of Owen [1975] and hence has a non-empty O -core. Choose $\bar{z} \in K_0(\bar{v})$ and define $z \in \mathbb{R}^N$ via

$$z(i) = \bar{z}(i) \cdot \frac{v(N)}{\bar{v}(N)} \quad \text{for all } i \in N \quad .$$

By Lemma 3 below, we have in particular $v(N)/\bar{v}(N) \geq \left(1 - \frac{1}{k}\right)$. Thus

$$\begin{aligned} z(S) &= \bar{z}(S) \cdot \frac{v(N)}{\bar{v}(N)} \geq \bar{v}(S) \cdot \frac{v(N)}{\bar{v}(N)} \\ &\geq v(S) \cdot \left(1 - \frac{1}{k}\right) \end{aligned}$$

□

Lemma 3: Under the conditions of the proof of Theorem 2, we have $v(S) \geq \left(1 - \frac{1}{k}\right)\bar{v}(S)$.

Proof: We use the fact (cf. Lovász and Plummer [1986], Chap. 7) that the linear program $LP(S)$ has an optimal solution $x \in \mathbb{R}^E$ such that for all $e \in E$,

$$x_e = 0 \quad \text{or} \quad x_e = \frac{1}{2} \quad \text{or} \quad x_e = 1 \quad .$$

Moreover, the set $\{e \in E | x_e = \frac{1}{2}\}$ is a union of a disjoint circuits of odd length. Let C be such a circuit. The contribution of C towards $\bar{v}(S)$ is

$$\frac{1}{2} \cdot \sum_{e \in C} w(e) .$$

We claim that C contains a matching M of weight

$$w(M) \geq \left(1 - \frac{1}{k}\right) \cdot \frac{1}{2} \sum_{e \in C} w(e) .$$

Let the edge $m \in C$ have minimal weight in C . Now $C \setminus \{m\}$ contains two disjoint matchings because $|C|$ is odd. So one of them has weight at least

$$\begin{aligned} \frac{1}{2} \left[\sum_{e \in C} w(e) - w(m) \right] &\geq \frac{1}{2} \sum_{e \in C} w(e) - \frac{1}{2} \frac{1}{|C|} \sum_{e \in C} w(e) \\ &\geq \left(1 - \frac{1}{k}\right) \cdot \frac{1}{2} \sum_{e \in C} w(e) . \end{aligned}$$

Thus, by replacing all odd circuits C with $x_{|C} \equiv \frac{1}{2}$ by an appropriate matching, we obtain a matching in S with weight at least $\left(1 - \frac{1}{k}\right) \bar{v}(S)$. $\square$

We remark that Theorem 2 is “best possible”. It is straightforward to see that the cardinality matching game (i.e., $w \equiv 1$) on an odd circuit of length k is not ε -balanced for any $\varepsilon < \frac{1}{k}$.

Corollary 4: (cf. Shapley and Shubik [1972]): If G is bipartite, then the matching game is balanced. $\square$

4 Bin Packing Games

We introduce the following combinatorial game: There are n items of sizes $a_1, \dots, a_n \in \mathbb{R}_+$ and m bins of sizes $b_1, \dots, b_m \in \mathbb{R}$, where we assume that each item i fits into any bin j , i.e., $a_i \leq b_j$ for all i and all j . The set N of *players* consists of

all items and all bins. In particular, $|N| = n + m$. A *coalition* $S \subseteq N$ comprises a set $A \subseteq \{1, \dots, n\}$ of items and a set $B \subseteq \{1, \dots, m\}$ of bins. The *value* $v(S)$ is defined as the maximum total size of items in A that can be packed into bins in B . So

$$v(S) = \max \sum_{j=1}^{|B|} \sum_{i \in A_j} a_i ,$$

where the maximum is taken over all pairwise disjoint subsets $A_1, \dots, A_{|B|}$ of A such that

$$\sum_{i \in A_j} a_i \leq b_j .$$

To illustrate a bin packing game, imagine that the “bins” are trucks of capacity b_j . The items produce a profit proportional to their size a_i if they are brought to a market place. The value $v(N)$ of the grand coalition thus represents the maximum profit achievable. Note that $v(N)$ is bounded by $\min\{\sum a_i, \sum b_j\}$.

How should $v(N)$ be allocated to the “owners” of the items and the “owners” of the bins?

We first show that we cannot guarantee ε -balancedness for general bin packing games unless $\varepsilon \geq \frac{1}{7}$.

Proposition 5: Let $\varepsilon \geq 0$ be such that every bin packing game is ε -balanced. Then $\varepsilon \geq \frac{1}{7}$.

Proof: Take 2 bins with capacity $b_1 = b_2 = 2$ and 5 items of size $a_1 = a_2 = 1 + \delta$, $a'_1 = a'_2 = a'_3 = 1$ for some $0 < \delta < 1$.

Consider the coalitions $S_1 = \{b_1, a'_1, a'_2\}$, $S_2 = \{b_2, a'_2, a'_3\}$, $S_3 = \{b_1, b_2, a_1, a_2, a'_1, a'_3\}$ and $S_4 = \{a_1, a_2\}$.

Then $\sum_{j=1}^4 1_{s_j} = 2 \cdot 1_N$ and, therefore, the balancedness conditions imply

$$(1 - \varepsilon) \sum_{j=1}^4 v(S_j) \leq 2 \cdot v(N) = 6 + 2\delta .$$

Now $v(S_1) = v(S_2) = 2$, $v(S_3) = 3 + \delta$, $v(S_4) = 0$, Letting $\delta \rightarrow 0$, we see that $\varepsilon \geq \frac{1}{7}$ must hold. $\square$

Theorem 6: Each bin packing game is $\frac{1}{2}$ -balanced.

Proof: For the items $a_1, \dots, a_n$ and bins $b_1, \dots, b_m$, we set $a = a_1 + \dots + a_n$ and $b = b_1 + \dots + b_m$. There are two cases to be distinguished.

Case 1: $a > v(N)$ (i.e. the grand coalition is short of truck drivers).

We allocate nothing to the item owners and assign to truck driver j the value

$$x_j = \frac{b_j}{b} v(N) .$$

We must show that $x(S) \geq \frac{1}{2}v(S)$ holds for any coalition S in the bin packing game. This will follow once we can demonstrate that there is an optimal bin packing in which every bin is at least half full because then $v(N) \geq \frac{1}{2}b$ and so

$$x(S) \geq \frac{1}{2} \sum_{j \in S} b_j \geq \frac{1}{2} v(S).$$

Consider an optimal bin packing and suppose bin j is less than half full. Because $a > v(N)$, there is an item a_i that is not in any bin. Because the current packing is optimal, we have $a_i \geq \frac{1}{2}b_j$. On the other hand, $a_i \leq b_j$ (the fundamental assumption in our model) implies that we would improve the current solution by replacing the content of bin j with a_i .

Case 2: $a = v(N)$.

In this case, we assign nothing to the truck owners and allocate to the owner of item i the value

$$x_i = a_i .$$

This allocation obviously satisfies $x(S) \geq v(S)$ for any coalition S . □

Our results show that the smallest ε for which every bin packing game is ε -balanced lies between $1/7$ and $1/2$. Are there sharper bounds? It should also be interesting to study the bin packing game in a more general framework where $a_i \leq b_j$ does not necessarily hold and the profit p_i gained from packing item a_i may be independent from its size.

5 TSP Games

TSP games (“Traveling Salesman” games) can be motivated by a repairman who starts from his home 0 and visits the customers $N = \{1, \dots, n\}$ before returning home (cf. Tamir [1989] and Potters et al. [1992]). Hence we consider a graph G on the vertex set $N \cup \{0\}$ whose edges correspond to links connecting customers. Each link e is weighted by its (non-negative) length $d(e)$. Taking N to be a set of players, the associated TSP game evaluates the cost $c(S)$ of any coalition $S \subseteq N$ as

$$c(S) = \text{length of a shortest tour through } S \cup \{0\} .$$

We seek to distribute the total cost $c(N)$ of the grand coalition among the players N . Similarly to the foregoing, we define the ε -core $K_\varepsilon(c)$ of the *cost game* c to consist of all vectors $x \in \mathbb{R}^N$ such that

- i) $x(N) = c(N)$
- ii) $x(S) \leq (1 + \varepsilon)c(S)$ for all $S \subset N$.

In the model we discuss in this section, we will assume throughout that the graph G is *undirected* and that the edge lengths satisfy the *triangle inequality*. Without going into details, let us remark here that one can exhibit an example of a TSP game with empty O -core whose underlying graph satisfies the above requirements (Tamir [1989]).

In fact, there are even euclidean TSP games with empty O -core (Faigle and Kern [1993]). (Recall that a *euclidean* TSP game, by definition, arises from choosing the vertex set in the plane $\mathbb{R}^n$ and considering the euclidean distances between pairs of points (see also below)).

In the following, we denote by $L = v(N)$ the length of the shortest tour and by T the length of a minimum spanning tree in G .

Theorem 7: If $L \leq (1 + \varepsilon)T$, then the TSP game c has a non-empty ε -core.

Proof: The MST (“Minimum Spanning Tree”) game $\bar{c}$ associated with G is given by

$$\bar{c}(S) = \text{length of an MST on } S \cup \{0\} .$$

It is well-known that $\bar{c}$ has a non-empty O -core (cf. Claus and Kleitman [1973]).

Choose $\bar{x} \in K_0(\bar{c})$ and define $x \in \mathbb{R}^N$ via

$$x(i) = \bar{x}(i) \frac{L}{T} \quad \text{for } i \in N .$$

We claim: $x \in K_\varepsilon(c)$.

Because $T = \bar{x}(N)$, we obtain $x(N) = L$. Moreover, $c(S) \geq \bar{c}(S)$ implies

$$x(S) = \bar{x}(S) \frac{L}{T} \leq \bar{c}(S)(1 + \varepsilon) \leq (1 + \varepsilon)c(S) . \quad \square$$

Note that $L \leq 2T$ always holds since a (in general non-optimal) tour can be constructed by the following standard procedure: double the edges of the optimal tree in order to obtain an eulerian graph on $N \cup \{0\}$. Trace the eulerian tour of length $2T$ and take "shortcuts" using the triangle inequality. There results a tour of length $\leq 2T$.

In many situations, the ε in the hypothesis of Theorem 6 is quite small. Consider, for example, the euclidean TSP and let $U := [0, 1]^2$ be the unit square with center $x_0 = (\frac{1}{2}, \frac{1}{2}) \in U$. Choose $x_1, \dots, x_n \in U$ and define $G = (V, E)$ to be the complete graph on $V = N \cup \{0\}$ with edge lengths

$$c(i, j) = \|x_i - x_j\|_2 .$$

It is known that with high probability

$$L \approx 0.72\sqrt{n} \quad \text{and} \quad T \approx 0.68\sqrt{n}$$

(cf. Kern [1989], Rhee and Talagrand [1989], Steele [1990], Goemans and Bertsimas [1991]). More precisely, it has been shown that there exist constants β_L and β_T such that for independent, uniformly distributed random variables $X_1, X_2, \dots$ in U ,

$$\text{Prob}(|L(X_1, \dots, X_n) - \beta_L\sqrt{n}| \geq \alpha\sqrt{n})$$

and

$$\text{Prob}(|T(X_1, \dots, X_n) - \beta_T\sqrt{n}| \geq \alpha\sqrt{n})$$

become exponentially small as $n \rightarrow \infty$ for any fixed $\alpha > 0$. Numerical experiments (Goemans and Bertsimas [1991]) suggest that $\beta_L \approx 0.72$ and $\beta_T \approx 0.68$.

Hence, for large n , we will have

$$L \leq 1.06 T$$

with high probability, which guarantees an ε -core with $\varepsilon \approx 0.06$.

Another possible allocation rule for the TSP game is motivated by the following relaxation of the TSP problem. As above, let c_{ij} denote the distance between nodes i and j . Define, for each coalition $S \subseteq N$

$$\begin{aligned} \tilde{c}(S) &:= \min \sum_{(i,j) \in E} c_{ij} x_{ij} \\ \sum_{(i,j) \in E} x_{ij} &= \begin{cases} 2 & \forall i \in S \cup \{0\} \\ 0 & \text{else} \end{cases} \\ \sum_{i \in T, i \notin T} x_{ij} &\geq 2 \quad \forall \emptyset \subset T \subset S \cup \{0\} \\ x_{ij} &\geq 0 \end{aligned}$$

Obviously, any optimal tour through $S \cup \{0\}$ satisfies these constraints (so called “degree constraints” and “subtour elimination constraints”). As observed already by Tamir [1989], [1991] and others, the game defined by $\tilde{c}$ has nonempty core. (In fact, a vector $\tilde{x}$ in the core of this game can be computed efficiently.) If $\tilde{x}$ is a vector in the core of the game defined by $\tilde{c}$, then

$$x(i) := \tilde{x}(i) \cdot \frac{L}{\tilde{c}(N)},$$

(where, again, L denotes the length of an optimum tour through $N \cup \{0\}$) defines an allocation for the TSP game. Obviously,

$$x(S) \leq \tilde{x}(S) \cdot \frac{L}{\tilde{c}(N)} \leq \tilde{c}(S) \cdot \frac{L}{\tilde{c}(N)} \leq c(S) \cdot \frac{1}{\tilde{c}(N)}.$$

It has been shown by Wolsey [1980] that the ratio $L/\tilde{c}(N)$ is at most $3/2$, assuming the triangle inequality for the distances c_{ij} . Thus, this allocation rule yields a vector x in the $\frac{1}{2}$ -core of the TSP-game. It has been conjectured that the above ratio is bounded by $4/3$. In the euclidean case, empirical results on random problems with uniformly distributed points in the unit square seem to imply

that, on the average,

$$c(N) \geq 0.70 \cdot \sqrt{n} ,$$

(cf. Goemans and Bertsimas [1991]). Thus, with high probability, a random TSP game has nonempty ε -core even for

$$\varepsilon \approx \frac{0.72}{0.70} \approx 0.03 .$$

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