DISCRETE APPLIED MATHEMATICS

ELSEVIER Discrete Applied Mathematics 99 (2000) 317–321

Not every 2-tough graph is Hamiltonian ☆

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Received 19 June 1997; received in revised form 27 January 1998; accepted 9 March 1999

The first two authors dedicate this paper to the memory of their dear friend and coauthor Henk Tau Veldman, who died October 12, 1998.

Abstract

We present $(\frac{9}{4}-\varepsilon)$-tough graphs without a Hamilton path for arbitrary $\varepsilon > 0$, thereby refuting a well-known conjecture due to Chvátal. We also present $(\frac{7}{4}-\varepsilon)$-tough chordal graphs without a Hamilton path for any $\varepsilon > 0$.

MSC: 05C45; 05C38; 05C35

Keywords: Hamiltonian graph; Traceable graph; Toughness; 2-tough graph; Chordal graph

1. Introduction

We use Bondy and Murty's book [5] for terminology and notation not defined here, and consider finite simple graphs only.

A graph G is *Hamiltonian* if it contains a Hamilton cycle (a cycle containing every vertex of G); G is *traceable* if G contains a Hamilton path (a path containing every vertex of G); G is *Hamiltonian-connected* if for every pair of distinct vertices x and y of G there is a Hamilton path with endvertices x and y.

The number of components of a graph G is denoted by $\omega(G)$. The graph G is *t-tough* ($t \in \mathbb{R}$, $t \geqslant 0$) if $|S| \geqslant t \cdot \omega(G-S)$ for every subset S of $V(G)$ with $\omega(G-S) > 1$. The *toughness* of G, denoted by $\tau(G)$, is the maximum value of t for which G is t-tough.

The concept of toughness of a graph was introduced by Chvátal [7]. Clearly, 1-toughness is a necessary condition for hamiltonicity, but it is not sufficient. In [7] the following conjecture is stated.

☆ Supported in part by NATO Collaborative Research Grant CRG 921251.

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PII: S0166-218X(99)00141-9

Conjecture 1 (Chvátal [7]). *There exists t_0 such that every t_0-tough graph is Hamiltonian.*

The stronger conjecture that every t-tough graph with $t > \frac{3}{2}$ is Hamiltonian, also occurring in [7], was first disproved by Thomassen (see [4]). Enomoto et al. [8] showed that every 2-tough graph contains a 2-factor (a 2-regular spanning subgraph), while for arbitrary $\varepsilon > 0$ there exist $(2-\varepsilon)$-tough graphs without a 2-factor, and hence without a Hamilton cycle. Therefore the following conjecture, usually attributed to Chvátal, appeared to be both reasonable and best possible.

Conjecture 2. *Every 2-tough graph is Hamiltonian.*

In [1] a construction of a nontraceable graph from non-Hamiltonian-connected building blocks was used to show that Conjecture 2 is equivalent to several other statements, some (seemingly) weaker, some (seemingly) stronger than Conjecture 2. This construction was inspired by examples of graphs of high toughness without 2-factors occurring in [3]. In the next section, we use the same construction to obtain $(\frac{9}{4}-\varepsilon)$-tough nontraceable graphs for arbitrary $\varepsilon > 0$, thereby refuting Conjecture 2. Conjecture 1 remains open.

2. Counterexamples to Conjecture 2

For a given graph H and two vertices x and y of H we define the graph $G(H,x,y,\ell,m)$ $(\ell,m \in \mathbb{N})$ as follows. Take m disjoint copies $H_1,\ldots,H_m$ of H, with x_i, y_i the vertices in H_i corresponding to the vertices x and y in H $(i=1,\ldots,m)$. Let F_m be the graph obtained from $H_1 \cup \cdots \cup H_m$ by adding all possible edges between pairs of vertices in $\{x_1,\ldots,x_m,y_1,\ldots,y_m\}$. Let $T = K_\ell$ and let $G(H,x,y,\ell,m)$ be the join $T \vee F_m$ of T and F_m.

The proof of the following theorem occurs almost literally in [1]. For convenience we repeat it here.

Theorem 3. *Let H be a graph and x, y two vertices of H which are not connected by a Hamilton path of H. If $m \geqslant 2\ell+3$, then $G(H,x,y,\ell,m)$ is nontraceable.*

Proof. Suppose $G(H,x,y,\ell,m)$ contains a Hamilton path P. The intersection of P and F_m consists of a collection $\mathscr{P}$ of at most $\ell+1$ disjoint paths, together containing all vertices in F_m. Since $m \geqslant 2(\ell+1)+1$, there is a subgraph H_{i_0} in F_m such that no endvertex of a path of $\mathscr{P}$ lies in H_{i_0}. Hence the intersection of P and H_{i_0} is a path with endvertices x_{i_0} and y_{i_0} that contains all vertices of H_{i_0}. This contradicts the fact that H_{i_0} is a copy of the graph H without a Hamilton path between x and y. □

Consider the graph L of Fig. 1.

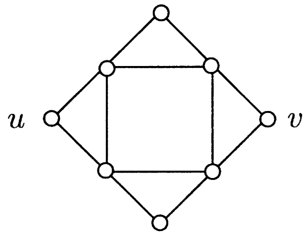

Fig. 1. The graph L.

Theorem 4. *For $\ell \geqslant 2$ and $m \geqslant 1$,*

$$\tau(G(L,u,v,\ell,m)) = \frac{\ell + 4m}{2m+1}.$$

Proof. Let $G = G(L,u,v,\ell,m)$ for some $\ell \geqslant 2$ and $m \geqslant 1$, and choose $S \subseteq V(G)$ such that $\omega(G-S) > 1$ and $\tau(G) = |S|/\omega(G-S)$. Obviously, $V(T) \subseteq S$. Define $S_i = S \cap V(L_i)$, $s_i = |S_i|$, and let ω_i be the number of components of $L_i - S_i$ that contain neither u_i nor v_i $(i = 1,\dots,m)$. Then

$$\tau(G) = \frac{\ell + \sum_{i=1}^m s_i}{c + \sum_{i=1}^m \omega_i} \geqslant \frac{\ell + \sum_{i=1}^m s_i}{1 + \sum_{i=1}^m \omega_i},$$

where

$$c = \begin{cases} 0 & \text{if } u_i, v_i \in S_i \text{ for all } i \in \{1,\dots,m\}, \\ 1 & \text{otherwise.} \end{cases}$$

We now show that

$$s_i \geqslant 2\omega_i \quad (i = 1,\dots,m).$$

First note that $\omega_i \leqslant 2$, since $L - \{u,v\}$ has independence number 2. Clearly $s_i \geqslant 2\omega_i$ if $\omega_i = 0$ or $\omega_i = 1$. By exhaustion it is readily checked that if $s_i \leqslant 3$, then $\omega_i \leqslant 1$. In other words, $s_i \geqslant 2\omega_i$ if $\omega_i = 2$.

It follows that

$$\tau(G) \geqslant \frac{\ell + 2\sum_{i=1}^m \omega_i}{1 + \sum_{i=1}^m \omega_i}.$$

Since $\ell \geqslant 2$, this lower bound for $\tau(G)$ is a nonincreasing function of $\sum_{i=1}^m \omega_i$, and is hence minimized if $\omega_i = 2$ for all $i \in \{1,\dots,m\}$. Thus

$$\tau(G) \geqslant \frac{\ell + 4m}{2m+1}.$$

Set $U = V(T) \cup U_1 \cup \cdots \cup U_m$, where U_i is the set of vertices of L_i having degree 4 in L_i $(i = 1,\dots,m)$. The proof is completed by observing that

$$\tau(G) \leqslant \frac{|U|}{\omega(G-U)} = \frac{\ell + 4m}{2m+1}. \qquad \square$$

Corollary 5. *For every $\varepsilon > 0$ there exists a $(\frac{9}{4} - \varepsilon)$-tough nontraceable graph.*

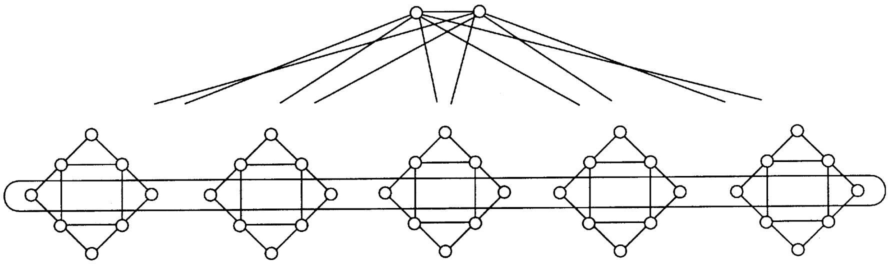

Fig. 2. The graph $G(L,u,v,2,5)$.

Proof. Clearly the graph L has no Hamilton path with endvertices u and v. Hence by Theorem 3 the graph $G(L,u,v,\ell,2\ell+3)$ is nontraceable for every ℓ. By Theorem 4 it has toughness $(9\ell+12)/(4\ell+7)$ for $\ell \geqslant 2$. The result follows. □

Remark 1. It is easily seen that Theorem 3 remains valid if "$m \geqslant 2\ell+3$" and "nontraceable" are replaced by "$m \geqslant 2\ell+1$" and "non-Hamiltonian", respectively. Thus the graph $G(L,u,v,2,5)$ is a non-Hamiltonian graph, which by Theorem 4 has toughness 2. This graph is sketched in Fig. 2. It follows that a smallest counterexample to Conjecture 2 has at most 42 vertices. Similarly, a smallest nontraceable 2-tough graph has at most 58 ($|V(G(L,u,v,2,7))|$) vertices.

Remark 2. A graph G is *neighborhood-connected* if the neighborhood of each vertex of G induces a connected subgraph of G. In [7] Chvátal also states the following weaker version of Conjecture 2: every 2-tough neighborhood-connected graph is Hamiltonian. Since all counterexamples to Conjecture 2 described above are neighborhood-connected, this weaker conjecture is also false.

Remark 3. Most of the ingredients used in the above counterexamples to Conjecture 2 were already present in [1]. It only remained to observe that using the specific graph L as our "building block" produced a graph with toughness at least 2.

3. Chordal graphs

A graph G is *chordal* if it contains no induced cycles of length at least 4. Chvátal [7] obtained $(\frac{3}{2}-\varepsilon)$-tough graphs without a 2-factor for arbitrary $\varepsilon > 0$. These examples are all chordal. Recently it was shown in [2] that every $\frac{3}{2}$-tough chordal graph has a 2-factor. Based on this, Kratsch [9] raised the question whether every $\frac{3}{2}$-tough chordal graph is Hamiltonian. Using Theorem 3 we now show that this conjecture, too, is false. A key observation in this context is that the graphs $G(H,x,y,\ell,m)$ are chordal whenever H is chordal, as is easily shown.

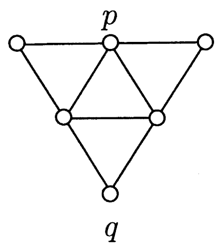

Fig. 3. The graph M.

Consider the graph M of Fig. 3. The graph M is chordal and has no Hamilton path with endvertices p and q. Hence by Theorem 3 the chordal graph $G(M,p,q,\ell,m)$ is nontraceable whenever $m \geqslant 2\ell + 3$. By arguments as used in the proof of Theorem 4 its toughness is $(\ell+3m)/(2m+1)$ if $\ell \geqslant 2$. Hence for $\ell \geqslant 2$ the graph $G(M,p,q,\ell,2\ell+3)$ is a chordal nontraceable graph with toughness $(7\ell+9)/(4\ell+7)$. We have thus obtained the following result.

Theorem 6. *For every* $\varepsilon > 0$ *there exists a* $(7/4-\varepsilon)$*-tough chordal nontraceable graph.*

On the other hand Chen et al. [6] recently proved that every 18-tough chordal graph is Hamiltonian, which means that Conjecture 1 is true when restricted to chordal graphs.

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