[13] J. P. Ostrowski, A. D. Lewis, R. M. Murray, and J. W. Burdick, "Nonholonomic mechanics and locomotion: The snakeboaxd example," in *Proc. IEEE Int. Conf. Robotics and Automation*, San Diego, CA, May 1994, pp. 2391–2397.
[14] J. Radford and J. Burdick, Proc. Conf. Mathematical Theory of Networks and Systems, submitted for publication.

Energy-Based Lyapunov Functions for Forced Hamiltonian Systems with Dissipation

Bernhard Maschke, Romeo Ortega, and Arjan J. van der Schaft

***Abstract*—In this paper, we propose a constructive procedure to modify the Hamiltonian function of forced Hamiltonian systems with dissipation in order to generate Lyapunov functions for nonzero equilibria. A key step in the procedure, which is motivated from energy-balance considerations standard in network modeling of physical systems, is to embed the system into a larger Hamiltonian system for which a series of Casimir functions can be easily constructed. Interestingly enough, for linear systems the resulting Lyapunov function is the incremental energy; thus our derivations provide a physical explanation to it. An easily verifiable necessary and sufficient condition for the applicability of the technique in the general nonlinear case is given. Some examples that illustrate the method are given.**

***Index Terms*—Casimirs, Hamiltonian systems, Lyapunov stability, sources.**

I. Problem Formulation

Network modeling of lumped-parameter physical systems [7] with independent storage elements leads to the following class of dynamical systems, called *port controlled Hamiltonian systems with dissipation* [6], [14], [15], [1]:

$$\Sigma\colon \begin{cases} \dot{x} = [J(x) - R(x)]\,\dfrac{\partial H}{\partial x}(x) + g(x)u \\ y = g^\top(x)\,\dfrac{\partial H}{\partial x}(x) \end{cases} \tag{1.1}$$

where $x \in \mathcal{X}$, an n-dimensional manifold, $u,\ y \in \mathcal{R}^m$. The state variables $x = [x_1, \cdots, x_n]^\top$ are the energy variables (i.e., the variables by which the energy of the system is defined), the smooth function $H(x_1, \cdots, x_n)\colon \mathcal{X} \to \mathcal{R}$ represents the total stored energy, and $u,\ y$ are the port power variables. The two $n \times n$ matrices $J(x)$ and $R(x)$ are called *structure matrices* and define the geometric structure of the state space of the energy variables. The matrix $J(x)$ corresponds to a power continuous interconnection in the network model; it is *skew-symmetric* and defines a generalized Poisson bracket on $\mathcal{X}$ (generalized because it need not satisfy the Jacobi identity [13]). The matrix $R(x)$ is a nonnegative *symmetric* matrix depending smoothly on x; it corresponds to the energy dissipating part of the network model and defines a symmetric bracket on the state space.

Manuscript received February 23, 1998; revised April 13, 1999. Recommended by Associate Editors, A. Bloch and P. Crouch.

B. Maschke and A. J. van der Schaft are with the Department of Signals, Systems and Control, Faculty of Mathematical Sciences, University of Twente, Enschede 7500 AE The Netherlands (e-mail: maschke@math.utwente.nl; a.j.vanderschaft@math.utwente.nl).

R. Ortega is with the Laboratoire des Signaux et Systèmes, CNRS-Supélec, Gif-sur-Yvette 91192 France (e-mail: rortega@lss.supelec.fr).

Publisher Item Identifier S 0018-9286(00)06315-7.

The port controlled Hamiltonian systems with dissipation (1.1) satisfy the following the power-balance equation:

$$\frac{d}{dt}H = -\frac{\partial^\top H}{\partial x}(x)R(x)\frac{\partial H}{\partial x}(x) + u^\top y \tag{1.2}$$

where $u^\top y$ is the power externally supplied to the system and the first term on the right-hand side represents the *energy dissipation* due to the resistive elements in the system.

While from the power-balance equation (1.2) the stability of the *uncontrolled* or *unforced* system (1.1) (for $u = 0$) may be analyzed from the properties of the Hamiltonian function $H(x)$, in the sequel we shall analyze the stability of the system (1.1) for a *constant*, but *nonzero*, input $\overline{u} \in \mathcal{R}^m$, leading to a *forced* (controlled) equilibrium $\overline{x} \in \mathcal{X}$. Such situations arise, e.g., in studies of the transient stability of synchronous generators in power systems [3]; see also [11] and [10]. Corresponding to $u = \overline{u}$, the forced equilibria $\overline{x}$ are solutions of

$$[J(\overline{x}) - R(\overline{x})]\,\frac{\partial H}{\partial x}(\overline{x}) + g(\overline{x})\overline{u} = 0. \tag{1.3}$$

In general, a forced equilibrium $\overline{x}$ will not be a minimum (nor an extremum) of H. Furthermore, inserting $u = \overline{u}$ in (1.2) yields

$$\frac{d}{dt}H = -\frac{\partial^\top H}{\partial x}(x)R(x)\frac{\partial H}{\partial x}(x) + \overline{u}^\top g^\top(x)\frac{\partial H}{\partial x}(x) \tag{1.4}$$

having a right-hand side that in general will not be nonpositive. Thus, in most cases, the Hamiltonian function can *not* be directly used as a Lyapunov function for investigating the stability of a forced equilibrium $\overline{x}$. Hence the problem comes up if, and how, we can construct *physically based* Lyapunov functions for *equilibria of forced physical systems* (1.1). Providing some (partial) solutions to this problem are the main contributions of our work.

II. A Lyapunov Function Based on Energy-Balance

One way of approaching the problem is to start from the power balance of the forced system (1.4) and to bring the second term on the right-hand side to the left-hand side, suggesting to look for candidate Lyapunov functions

$$H(x(t)) - \overline{u}^\top \int_0^t y(\tau)\,d\tau. \tag{2.1}$$

To check whether (2.1) can be used as a Lyapunov function, the first basic question is if we can write $\overline{u}^\top \int_0^t y(\tau)\,d\tau$ as a function of the state $x(t)$. From a control theoretic point of view, this question suggests to consider a cascade of Σ with input $\overline{u}$, followed by the integration of y, and to look for Lyapunov functions of the composed system

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\,\frac{\partial H}{\partial x}(x) + g(x)\overline{u} \\ \dot{\zeta} &= g^\top(x)\,\frac{\partial H}{\partial x}(x), \qquad \zeta \in \mathcal{R}^m. \end{aligned} \tag{2.2}$$

Note that (2.2) can be rewritten as an unforced Hamiltonian system with dissipation

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \left(\begin{bmatrix} J(x) & -g(x) \\ g^\top(x) & 0 \end{bmatrix} - \begin{bmatrix} R(x) & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \dfrac{\partial H_a}{\partial x} \\ \dfrac{\partial H_a}{\partial \zeta} \end{bmatrix} \tag{2.3}$$

with $H_a(x,\ \zeta)$ the *augmented* energy function

$$H_a(x,\ \zeta) \triangleq H(x) + H_s(\zeta), \qquad H_s(\zeta) \triangleq -\overline{u}^\top \zeta. \tag{2.4}$$

Writing $\overline{u}^\top \int_0^t y(\tau)\,d\tau$ as a function of $x(t)$ then corresponds to expressing $\zeta(t)$ as a function of $x(t)$ along the dynamics (2.3). This is the starting point of our approach. To motivate our subsequent developments, we first present two simple examples in Section III. From these examples, it follows that in order to cope with the general problem, we have to modify the dynamical system (2.3) to a more general form. The treatment of this more general form leading to a solution to the general problem shall be given in Sections IV–VI. Conditions that ensure that the indicated first attempt already "works" shall be given in Section VII, along with some other examples.

Remark 2.1: From a modeling perspective, (2.3) corresponds to viewing Σ for constant $u = \overline{u}$ as the *interconnection* of Σ with a *source system*

$$\begin{aligned} \dot{\zeta} &= u_s \\ y_s &= \frac{\partial H_s}{\partial \zeta} \end{aligned} \tag{2.5}$$

with $H_s(\zeta) = -\overline{u}^\top \zeta$ the (unbounded) energy of the source system, via the interconnection constraints

$$\begin{aligned} u_s &= y \\ u &= -y_s. \end{aligned} \tag{2.6}$$

Remark 2.2: Notice that the term $\overline{u}^\top \int_0^t y(\tau)\, d\tau$ is the energy externally supplied to the system Σ and withdrawn from the source system. Hence the new Lyapunov function (2.1) that we propose is intimately related with an *energy balance*, since it is exactly the difference between the energy of the system and the supplied energy.

III. Two Motivating Examples

A. A Series RLC Circuit

Consider the linear time-invariant circuit consisting of the series interconnection of a resistor (with resistance R_2), an inductor (with inductance L) a capacitor (with capacitance C), and a constant voltage source $\overline{u}$. The total electromagnetic energy of the circuit is $H(x) = (1/2)x^\top Q x$, with $x = [x_1,\ x_2]^\top = [q_C,\ \phi_L]^\top$, where q_C, ϕ_L are the capacitor's charge and the inductor's flux, respectively, and $Q = \text{Diag}(1/C,\, 1/L)$.

The dynamical model of the circuit can be written in the form of a port controlled Hamiltonian system with dissipation (1.1) with the input being the voltage delivered by the source $u = \overline{u}$, the output being the associated current $y = (1/L)x_2$, and the structure matrices

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ 0 & R_2 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The equilibrium of this system is unique and given by $\overline{x} = [C\overline{u},\, 0]^\top$. The candidate Lyapunov function (2.1) takes the form

$$\frac{1}{2}\, x^\top Q x - \frac{1}{L}\, \overline{u} \int_0^t x_2(\tau)\, d\tau = \frac{1}{2}\, x^\top Q x - \overline{u} x_1$$

and is actually exactly (up to a constant) the standard incremental Lyapunov function used for linear systems

$$W(x) \triangleq \frac{1}{2}\, x^\top Q x - \overline{u} x_1 + \frac{C}{2}\, \overline{u}^2 = \frac{1}{2}\, (x - \overline{x})^\top Q (x - \overline{x}). \tag{3.1}$$

Let us now view this from the perspective of the cascaded system (2.2), which in this example takes the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -R_2 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{C} x_1 \\ \frac{1}{L} x_2 \\ -\overline{u} \end{bmatrix}. \tag{3.2}$$

As pointed out above, if we can express $\zeta(t)$ as a function of $x(t)$, then we can write $\overline{u}^\top \int_0^t y(\tau)\, d\tau$ as a function of x. This, in turn, is true if there exists a function of the form

$$F(x, \zeta) = C(x) - \zeta$$

which is a "Casimir function"[1] [4] for the combined structure matrix in (3.2). Being a "Casimir function" for (3.2) means that the time-derivative of F is zero along the solutions of (3.2) for any energy function H, and thus for any value of the constants $C, L, \overline{u}$. Since the system (3.2) is linear, we can take $C(x)$ as a linear function $C(x) = k_1 x_1 + k_2 x_2$, and one may compute the coefficients k_1, k_2 from

$$[k_1,\ k_2,\ -1] \begin{bmatrix} 0 & 1 & 0 \\ -1 & -R_2 & -1 \\ 0 & 1 & 0 \end{bmatrix} = 0$$

which yields the unique solution $k_1 = 1,\ k_2 = 0$. Hence, along trajectories of (3.2), we have $\zeta = x_1 + c$, with c some constant, and from (2.4) we get the Lyapunov candidate function

$$H_a(x, \zeta)|_{\zeta = x_1 + c} = \tfrac{1}{2} x^\top Q x - \overline{u}(x_1 + c)$$

which, setting $c = (C/2)\overline{u}$, reduces to (3.1).

B. A Parallel RLC Circuit

As an example where the cascaded systems approach does not work, let us consider the RLC circuit obtained by modifying the preceeding example by connecting the resistor in parallel with the capacitor. Now the symmetric structure matrix $R(x)$ has changed into

$$R = \begin{bmatrix} \frac{1}{R_2} & 0 \\ 0 & 0 \end{bmatrix}.$$

Repeating the arguments used for the previous example, we end up with a system of equations

$$[k_1,\ k_2,\ -1] \begin{bmatrix} \frac{-1}{R_2} & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = 0$$

which clearly does not have a solution in k_1, k_2. In the next section, we will show how to overcome this problem by embedding the system into a system (2.3) with suitably *modified* interconnection and dissipation structure.

Remark 3.1: An important observation is that the equilibrium of the parallel RLC circuit is given by $\overline{x} = [C\overline{u},\, (L/R_2)\overline{u}]^\top$. Hence, in contrast to the series RLC, in this circuit the equilibrium current in the resistor is nonzero. Consequently, it drains an infinite amount of energy from the source and, in view of Remark 2.2, (2.1) is not bounded from below.

IV. System Embedding

Key to our developments is the static relation (1.3) describing the forced equilibria. Since we want to consider forced equilibria for every $\overline{u}$, it is logical to assume that $\text{Im}\{g(x)\} \subset \text{Im}\{J(x) - R(x)\}$. For simplicity we make throughout the following stronger assumption.

Assumption A: $[J(x) - R(x)]$ is invertible for every $x \in \mathcal{X}$. □□□

Consider the equation (1.3) in the variable $v = (\partial H/\partial x)(x)$. By Assumption A, it has the unique solution $v = K(x)\overline{u}$, with

$$K(x) = -[J(x) - R(x)]^{-1} g(x). \tag{4.1}$$

[1]Note that these "Casimir functions" are actually extensions of the usual Casimir functions associated with a Poisson bracket [4] to the nonskew-symmetric bracket defined by the combined structure matrix $J(x) - R(x)$.

Let us now consider the following port controlled Hamiltonian system with dissipation

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = [J_a(x) - R_a(x)] \begin{bmatrix} \frac{\partial H_a}{\partial x} \\ \frac{\partial H_a}{\partial \zeta} \end{bmatrix} \tag{4.2}$$

on the augmented state space $(x, \zeta) \in \mathcal{X} \times \mathcal{R}^m$, endowed with the structure matrices

$$J_a(x) \triangleq \begin{bmatrix} J(x) & J(x)K(x) \\ -(J(x)K(x))^\top & J_s(x) \end{bmatrix}$$

and

$$R_a(x) \triangleq \begin{bmatrix} R(x) & R(x)K(x) \\ (R(x)K(x))^\top & R_s(x) \end{bmatrix}$$

with Hamiltonian function $H_a(x, \zeta)$ defined by (2.4) and with $J_s(x) = -J_s^\top(x)$, and $R_s(x) = R_s^\top(x)$ yet to be determined. Note that

$$R_a(x) = \begin{bmatrix} I \\ K^\top(x) \end{bmatrix} R(x) [I \quad K(x)]$$

and thus, since by assumption $R(x) \geq 0$, also $R_a(x) \geq 0$.

Considering that $H_s(\zeta) = -\overline{u}^\top \zeta$ is linear with respect to ζ and that, by (4.1), $R(x)K(x) = J(x)K(x) + g(x)$, it may be seen that the x-dynamics is the same as in the forced system (1.1). Thus the x-dynamics of Σ for $u = \overline{u}$ has been *embedded* in the dynamics (4.2) in the same way as it was in the augmented system (2.3). Comparing the two embedding systems (2.3) and (4.2), one sees that they differ only in their structure matrices.

V. Construction of the Lyapunov Function

The next question is how to determine $J_s(x) = -J_s^\top(x)$ and $R_s(x) = R_s^\top(x) \geq 0$. This is guided by (4.1). Indeed, the m-dimensional linear spaces

$$P(x) = \left\{ \begin{bmatrix} -K(x)u \\ u \end{bmatrix} \middle| u \in \mathcal{R}^m \right\} \tag{5.1}$$

are, by construction, in the kernel of the matrix $[J(x),\ J(x)K(x)]$ defined by the first n rows of J_a in (4.2). We now define $J_s(x)$ in such a manner that $P(x)$ is in the kernel of the *whole* matrix J_a, by setting

$$J_s(x) \triangleq K^\top(x)J(x)K(x). \tag{5.2}$$

Clearly J_s satisfies $J_s(x) = -J_s^\top(x)$. In the same way, we note that $P(x)$ is in the kernel of the first n rows of R_a in (4.2), while it is in the kernel of the whole matrix R_a if we choose

$$R_s(x) \triangleq K^\top(x)R(x)K(x). \tag{5.3}$$

Then $R_s(x) = R_s^\top(x) \geq 0$. Now we are ready to deliver the *coup de grâce*.

Assume that there exist smooth functions $C_j\colon \mathcal{X} \to \mathcal{R}$, $j \in \overline{m} \triangleq \{1, \cdots, m\}$, such that

$$K_{ij}(x) = \frac{\partial C_j}{\partial x_i}(x), \qquad i \in \overline{n} \triangleq \{1, \cdots, n\},\ j \in \overline{m}. \tag{5.4}$$

Then it immediately follows that the functions

$$\zeta_j - C_j(x), \qquad i \in \overline{m} \tag{5.5}$$

are *constant* along the trajectories of (4.2), with J_s and R_s as defined in (5.2), respectively (5.3). Indeed, we can write

$$\begin{aligned} \frac{d}{dt}[\zeta_j - C_j(x)] &= \left[-\frac{\partial^\top C_i}{\partial x}(x),\ e_j^\top\right](J_a(x) - R_a(x)) \begin{bmatrix} \frac{\partial H_a}{\partial x} \\ \frac{\partial H_a}{\partial \zeta} \end{bmatrix} \end{aligned} \tag{5.6}$$

with e_j the jth basis vector in $\mathcal{R}^m$. Since the $(n+m)$-dimensional column vector $[(\partial C_j^\top/\partial x)(x), -e_j^\top]^\top$ is by (5.4) contained in $P(x)$, it is by construction and definition of J_s and R_s contained in the kernels of J_a and R_a. Thus the expression in (5.6) is zero (for all Hamiltonians H_a). Hence, along trajections of (4.2), we can express

$$\zeta_j = C_j(x) + c_j, \qquad j \in \overline{m} \tag{5.7}$$

where the constants $c_1, \cdots, c_m$ depend on the initial conditions of ζ (and can be set to zero). Thus the dynamics of

$$\dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)\overline{u}$$

is copied on every submanifold of $\mathcal{X} \times \mathcal{R}^m$ defined by (5.7). The total energy of the augmented system

$$H_a(x, \zeta) = H(x) - \overline{u}^\top \zeta$$

restricted to such a submanifold is given as

$$H_r(x) \triangleq H_a(x, C(x) + c) = H(x) - \sum_{j=1}^m \overline{u}_j (C_j(x) + c_j) \tag{5.8}$$

while the dynamics restricted to such a submanifold is given by

$$\dot{x} = [J(x) - R(x)] \frac{\partial H_r}{\partial x}(x). \tag{5.9}$$

Note that by (5.4)

$$\frac{\partial H_r}{\partial x}(x) = \frac{\partial H}{\partial x}(x) - \sum_{j=1}^m \overline{u}_j \frac{\partial C_j}{\partial x}(x) = \frac{\partial H}{\partial x}(x) - K(x)\overline{u}. \tag{5.10}$$

Hence, premultiplying by $[J(x) - R(x)]$ and using (4.1)

$$[J(x) - R(x)] \frac{\partial H_r}{\partial x}(x) = [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)\overline{u}. \tag{5.11}$$

Consequently, by (1.3) and Assumption A, the unique forced equilibrium $\overline{x}$ corresponding to $\overline{u}$ is an *extremum* of H_r [that is, $(\partial H_r/\partial x)(\overline{x}) = 0$].

Remark 5.1: From the derivations above, it follows that the functions $\zeta_j - C_j(x)$ defined on the augmented state space $\Xi \times \mathcal{R}^m$ are *Casimirs* of the generalized Poisson bracket defined by J_a [4]. Furthermore, the functions $\zeta_j - C_j(x)$ are also "Casimirs" with respect to the symmetric bracket corresponding to R_a.

VI. Main Result

Let us summarize the developments above in the following theorem.

Theorem 6.1: Consider Σ for constant $u = \overline{u}$, that is

$$\Sigma\colon \quad \dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)\overline{u} \tag{6.1}$$

with Assumption A. Define $K(x)$ by (4.1) and assume the functions K_{ij} satisfy

$$\frac{\partial K_{ij}}{\partial x_k} = \frac{\partial K_{kj}}{\partial x_i}, \qquad i, k \in \overline{n} \tag{6.2}$$

for $j \in \overline{m}$. Then, there exist locally smooth functions $C_1, \cdots, C_m$ satisfying (5.4), and the dynamics (6.1) can be alternatively represented by

$$\dot{x} = [J(x) - R(x)] \frac{\partial H_r}{\partial x}(x) \tag{6.3}$$

where

$$H_r(x) \triangleq H(x) - \sum_{j=1}^{m} \overline{u}_j (C_j(x) + c_j).$$

The function $H_r(x)$ has an extremum at $\overline{x}$, which is an equilibrium of (6.1). Further, we have

$$\frac{d}{dt} H_r = -\frac{\partial^\top H_r}{\partial x}(x) R(x) \frac{\partial H_r}{\partial x}(x) \le 0 \tag{6.4}$$

and thus H_r qualifies as a *Lyapunov function* for the forced dynamics (6.1) provided we can show that H_r not only has an extremum at $\overline{x}$ but even a *minimum*.

Proof: In view of the developments of the previous section, to complete the proof it only remains to show that, under the given conditions, there exist smooth functions $C_1, \cdots, C_m$, satisfying (5.4). This follows immediately from (6.2) and Poincaré's lemma. □□□

The corollary below follows immediately from Theorem 6.1 and standard Lyapunov stability theory; see, e.g., [2].

Corollary 6.1: Assume that H_r has a strict local minimum at $\overline{x}$, that is, there exists an open neighborhood $\mathcal{B}$ of $\overline{x}$ such that $H_r(x) > H_r(\overline{x})$ for all $x \in \mathcal{B}$. Furthermore, assume that the largest invariant set under the dynamics (6.3) contained in

$$\left\{ x \in \mathcal{X} \cap \mathcal{B} \,\middle|\, \frac{\partial^\top H_r}{\partial x}(x) R(x) \frac{\partial H_r}{\partial x}(x) = 0 \right\}$$

equals $\{\overline{x}\}$. Then, $\overline{x}$ is a locally asymptotically stable equilibrium of the forced system (6.1).

Remark 6.1: Note that if $\mathcal{X}$ is, e.g., simply connected, then the functions $C_1, \cdots, C_m$ satisfying (5.4) exist *globally* if (6.2) is satisfied.

Remark 6.2: An equivalent way to analyze the stability of the equilibrium $\overline{x}$ of the forced system (6.1) by means of the Lyapunov function H_r is to look at the stability of the equilibrium $(\overline{x}, \overline{\zeta})$ with $\overline{\zeta}_j = C_j(\overline{x}), \; j \in \overline{m}$, of the embedding system (4.2) by means of a candidate Lyapunov function of the form

$$\tilde{H}(x, \zeta) \triangleq H(x) - \overline{u}^\top \zeta + \Phi(\zeta_1 - C_1(x), \cdots, \zeta_m - C_m(x))$$

where the function Φ, depending on the Casimirs $\zeta_j - C_j(x), \; j \in \overline{m}$, is still to be determined. This approach is similar to what is called the energy–Casimir method in mechanics (see, e.g., [5] and the references therein). Note that, restricted to any submanifold given by (5.7), the function $\tilde{H}(x, \zeta)$ reduces to the function $H_r(x)$.

Remark 5.2: The integrability condition (6.2) can be geometrically formulated as follows. The subspaces $P(x)$ defined in (5.1) define a codistribution P on the augmented state-space $\mathcal{X} \times \mathcal{R}^m$. It can be seen that condition (6.2) is satisfied *if and only if* P is involutive.

VII. EXAMPLES

A. Linear Systems

If J, R, and g are *constant* matrices, then also K is a constant matrix, and the existence of functions $C_1, \cdots, C_m$ satisfying (5.4) is automatic. [In fact $C_j(x)$ is given as the *linear* function $K_{1j}x_1 + \cdots + K_{nj}x_n$.] In particular, for linear systems Σ with

$$H(x) = \tfrac{1}{2} x^\top Q x, \qquad Q = Q^\top.$$

Theorem 3.2 results in a linear forced dynamics

$$\dot{x} = (J - R) \frac{\partial H_r}{\partial x}(x)$$

with (since $K\overline{u} = Q\overline{x}$)

$$H_r(x) = \tfrac{1}{2} x^\top Q x - x^\top K \overline{u} + c = \tfrac{1}{2}(x - \overline{x}^\top) Q (x - \overline{x}) + c. \tag{7.1}$$

Hence we have recovered in this special case the incremental Lyapunov function, which is normally used. Furthermore, we have given an interpretation in terms of energy balance.

B. A Parallel RLC Circuit (cont.)

Let us come back to the parallel RLC circuit studied in Section III-B. The embedding system (4.2) now takes the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\zeta} \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 & \frac{1}{R_2} \\ -1 & 0 & -1 \\ \frac{-1}{R_2} & 1 & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{R_2} & 0 & \frac{1}{R_2} \\ 0 & 0 & 0 \\ \frac{1}{R_2} & 0 & \frac{1}{R_2} \end{bmatrix} \right) \cdot \begin{bmatrix} \frac{\partial H_a}{\partial x_1} \\ \frac{\partial H_a}{\partial x_2} \\ \frac{\partial H_a}{\partial \zeta} \end{bmatrix}$$

with $H_a(x, \zeta) = H(x) - \overline{u}\zeta$. The symmetric and skew-symmetric structure matrices admit the following Casimir function:

$$F(x, \zeta) = x_1 + \frac{1}{R_2} x_2 - \zeta. \tag{7.2}$$

Consequently the corresponding Lyapunov function is

$$W(x) = \frac{1}{2C_1} x_1^2 + \frac{1}{2L} x_2^2 - \overline{u} \left(x_1 + \frac{1}{R_2} x_2 \right) + \frac{\overline{u}^2}{2} \left(C_1 + \frac{L}{R_2^2} \right).$$

C. Mechanical Systems

Consider a mechanical system with damping and actuated by external forces u

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \left(\begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & D(q) \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} u \tag{7.3}$$

$$y = B^\top(q) \frac{\partial H}{\partial p}$$

with generalized configuration coordinates $q = [q_1, \cdots, q_k]^\top$ and generalized momenta $p = [p_1, \cdots, p_k]^\top$. The outputs $y \in \mathcal{R}^m$ are the generalized velocities corresponding to the generalized external forces $u \in \mathcal{R}^m$. Let $\overline{u}$ be a constant actuating force. It follows that

$$K(q, p) = -\begin{bmatrix} 0 & I_k \\ -I_k & -D(q) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B(q) \end{bmatrix} = \begin{bmatrix} B(q) \\ 0 \end{bmatrix} \tag{7.4}$$

and hence $J_s = 0$ and $R_s = 0$. Furthermore, the integrability conditions (5.4) boil down to the existence of functions $C_1, \cdots, C_m$ such that

$$B_{ij}(q) = \frac{\partial C_j}{\partial q_i}(q), \qquad i, j \in \overline{m}. \tag{7.5}$$

Condition (7.5) means that the input vector fields in (7.3) are actually *Hamiltonian* vector fields with Hamiltonians $C_1(q), \cdots, C_m(q)$. The candidate Lyapunov function is given as $H(q, p) - \sum_{i=1}^m \overline{u}_i C_i(q)$ and in the case where H is the sum of a quadratic kinetic energy and a potential energy $V(q)$, the stability analysis reduces to checking the positive definiteness of $V(q) - \sum_{i=1}^m \overline{u}_i C_i(q)$.

VIII. Conclusion

In this paper, we have proposed a construction of candidate Lyapunov functions for port controlled Hamiltonian systems with dissipation subject to constant inputs. The construction involves the embedding of the forced system into a higher dimensional system followed by its reduction using Casimir functions. The integrability conditions for finding Casimirs, may be interpreted as the input vector fields of the forced system being Hamiltonian with dissipation [8]. For further developments on the role of Casimir functions in the synthesis of stabilizing controllers of physical systems, we refer to [12], [9], and [15].

Acknowledgment

B. Maschke wishes to acknowledge that he is on leave from the Laboratoire d'Automatisme Industriel of the Conservatoire National des Arts et Métiers with the support of the Department of Physics and Electronics of the CNAM and the Faculty of Mathematical Sciences of the University of Twente.

References

[1] M. Dalsmo and A. J. van der Schaft, "On representations and integrability of mathematical structures in energy-conserving physical systems," *SIAM J. Optim. Contr.*, vol. 37, no. 1, pp. 54–92, 1999.

[2] H. Khalil, *Nonlinear Systems*, 2nd ed. Englewood Cliffs, NJ: Prentice-Hall, 1996.

[3] P. Kundur, *Power System Stability and Control*. New York: McGraw-Hill, 1994.

[4] P. Libermann and C. M. Marle, *Symplectic Geometry and Analytical Mechanics*. Dordrecht, The Netherlands: Reidel, 1987.

[5] J. E. Marsden, *Lecture Notes on Mechanics*. Cambridge, U.K.: Cambridge Univ. Press, 1992.

[6] B. M. J. Maschke, "Interconnection and structure in physical systems' dynamics," in *Proc. 4th IFAC Symp. Nonlinear Control Systems Design (NOLCOS'98)*, Enschede, The Netherlands, July 1–3, 1998, pp. 291–296.

[7] B. Maschke, A. van der Schaft, and P. Breedveld, "An intrinsic Hamiltonian formulation of network dynamics: Non-standard Poisson structures and gyrators," *J. Frank. Inst.*, vol. 329, pp. 923–926, 1992.

[8] B. M. J. Maschke, R. Ortega, A. J. van der Schaft, and G. Escobar, "An energy based derivation of Lyapunov function for forced systems with application to stabilizing control," in *Proc. IFAC World Congress*, vol. E, Beijing, July 1999, pp. 409–414.

[9] R. Ortega, A. J. van der Schaft, B. M. J. Maschke, and G. Escobar, submitted for publication.

[10] R. Ortega, A. Stanković, and P. Stefanov, "A passivation approach to power systems stabilization," in *Proc. 4th IFAC Symp. Nonlinear Control Systems Design (NOLCOS'98)*, Enschede, The Netherlands, July 1–3, 1998, pp. 314–318.

[11] J. V. Sarlashkar, "Hamilton/Lagrange formalisms in stability analysis of detailed power system models," Ph.D. dissertation, Univ. of Wisconsin, 1996.

[12] S. Stramigioli, B. M. J. Maschke, and A. J. van der Schaft, "Passive output feedback and port interconnection," in *Proc. 4th IFAC Symp. Nonlinear Control Systems Design (NOLCOS'98)*, Enschede, The Netherlands, July 1–3, 1998, pp. 613–618.

[13] A. J. van der Schaft and B. Maschke, "On the Hamiltonian formulation of nonholonomic mechanical systems," *Rep. Math. Phys.*, vol. 34, pp. 225–233, 1994.

[14] ——, "The Hamiltonian formulation of energy-conserving physical systems with external ports," *Archiv für Elektronik und Übertragungstechnik*, vol. 49, pp. 362–371, 1995.

[15] A. J. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*, 2nd revised and enlarged ed. London: Springer-Verlag, 2000. Springer Communications and Control Engineering Series.