Journal of Computational and Applied Mathematics 49 (1993) 289–296
North-Holland

CAM 1414

Random walk polynomials and random walk measures

Erik A. Van Doorn and Pauline Schrijner
Faculty of Applied Mathematics, University of Twente, Enschede, Netherlands

Received 18 February 1992
Revised 2 April 1992

Abstract

Van Doorn, E.A. and P. Schrijner, Random walk polynomials and random walk measures, Journal of Computational and Applied Mathematics 49 (1993) 289–296.

Random walk polynomials and random walk measures play a prominent role in the analysis of a class of Markov chains called random walks. Without any reference to random walks, however, a random walk polynomial sequence can be defined (and will be defined in this paper) as a polynomial sequence $\{P_n(x)\}$ which is orthogonal with respect to a measure on $[-1, 1]$ and which is such that the parameters α_n in the recurrence relations $P_{n+1}(x)=(x-\alpha_n)P_n(x)-\beta_n P_{n-1}(x)$ are nonnegative. Any measure with respect to which a random walk polynomial sequence is orthogonal is a random walk measure. We collect some properties of random walk measures and polynomials, and use these findings to obtain a limit theorem for random walk measures which is of interest in the study of random walks. We conclude with a conjecture on random walk measures involving Christoffel functions.

Keywords: Orthogonal polynomials; random walks

1. Introduction

We use the term *measure* to designate a Borel measure on $\mathbb{R}$, with infinite support, total mass 1, and finite moments of all positive orders. Before defining the particular type of measures called random walk measures and the associated random walk polynomials, we recall some basic facts from the theory of orthogonal polynomials, see, e.g., [2].

Let ψ then be a measure and $\{P_n(x)\}$ the (unique) monic orthogonal polynomial sequence (OPS) with respect to ψ. Then $\{P_n(x)\}$ satisfies the recurrence relation

$$\begin{aligned} &P_{n+1}(x)=(x-\alpha_n)P_n(x)-\beta_n P_{n-1}(x), \quad n=1,2,\ldots, \\ &P_0(x)=1, \qquad P_1(x)=x-\alpha_0, \end{aligned} \tag{1.1}$$

Correspondence to: Dr. E.A. Van Doorn, Faculty of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, Netherlands.

SSDI 0377-0427(93)E0041-J

where, for $n = 0, 1, \ldots,$

$$\alpha_n = \frac{\int_{-\infty}^{\infty} x P_n^2(x)\, \mathrm{d}\psi(x)}{\int_{-\infty}^{\infty} P_n^2(x)\, \mathrm{d}\psi(x)} \tag{1.2}$$

and

$$\beta_{n+1} = \frac{\int_{-\infty}^{\infty} P_{n+1}^2(x)\, \mathrm{d}\psi(x)}{\int_{-\infty}^{\infty} P_n^2(x)\, \mathrm{d}\psi(x)} > 0. \tag{1.3}$$

Conversely, by Favard's theorem, if $\{P_n(x)\}$ satisfies a recurrence relation of the type (1.1), where $\alpha_n \in \mathbb{R}$ and $\beta_{n+1} > 0$, $n = 0, 1, \ldots,$ then $\{P_n(x)\}$ is the OPS with respect to a measure ψ. In general this measure ψ need not be unique, but we shall encounter measures with compact supports only, in which case the correspondence between measures and orthogonal polynomial sequences is one-to-one.

In the following definition α_n denotes the quantity given by (1.2).

Definition 1.1. The measure ψ is a *random walk measure*, and the corresponding OPS $\{P_n(x)\}$ is a *random walk polynomial sequence* (RWPS), if supp $(\psi) \subset [-1, 1]$ and $\alpha_n \geqslant 0$, $n = 0, 1, \ldots$.

We note that $\alpha_n \geqslant 0$ for all n means

$$\int_{-\infty}^{\infty} x P_n^2(x)\, \mathrm{d}\psi(x) \geqslant 0, \quad n = 0, 1, \ldots, \tag{1.4}$$

which suggests that for ψ to be a random walk measure the mass of ψ on the negative axis should not outweigh (in some sense) the mass on the positive axis. Indeed, some obvious sufficient conditions for ψ to be a random walk measure are

$$\mathrm{supp}(\psi) \subset [0, 1] \quad \text{and} \quad \psi \text{ is a symmetric measure on } [-1, 1].$$

In general, however, it may be difficult to establish whether a given measure is a random walk measure. On the other hand, it is easy to establish whether an OPS, that is, a polynomial sequence satisfying a recurrence relation of the type (1.1) with $\beta_n > 0$, is an RWPS, because of the following theorem.

Theorem 1.2. *The OPS $\{P_n(x)\}$ is an RWPS if and only if the sequence $\{Q_n(x)\}$, where $Q_n(x) \equiv P_n(x)/P_n(1)$, satisfies a recurrence of the type*

$$\begin{aligned} xQ_n(x) &= q_n Q_{n-1}(x) + r_n Q_n(x) + p_n Q_{n+1}(x), \quad n = 1, 2, \ldots, \\ Q_0(x) &= 1, \qquad p_0 Q_1(x) = x - r_0 \end{aligned} \tag{1.5}$$

(*so that $p_0 + r_0 = 1$ and $p_n + q_n + r_n = 1$, $n = 1, 2, \ldots$*), *where*

$$p_n,\ q_{n+1} > 0 \quad \text{and} \quad r_n \geqslant 0, \qquad n = 0, 1, \ldots . \tag{1.6}$$

Proof. Let $\{P_n(x)\}$, satisfying (1.1), be an RWPS with respect to the random walk measure ψ. Since $\text{supp}(\psi) \subset [-1, 1]$, a well-known result on orthogonal polynomials tells us that $P_n(1) > 0$ for all n. Moreover, $Q_n(x) \equiv P_n(x)/P_n(1)$ satisfies the recurrence relation

$$xQ_n(x) = \beta_n \frac{P_{n-1}(1)}{P_n(1)} Q_{n-1}(x) + \alpha_n Q_n(x) + \frac{P_{n+1}(1)}{P_n(1)} Q_{n+1}(x), \quad n = 1, 2, \ldots,$$

$$Q_0(x) = 1, \qquad P_1(1)Q_1(x) = x - \alpha_0,$$

which is of the type (1.5) and (1.6), since $\alpha_n \geqslant 0$.

Conversely, let $\{Q_n(x)\}$ satisfy (1.5) and (1.6). Then the corresponding monic polynomials $P_n(x)$ satisfy (1.1) with

$$\alpha_n = r_n \geqslant 0 \quad \text{and} \quad \beta_{n+1} = p_n q_{n+1} > 0, \qquad n = 0, 1, \ldots .$$

It remains to be shown that $\text{supp}(\psi) \subset [-1, 1]$, where ψ is the orthogonalizing measure for $\{P_n(x)\}$. But in view of [2, Theorem II.4.5], we can apply [8, Theorem 2] with $\chi_j = p_{j-1}$, $j \geqslant 2$, which yields

$$\inf \text{supp}(\psi) \geqslant \inf_j \{2r_j - 1\} \geqslant -1.$$

Defining $q_0 \equiv 0$ and using analogous results involving sup supp(ψ), one readily obtains

$$\sup \text{supp}(\psi) \leqslant \sup_j \{p_j + q_j + r_j\} = 1,$$

which completes the proof. □

Random walk polynomials and random walk measures play a prominent role in the analysis of a class of Markov chains in discrete time called random walks. In particular, Karlin and McGregor [3] have shown that the transition probabilities of a random walk can be represented in terms of a random walk measure and the associated RWPS. In this paper we shall not be concerned with random walks per se. Instead, we mention in Section 2 a theorem on random walk measures which is a direct consequence of Karlin and McGregor's representation formula. This theorem then, as well as some other properties of random walk measures obtained in Section 2, will provide the basis for the analysis in Section 3, where we study the limiting behaviour as $n \to \infty$ of

$$L_n(f, \psi) \equiv \frac{\int_{-1}^{1} x^n f(x) \, \mathrm{d}\psi(x)}{\int_{-1}^{1} x^n \, \mathrm{d}\psi(x)}, \tag{1.7}$$

where ψ is a random walk measure and f is a continuous function. The limiting behaviour of functionals of the type (1.7) plays a key role in the analysis of certain aspects of random walks such as ratio limits and quasi limiting behaviour, but we will not discuss those aspects here. Also the implications of our results for random walks will be elaborated elsewhere, see [9]. We conclude Section 3 with a conjecture on random walk measures involving Christoffel functions.

2. Some properties of random walk measures

Let ψ be a random walk measure and $\{P_n(x)\}$ the associated monic RWPS. We let α_n be as in (1.2) and define

$$\xi \equiv \inf \operatorname{supp}(\psi), \qquad \eta \equiv \sup \operatorname{supp}(\psi),$$

so that $-1 \leqslant \xi < \eta \leqslant 1$, since ψ is a measure on $[-1, 1]$ with infinite support. We also note that

$$\eta > 0, \tag{2.1}$$

as a consequence of (1.4) (with $n = 0$).

We shall have use for the following result, which is an immediate consequence of Karlin and McGregor's representation formula for the transition probabilities of a random walk, see [3].

Lemma 2.1. *For all i, j and* $n = 0, 1, \ldots,$ *one has*

$$\int_{-1}^{1} x^n P_i(x) P_j(x) \, \mathrm{d}\psi(x) \geqslant 0. \tag{2.2}$$

It follows in particular that

$$\int_{-1}^{1} x^{2n+1} \, \mathrm{d}\psi(x) \left(= \int_{\xi}^{\eta} x^{2n+1} \, \mathrm{d}\psi(x) \right) \geqslant 0, \quad n = 0, 1, \ldots, \tag{2.3}$$

which is readily seen to lead to

$$\xi \geqslant -\eta. \tag{2.4}$$

But we can do better as follows.

Lemma 2.2. $\xi + \eta \geqslant 2 \inf_n \{\alpha_n\}$.

Proof. Apply [7, Theorem 13]. □

In the case that $\xi = -\eta$ and ψ is not symmetric (i.e., $\alpha_n > 0$ for some n), ψ cannot have an atom at ξ, as the first part of the next lemma shows.

Lemma 2.3. *If* ψ *is not symmetric, then*

$$\psi(\{-\eta\}) = 0 \tag{2.5}$$

and

$$\int_{-1}^{1} x^n \, \mathrm{d}\psi(x) > 0, \quad \textit{for n sufficiently large}. \tag{2.6}$$

The statement (2.5) may be obtained as a corollary to [11, Theorem 5.2], while (2.6) follows from Karlin and McGregor's representation formula and an elementary probabilistic result.

In Section 3 we shall encounter the quantities

$$C_n(\psi) \equiv \frac{\int_{-1}^{0} (-x)^n \, \mathrm{d}\psi(x)}{\int_{0}^{1} x^n \, \mathrm{d}\psi(x)}, \quad n = 0, 1, \ldots, \tag{2.7}$$

some pertinent properties of which will be collected next. From (2.1) it follows that the denominator in (2.7) is positive, so that $C_n(\psi)$ exists and is nonnegative for all n. To obtain upper bounds on $C_n(\psi)$ we must distinguish between odd and even n. Indeed (2.3) immediately gives us

$$C_{2n+1}(\psi) \leqslant 1, \quad n = 0, 1, \dots . \tag{2.8}$$

But we can do slightly better as follows.

Lemma 2.4. *If ψ is not symmetric, then there exists a $\delta > 0$, such that $0 \leqslant C_{2n+1}(\psi) \leqslant 1 - \delta$ for n sufficiently large.*

The proof of this lemma, which will be omitted, may be based on probabilistic arguments similar to those used in the proof of [4, Theorem 2.1].

For even n we have the following result.

Lemma 2.5. $0 \leqslant C_{2n}(\psi) \leqslant \{\int_0^1 x \, d\psi(x)\}^{-1}$, $n = 0, 1, \dots$.

Proof. Hölder's inequality tells us that

$$\left\{ \int |x|^k \, d\psi(x) \right\}^{1/k} \leqslant \left\{ \int |x|^{k+1} \, d\psi(x) \right\}^{1/k+1},$$

for any interval of integration and any $k = 1, 2, \dots$. Using this result twice together with (2.8), we conclude

$$0 \leqslant C_{2n}(\psi) \equiv \frac{\int_{-1}^{0} x^{2n} \, d\psi(x)}{\int_0^1 x^{2n} \, d\psi(x)} \leqslant \frac{\left\{ \int_{-1}^{0} -x^{2n+1} \, d\psi(x) \right\}^{2n/2n+1}}{\int_0^1 x^{2n} \, d\psi(x)}$$

$$= \left(C_{2n+1}(\psi)\right)^{2n/2n+1} \frac{\left\{ \int_0^1 x^{2n+1} \, d\psi(x) \right\}^{2n/2n+1}}{\int_0^1 x^{2n} \, d\psi(x)} \leqslant \frac{\left\{ \int_0^1 x^{2n+1} \, d\psi(x) \right\}^{2n/2n+1}}{\int_0^1 x^{2n} \, d\psi(x)}$$

$$\leqslant \frac{\left\{ \int_0^1 x^{2n+1} \, d\psi(x) \right\}^{2n/2n+1}}{\int_0^1 x^{2n+1} \, d\psi(x)} = \left\{ \int_0^1 x^{2n+1} \, d\psi(x) \right\}^{-1/2n+1} \leqslant \left\{ \int_0^1 x \, d\psi(x) \right\}^{-1},$$

as required. □

Evidently, $\{\int_0^1 x \, d\psi(x)\}^{-1} > 1$, but we conjecture that $C_{2n}(\psi) \leqslant 1$ for n sufficiently large. Actually the conjecture we shall put forward in Section 3 is stronger than the present one.

We have now gathered sufficient information on random walk measures to commence our study of the functionals $L_n(f, \psi)$ of (1.7). For further properties of random walk measures and random walk polynomials we refer to [3,9–11].

3. A limit theorem for random walk measures

Our starting points and notation are identical to those of Section 2, but in addition we assume that the random walk measure ψ is not symmetric. Our aim is to find conditions for the existence of

$$\lim_{n\to\infty} L_n(f, \psi), \tag{3.1}$$

where $L_n(f, \psi)$ is defined in (1.7), for all continuous functions f on $[-1, 1]$. We shall assume that $f(\eta) \neq f(-\eta)$, which is no restriction of generality, since it can be shown that (3.1) always exists if $f(\eta) = f(-\eta)$, see [9]. Note that, for n sufficiently large, the denominator of (1.7) is positive (by (2.6)), so that $L_n(f, \psi)$ is defined.

We start off our analysis by writing

$$L_n(f, \psi) = \frac{A_n(f, \psi) + B_n(f, \psi)}{1 + (-1)^n C_n(\psi)}, \tag{3.2}$$

where $C_n(\psi)$ is given by (2.7),

$$A_n(f, \psi) \equiv \frac{\int_{-1}^{0} x^n f(x)\, \mathrm{d}\psi(x)}{\int_0^1 x^n\, \mathrm{d}\psi(x)} \quad \text{and} \quad B_n(f, \psi) \equiv \frac{\int_0^1 x^n f(x)\, \mathrm{d}\psi(x)}{\int_0^1 x^n\, \mathrm{d}\psi(x)}.$$

Before stating our main result we mention two auxiliary lemmas, which may be proved by arguments similar to those used in the proof of [3, Lemma 3].

Lemma 3.1. $\lim_{n\to\infty} B_n(f, \psi) = f(\eta)$.

Lemma 3.2. *If $\{n_k\}$ is a subsequence of the sequence of positive integers such that $(-1)^{n_k} C_{n_k}(\psi) \to c$ as $k \to \infty$, then $A_{n_k}(f, \psi) \to c f(-\eta)$ as $k \to \infty$.*

Theorem 3.3. *Let f be a continuous function on $[-1, 1]$ such that $f(\eta) \neq f(-\eta)$, and let ψ be a random walk measure which is not symmetric. Then $\{L_n(f, \psi)\}$ converges as $n \to \infty$ if and only if $C_n(\psi) \to 0$ as $n \to \infty$, in which case $\lim_{n\to\infty} L_n(f, \psi) = f(\eta)$.*

Proof. First suppose $C_n(\psi) \to 0$ as $n \to \infty$. The previous two lemmas and (3.2) then imply that $\lim_{n\to\infty} L_n(f, \psi) = f(\eta)$.

Next suppose that $C_n(\psi) \nrightarrow 0$ as $n \to 0$. Then, by Lemmas 2.4 and 2.5, there must be distinct numbers c_1 and c_2, with $-1 < c_1 \leqslant 0$ and $0 \leqslant c_2 < \infty$, and subsequences $\{n_k^{(i)}\}$, $i = 1, 2$, of the natural numbers, such that

$$(-1)^{n_k^{(i)}} C_{n_k^{(i)}}(\psi) \to c_i, \quad \text{as } k \to \infty,\ i = 1, 2.$$

As a consequence, if $\lim_{n\to\infty} L_n(f, \psi)$ exists, (3.2) and Lemmas 3.1 and 3.2 imply that

$$\frac{c_1 f(-\eta) + f(\eta)}{1 + c_1} = \frac{c_2 f(-\eta) + f(\eta)}{1 + c_2},$$

that is, $f(\eta) = f(-\eta)$. But this is a contradiction, so $\{L_n(f, \psi)\}$ does not converge as $n \to \infty$. □

The above theorem gives us a necessary and sufficient condition for the existence of the limits as $n \to \infty$ of $L_n(f, \psi)$ in terms of ψ. For the applications we have in mind, however, it would be desirable to have a criterion in terms of the parameters of the recurrence relation (1.1) (or (1.5)) for the associated RWPS, or at least in terms of these polynomials themselves. As a preparatory result in this direction, we first note the following lemma, the (easy) proof of which will be omitted.

Lemma 3.4. *If $\xi > -\eta$, then $C_n(\psi) \to 0$ as $n \to \infty$.*

With the help of Lemma 2.2 we now obtain a sufficient condition for $\lim_{n\to\infty} C_n(\psi) = 0$ in terms of the parameters α_n of (1.2), as follows.

Corollary 3.5. *If $\alpha_n \geqslant \delta > 0$ for all n, then $C_n(\psi) \to 0$ as $n \to \infty$.*

Another sufficient condition is stated in the next theorem.

Theorem 3.6. *If $P_n^2(\eta)/P_n^2(-\eta) \to 0$ as $n \to \infty$, then $C_n(\psi) \to 0$ as $n \to \infty$.*

Proof. Suppose that $C_n(\psi) \nrightarrow 0$ as $n \to \infty$. Then, by Lemmas 2.4 and 2.5, there must exist a subsequence $\{n_k\}$ of the natural numbers, and a number $c \neq 0$ with $-1 < c < \infty$, such that $(-1)^{n_k} C_{n_k}(\psi) \to c$ as $k \to \infty$. It follows by (3.2) and Lemmas 3.1 and 3.2 that

$$L_{n_k}(P_i, \psi) \to \frac{cP_i(-\eta) + P_i(\eta)}{1+c}, \quad \text{as } k \to \infty, \tag{3.3}$$

for all $i = 0, 1, \ldots$. We subsequently note that $(-1)^i P_i(-\eta) > 0$ by a well-known result on orthogonal polynomials, since $-\eta \leqslant \xi$. Because $L_{n_k}(P_i, \psi)$ exists and is nonnegative for n_k sufficiently large, the limit in (3.3) must be nonnegative for all i, from which it follows that $P_i^2(\eta)/P_i^2(-\eta) \nrightarrow 0$ as $i \to \infty$. □

Remark 3.7. Generalizing a result by Karlin and McGregor [3], it is not difficult to show that the sequence $\{P_n^2(\eta)/P_n^2(-\eta)\}_n$ is nonincreasing, so that its limit actually exists. The same generalization of Karlin and McGregor's result allows one to prove that the condition in Corollary 3.5 actually implies the condition in Theorem 3.6.

We conjecture that the sufficient condition for $\lim_{n\to\infty} C_n(\psi) = 0$ in the above theorem is also a necessary condition. In fact, we conjecture the validity of the following stronger statement.

Conjecture 3.8. $\lim_{n\to\infty} C_n(\psi)$ *exists and equals* $\lim_{n\to\infty} P_n^2(\eta)/P_n^2(-\eta)$.

To motivate this conjecture, we first note that it is obviously true when ψ is a symmetric random walk measure. Thus suppose that ψ is not symmetric. Then, by (2.5), $\psi(\{-\eta\}) = 0$ and it follows by a famous result on orthogonal polynomials [6, Corollary 2.6] that

$$\rho_n(-\eta) \to 0, \quad \text{as } n \to \infty.$$

Here $\rho_n(x)$ is a Christoffel function, defined by

$$\rho_n(x) = \left\{ \sum_{i=0}^{n-1} p_i^2(x) \right\}^{-1},$$

where $\{p_n(x)\}$ is the orthonormal RWPS associated with ψ, cf. [5]. Next, using *Stolz' criterion* (see, e.g., [1, p.414]), we obtain

$$\lim_{n\to\infty} \frac{P_n^2(\eta)}{P_n^2(-\eta)} = \lim_{n\to\infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)}.$$

In a number of cases, e.g., when $\psi(\{\eta\}) > 0$, the limiting behaviour of the Christoffel functions is known and consistent with our conjecture.

We refer to [9] for more information on the probabilistic consequences of our findings.

References

[1] T.J.I'a Bromwich, *An Introduction to the Theory of Infinite Series* (Macmillan, London, 2nd ed., 1965).

[2] T.S. Chihara, *An Introduction to Orthogonal Polynomials* (Gordon and Breach, New York, 1978).

[3] S. Karlin and J.L. McGregor, Random walks, *Illinois J. Math.* **3** (1959) 66–81.

[4] G. Kersting, Strong ratio limit property and R-recurrence of reversible Markov chains, *Z. Wahrsch. Verw. Gebiete* **30** (1974) 343–356.

[5] P. Nevai, Géza Freud, orthogonal polynomials and Christoffel functions. A case study, *J. Approx. Theory* **48** (1986) 3–167.

[6] J.A. Shohat and J.D. Tamarkin, *The Problem of Moments,* Math. Surveys **I** (Amer. Mathematical Soc., Providence, RI, rev. ed., 1963).

[7] E.A. van Doorn, On oscillation properties and the interval of orthogonality of orthogonal polynomials, *SIAM J. Math. Anal.* **15** (1984) 1031–1042.

[8] E.A. van Doorn, Representations and bounds for zeros of orthogonal polynomials and eigenvalues of sign-symmetric tri-diagonal matrices, *J. Approx. Theory* **51** (1987) 254–266.

[9] E.A. van Doorn and P. Schrijner, Ratio limits and quasi-limiting distributions for random walks, submitted, 1993.

[10] T.A. Whitehurst, On random walks and orthogonal polynomials, Ph.D. Thesis, Indiana Univ., 1978.

[11] T.A. Whitehurst, An application of orthogonal polynomials to random walks, *Pacific J. Math.* **99** (1982) 205–213.