Energy considerations concerning current loops and magnetic objects

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(Received 16 July 1979; accepted 7 November 1979)

In the thermodynamics of compound magnetic systems there is an ambiguity in defining the free energies connected to the constituent parts or subsystems. It is argued that the choice, usually made in defining the energy of a magnetized body, leads to an expression for the energy of a current loop or coil of the form $U = (1/2)Li^2 + i\phi_a$, where i and ϕ_a (an externally applied flux, coupled to the loop) are considered as independent variables. With this expression a convention to decompose compound magnetic systems into subsystems can be given, which fits to the rules applied for nonmagnetic systems. Analogous to the case of a coil, an expression for the energy of a charged particle in a magnetic field can be derived which results in an expression for the Hamiltonian, which is generally applicable.

I. INTRODUCTION

In the thermodynamics of magnetic systems there is an ambiguity in defining the energy, or the free energy, of a magnetized body. If a body is magnetized with the help of a magnetic field, generated by a current coil, the work done by the source of the coil in this process is

$$A = (1/2)Li^2 + \int_{\text{vol}} dV \int_0^{M_0} \mu_0 \mathbf{H}_a \cdot d\mathbf{M}, \quad (1a)$$

where A is the work, i the current through the coil, L the self-inductance coefficient of the coil, $\mathbf{H}_a$ the magnetic field generated by the coil, in absence of the body, and $\mathbf{M}$ the magnetization of the body. If, on the other hand, the body is magnetized by placing it in the field of a permanent magnet, the (mechanical) work done equals

$$A' = - \int_{\text{vol}} dV \int_0^{H_{a0}} \mathbf{M} \cdot d\mathbf{H}_a, \quad (1b)$$

where $\mathbf{H}_a$ is the field of the permanent magnet—in absence of the magnetized body. The derivation of these expressions can be found in standard textbooks (e.g., Becker and Döring¹).

The expressions (1a) and (1b) both contain a term, with an integral over the volume of the magnetized body, which suggests a (free) energy contribution localized in, or connected to, the magnetized body. However, there is no way to prove that one of the choices is correct in a physically significant way, i.e., by means of an experiment.^{2,3} The acceptance of the term in expression (1a), which is generally made, occurs on the basis of convenience.⁴ So there seems to be no way to decompose a system of interacting magnetic objects into subsystems other than by convention. Nevertheless, there is a strong motivation from system theory to investigate the merits of such a convention, i.e., whether or not there is a rule to decompose systems into subsystems in a well-defined way, that is generally applicable and consistent with physics. (If such a rule exists and proves to be significantly more convenient than the alternatives, there always exists a motivation to look further for physical significance, if any.) The purpose of this paper is to show that a convention to ascribe energies to parts of complicated magnetic systems can be given and that this convention leads to an energy expression for a current loop that is not generally accepted or known.

In order to simplify the treatment we will assume that $\mathbf{H}_a$ and $\mathbf{M}$ are constant over the region of space of the body so that the integration over the volume in Eq. (1a), for example, can be carried out to give $\mu_0 \int_0^{I_0} \mathbf{H}_a \cdot d\mathbf{I}$, where $\mathbf{I}$ is the total magnetic moment of the body. Furthermore, we will assume that the process, leading to Eq. (1) takes place at constant temperature and that the conditions of reversible thermodynamics are fulfilled. In that case, the work A or A' will be equal to the increase of the free energy $\Delta F = \Delta(U - TS)$ of the system. In the case of an empty coil, which we consider as an ideal lumped element, we will also use the term "free energy" at some places, although there is no meaningful difference between the energy and the free energy then.

The two expressions mentioned above can be rewritten into

$$F = \Delta F = (1/2)Li^2 + \mu_0 \int_0^{I_0} \mathbf{H}_a \cdot d\mathbf{I} \quad (2a)$$

and

$$\Delta F' = -\mu_0 \int_0^{H_{a0}} \mathbf{I} \cdot d\mathbf{H}_a$$

or

$$F' = U_{\text{pm}} - \mu_0 \int_0^{H_{a0}} \mathbf{I} \cdot d\mathbf{H}_a, \quad (2b)$$

where U_{pm} is the energy of the permanent magnet, considered to be constant in the process (see Sec. III).

As we have said, the term

$$\mu_0 \int_0^{I_0} \mathbf{H}_a \cdot d\mathbf{I} \quad (3)$$

is generally accepted in literature as representing the free energy of the magnetized body, a choice based on convenience, which means that the term easily fits into the framework of thermodynamics, as is expressed in the Gibbs equation

$$dU = TdS - pdV + \dots + \mu_0 \mathbf{H}_a \cdot d\mathbf{I}. \quad (4)$$

It has been shown,⁵ however, that the term (3), as well as the term

$$-\mu_0 \int_0^{H_{a0}} \mathbf{I} \cdot d\mathbf{H}_a \quad (5)$$

Table I. Four expressions for mutually interacting magnetic subsystems: (1) two interacting coils; (2) a coil in interaction with a magnetized body; (3) a permanent magnet in interaction with a magnetized body; (4) two interacting permanent magnets. In (1) the subscripts refer to the first and second coil, respectively, and m is the coefficient of mutual inductance; in (4) the subscripts refer to the first and second permanent magnet, respectively.

$$\begin{aligned} (1): F &= (1/2) L_1 i_1^2 + m i_1 i_2 + (1/2) L_2 i_2^2 \\ (2): F &= (1/2) L i^2 + \mu_0 \int \mathbf{H}_a \cdot d\mathbf{I} \\ (3): F &= U_{pm} - \mu_0 \int \mathbf{I} \cdot d\mathbf{H}_a \\ (4): F &= U_{pm1} - \mu_0 \mathbf{H}_{a01} \cdot \mathbf{I}_{02} + U_{pm2} \end{aligned}$$

from Eq. (2b), lead to the same results in thermodynamics (Appendix A).

On the basis of expression (3), Eq. (2b) can be rewritten as

$$F' = U_{pm} - \mu_0 \mathbf{I}_0 \cdot \mathbf{H}_{a0} + \mu_0 \int_0^{I_0} \mathbf{H}_a \cdot d\mathbf{I} \quad (2c)$$

by means of partial integration, and a new term appears that suggests a potential energy of one body with respect to the other. However, if this vision is accepted, one can wonder why such a term is missed in Eq. (2a), and the confusion is still greater if one remembers the energy expression of a system of interacting coils (see Table I) which also contains a cross term but with a different sign. Table I gives an overview of (free) energy expressions of several combinations of interacting magnetic objects. One can note the general lack of systematics. It will be shown in Sec. II that the lack of systematics is caused by the fact that the contribution to the energy of a current-carrying coil is generally defined as $U = (1/2) Li^2$ and that this choice is not in line with the choice of the term (3). In fact, it will be shown, that a consequence of the choice of (3) is that the energy contribution ascribed to a current-carrying coil, in an interacting system, must be noted as

$$F = (1/2) Li^2 + i\phi_a, \quad (6)$$

where ϕ_a is the flux, coupled to the coil, as a consequence of applied fields, not generated by the coil itself. A preview on Table II, shows that Eq. (6) leads to a systematic notation of (free) energies of interacting systems into energies of subsystems and "potential energy" terms.

II. GIBBS EQUATION FOR A CURRENT-CARRYING COIL

The choice of (3) as being the free energy of a magnetized object suggests the acceptance of a magnetic field as an environment or, as a variable of the intensity type in thermodynamics. This means that if a coil is treated on the same footing, there must be an extensity variable, comparable to the magnetic moment in (3). Indeed, if one writes down Eq. (3), translated to a lumped current loop one finds

$$\mu_0 \mathbf{H}_a \cdot d\mathbf{I} = \mu_0 \mathbf{H}_a O di = \phi_a di, \quad (7)$$

Table II. As in Table I, but with a reordering of terms in order to show the energy form (6) for a lumped coil. In (1), $i_1 \phi_{a2} = m i_1 i_2 = i_2 \phi_{a1}$ as a consequence of reciprocity; in (2) $i\phi_a = \mu_0 \mathbf{H}_{a0} \cdot \mathbf{I}$ for the same reason.

$$\begin{aligned} (1): F &= [(1/2) L_1 i_1^2 + i_1 \phi_{a2}] - m i_1 i_2 + [(1/2) L_2 i_2^2 + i_2 \phi_{a1}] \\ (2): F &= [(1/2) Li^2 + i\phi_a] - \mu_0 \mathbf{H}_{a0} \cdot \mathbf{I}_0 + \mu_0 \int \mathbf{H}_a \cdot d\mathbf{I} \\ (3): F &= U_{pm} - \mu_0 \mathbf{H}_{a0} \cdot \mathbf{I}_0 + \mu_0 \int \mathbf{H}_a \cdot d\mathbf{I} \\ (4): F &= U_{pm1} - \mu_0 \mathbf{H}_{a01} \cdot \mathbf{I}_{02} + U_{pm2} \end{aligned}$$

where $\mathbf{I}$ is the magnetic moment of the "body" ($\mathbf{H}_a$ considered as homogeneous) and iO the magnetic moment of the loop (O is the surface enclosed by the loop).

On the other hand, the Gibbs equation for a system containing a lumped coil has a term of the form $i d\phi_k$, where ϕ_k is the total magnetic flux coupled to the coil. This term, typical for electromechanics,⁶ expresses the work that is delivered to the system from a current source, and this term does not generally appear in formula (4), which is from the field of pure thermodynamics. However, in a general theory, which considers energies of systems containing lumped elements as well as elements considered in thermodynamics the terms can be combined.⁷ But in combining these elements there seems to arise a confusing situation, when a current loop is considered as a system, with a current source and at the same time an externally applied field H_a as "environments," which can do work on the system in principle. The Gibbs equation for this case would be something like

$$dU = i d\phi_k + \phi_a di,$$

with $i d\phi_k$ as the work delivered by the source and $\phi_a di$ as the work delivered by the field [cf. Eq. (7)]. This equation is obviously wrong from the standpoint of thermodynamics, but there is an alternative way to express the fact that there is only one single current in the loop. Therefore an independent variable i^* is introduced instead of i in the second right-hand term $\phi_a di$ and next it is stated that physically i equals i^* . In this way two new expressions are created,

$$dU = i d\phi_k + \phi_a di^* \quad (8)$$

and

$$i = i^*. \quad (9)$$

So, it is stated that on the level of the Gibbs equation i and i^* should be considered as independent variables, while they are related via a constitutive equation (9) and in this way the paradox is solved. A further treatment, along the lines of the formalism of thermodynamics, is straightforward.

The second constitutive equation reads

$$\phi_k = \phi_a + Li. \quad (10)$$

Integration of (8) leads to

$$U = -(1/2) Li^{*2} + i^* \phi_k. \quad (11)$$

Indeed one finds

$$\left(\frac{\partial U}{\partial \phi_k} \right)_{i^*} = i^* = i$$

and

$$\left(\frac{\partial U}{\partial i^*} \right)_{\phi_k} = -Li^* + \phi_k = \phi_a.$$

Expression (11) gives the energy in the "extensity" parameters i^* and ϕ_k . Expressed in i and ϕ_a we have

$$U = (1/2) Li^2 + i\phi_a, \quad (12)$$

which is identical to Eq. (6).

As we have stated before, we may write $F = U$ in the case of a coil. If expression (6) is introduced into Table I we can rewrite the expressions into the form shown in Table II. In this way a systematic notation for the energy is introduced, as if the system is decomposed into subsystems interacting via potential energy terms (containing the information

about the mutual force) and this notation is based on an expression for the self-energy of a coil (6), which itself is the result of fitting the treatment of a lumped coil into the framework of thermodynamics. After all, Eq. (6) can also be found in a very simple way. If one considers an unenergized coil, placed in an externally applied magnetic field and then energizes the coil, keeping the applied field constant, the work done by the source is $(1/2)Li^2$. On the other hand, it follows from a paraphrase of the first law of thermodynamics that this work comes to the benefit of the thermodynamic potential of relevance, in this case the generalized "magnetic enthalpy" $F - i\phi_a$.

So we find $(1/2)Li^2 = F - i\phi_a$ and therefore $F = (1/2)Li^2 + i\phi_a$.

III. DIAMAGNETISM

One can feel uneasy about the fact that in Tables I and II the energies of the permanent magnets are presented as constant whether they are placed in a magnetic field or not. This looks contradictory to the fact that a permanent magnet may be considered as a conglomerate of current loops and in the energy expression of a current loop there is a large contribution $i\phi_a$. This term is present indeed in the case of a permanent magnet but is compensated by a change in the kinetic energy of the electrons as a consequence of the always present diamagnetic change of the microcurrents. To show this, let it first be stated that expression (12) is incomplete in fact, because a kinetic energy term is missing. As long as the charges which build up a current are considered as massless Eq. (12) is correct, but for a real current we must write

$$U = 1/2 (L + L^*)i^2 + i\phi_a, \quad (13)$$

where L^* is a pseudocoefficient of self-inductance representing mass inertia. As shown in Appendix B, L^* is negligible for a macroscopic current, but will dominate over L in the case of an electron in an atomic orbit (considered in a semiclassical way). In the latter case Eq. (13) can also be read as

$$U = (1/2) (m^* + m)v^2 + i\phi_a, \quad (14)$$

where m^* is a pseudomass as a consequence of electromagnetic inertia [in fact $(1/2)Li^2 = (1/2)m^*v^2$ and $(1/2)L^*i^2 = (1/2)mv^2$] and negligible compared to m in this case.

When a permanent magnet is placed in an external field the diamagnetic changes in the electron-orbital currents are small; if the microcurrents have a value i_0 without an applied field and $i_0 + \Delta i_0$ in an applied field we have (see Appendix B)

$$\Delta i_0 = -\phi_a/(L + L^*) \quad \text{with} \quad \Delta i_0 \ll i_0. \quad (15)$$

Further we have

$$\begin{aligned} U &= (1/2) (L + L^*) (i_0 + \Delta i_0)^2 + (i_0 + \Delta i_0)\phi_a \\ &= (1/2) (L + L^*)i_0^2 + (L + L^*)i_0\Delta i_0 \\ &\quad + i_0\phi_a + (1/2) (L + L^*)\Delta i_0^2 + \phi_a\Delta i_0 \\ &= U_0 + (L + L^*)i_0\Delta i_0 + i_0\phi_a, \end{aligned} \quad (16)$$

where U_0 is the starting energy and the terms of second order in Δi_0 are omitted. [The term $\phi_a\Delta i_0$ is of second order as a consequence of Eq. (15).] It is seen that, as a consequence of Eq. (15), we have $U = U_0$; the appearance of the

term $i_0\phi_a$ is compensated by the change in kinetic energy $(L + L^*)i_0\Delta i_0 = (m^* + m)v_0\Delta v_0$.

IV. ENERGY OF A CHARGED PARTICLE IN A MAGNETIC FIELD

Since we have made, in Sec. III, a step from macroscopy to microscopy it is straightforward to consider a free charged particle in a magnetic field. For this purpose we rewrite Eq. (14) by changing the term $i\phi_a$:

$$\begin{aligned} i\phi_a &= \frac{qv}{2\pi r} \int \mathbf{B}_a \cdot d\sigma = \frac{qv}{2\pi r} \int (\nabla \times \mathbf{A}) \cdot d\sigma \\ &= \frac{qv}{2\pi r} \oint \mathbf{A} \cdot d\mathbf{s} = q\mathbf{v} \cdot \mathbf{A}. \end{aligned}$$

Here we have considered a circular loop with radius r , small enough to take for $\mathbf{A}$ a constant vector potential.

In this way Eq. (14) transforms into

$$U = (1/2) (m^* + m)v^2 + q\mathbf{v} \cdot \mathbf{A}. \quad (17)$$

If we also take Eq. (17) as the energy expression for a charged particle in the sense of classical dynamics, i.e., with U as the momentary value of the energy when the particle is at vector potential $\mathbf{A}$, and we compare Eq. (17) with Eq. (13),

$$U = (1/2) (L + L^*)i^2 + i\phi_a,$$

we can find constitutive equations connected to Eq. (17) by analogy. Equation (9) transforms into

$$v = v^* \quad (18)$$

and Eq. (10), corrected for L^* , transforms into

$$\mathbf{p} = (m + m^*)\mathbf{v} + q\mathbf{A}, \quad (19)$$

in which we recognize the generalized impuls from Lagrangian mechanics.

Further we have

$$dU = \mathbf{v} \cdot d\mathbf{p} + q\mathbf{A} \cdot d\mathbf{v}^* \quad (20)$$

as the counterpart of Eq. (8), and

$$U = -(1/2) (m + m^*)v^2 + \mathbf{v}^* \cdot \mathbf{p} \quad (21)$$

as the counterpart of Eq. (11).

In the Lagrange theory of mechanics, the potential energy expression to be used in case of a charged particle in a magnetic field is found to be⁸

$$V = -q\mathbf{v} \cdot \mathbf{A}. \quad (22)$$

Neglecting m^* in the following and using Eq. (19), this means that Eq. (17) can be written as

$$U = \left(\frac{\mathbf{p} - q\mathbf{A}}{2m} \right)^2 - V. \quad (23)$$

In the first term on the right-hand side we recognize the Hamiltonian H of the system. So we have

$$H = U + V. \quad (24)$$

This means that the choice of the energy expression (17), which ultimately finds its roots in the choice of Eq. (3), leads to an interpretation of the Hamiltonian which is valid under all circumstances, whether or not there is a magnetic field, namely the sum of the internal or self-energy (containing the classical kinetic energy) and the potential energy.

Table III. As Table I, but with a reordering of terms in order to show the energy form $(1/2)Li^2$ for a lumped coil, and consequently expression (5) as the energy of a magnetized body. In (3) and (4) the energy expressions for a permanent magnet are redefined.

(1): $F = (1/2)L_1 i_1^2 + m_1 i_1 z_1 + (1/2)L_2 i_2^2$
(2): $F = (1/2)Li^2 + \mu_0 \mathbf{H}_{a0} \cdot \mathbf{I}_0 - \mu_0 \int \mathbf{I} \cdot d\mathbf{H}_a$
(3): $F = (U_{pm} - \mu_0 \mathbf{H}_{a0} \cdot \mathbf{I}_0) + \mu_0 \mathbf{H}_{a0} \cdot \mathbf{I}_0 - \mu_0 \int \mathbf{I} \cdot d\mathbf{H}_a$
(4): $F = (U_{pm1} - \mu_0 \mathbf{H}_{a02} \cdot \mathbf{I}_{01}) + \mu_0 \mathbf{H}_{a01} \cdot \mathbf{I}_{02}$ $+ (U_{pm2} - \mu_0 \mathbf{H}_{a01} \cdot \mathbf{I}_{02})$

V. DISCUSSION AND CONCLUSIONS

It has been shown that a consequent exploitation of the convention of using the term (3) meets the wish for a general formalism in which magnetics is not an exceptional case. The most important step in the process is the introduction of the term $i\phi_a$ in the expression for the energy of a coil or current loop. It can be shown (Appendix C) that this term is typical for the rotational character of the magnetic field and that an analogous term in electrostatics does not exist.

Although we have shown that Eq. (3) leads to an elegant formalism our argument is not complete since we have not shown where an acceptance of Eq. (4) would have lead us. In Table III we have worked this out, and it is seen that apart from the inconveniences in thermodynamics, a systematic formalism is only present if the energy of a diamagnet is redefined appropriately and a positive cross term is accepted. Unlike other cases of interacting systems this cross term does not represent a "potential energy" in the sense of mechanics, i.e., derived from the interacting forces. That such a convention may easily lead to wrong conclusions about the interacting forces is pointed out by Feynman,⁹ for example. Furthermore, a result like Eq. (24) is missing, so that we conclude that there is only support for the choice of Eqs. (3), (6), and (17) as a coherent set of definitions for the energies of magnetic (sub)systems.

APPENDIX A

From $F = \mu_0 \int \mathbf{H} \cdot d\mathbf{I}$ the entropy can be calculated, since

$$\begin{aligned} S &= - \left(\frac{\partial F}{\partial T} \right)_I = -\mu_0 \int_{T=\text{const}}^{I_0} \left(\frac{\partial \mathbf{H}}{\partial T} \right)_I \cdot d\mathbf{I} \\ &= -\mu_0 \int_{T=\text{const}}^{I_0} \left(\frac{\partial \mathbf{H}}{\partial T} \right)_I \left(\frac{\partial I}{\partial H} \right)_T dH \\ &= \mu_0 \int_0^{H_0} \left(\frac{\partial I}{\partial T} \right)_H \cdot d\mathbf{H}. \end{aligned}$$

The same result follows from

$$F = -\mu_0 \int_0^{H_0} \mathbf{I} \cdot d\mathbf{H}$$

and

$$S = - \left(\frac{\partial F}{\partial T} \right)_H.$$

APPENDIX B

Consider a circular current loop with radius r and N current carriers in it. As a consequence of a flux change through the loop the current carriers will be accelerated following $\Delta v = a\Delta t$, where $a = qE/m$ and $2\pi rE = -\Delta\phi/\Delta t$ ($\Delta\phi$ is the flux change). Therefore we have

$$\Delta v = - (q/2\pi r m) \Delta\phi.$$

Since $i = Nqv/2\pi r$, it follows that

$$\Delta i = - (Nq^2/4\pi^2 r^2 m) \Delta\phi,$$

from which a pseudoinductance coefficient

$$L^* = 4\pi^2 r^2 m / Nq^2 = 2\pi r m / nq^2 O$$

(n is the density of current carriers and O the cross-section surface of the wire) can be defined, while for L we have the following formula¹⁰:

$$L = 4\pi r [\ln(8r/a) - 2] \times 10^{-7}, \quad a \ll r,$$

(a is the radius of the cross section of the wire: $O = \pi a^2$). For a macroscopic loop (take $r = 10^{-2}$ m, $O = 10^{-6}$ m², $m = 9.1 \times 10^{-31}$ kg, $q = -1.6 \times 10^{-19}$ C, and $n = 8.5 \times 10^{28}$ m⁻³) we find $L^*/L \sim 10^{-9}$ while for an atomic loop (take $r = 10^{-11}$ m, $N = 1$, $a = 3 \times 10^{-15}$ m and $r = 5 \times 10^{-11}$ m) it is found $L^*/L \sim 10^4$.

With $\Delta\phi$, the total flux change $\Delta\phi_a + L\Delta i$ is meant, with $\Delta\phi_a$ the flux change caused by the external field change. So we find $\Delta i = -\Delta\phi_a/(L + L^*)$, which for an atomic current reduces to $\Delta i = -\Delta\phi_a/L^*$ or

$$\Delta i = - (Nq^2/4\pi^2 r^2 m) \Delta\phi_a.$$

With $i = Nqv/2\pi r$ we find

$$\begin{aligned} \frac{\Delta i}{i} &= - \frac{q}{2\pi r m v} \Delta\phi_a = \frac{-qr}{2mv} \Delta B_a \\ &= - \frac{q^2 r^2}{4m\mu} \Delta B_a \approx -10^{-3} \Delta B_a \end{aligned}$$

($\mu = qrv/2$ is the magnetic moment of the electron).

APPENDIX C

A magnetic field from a current loop can be written as

$$\mathbf{H}(\mathbf{r}) = \mathbf{m}(\mathbf{r}) - \nabla\psi(\mathbf{r}),$$

with $\nabla \times \mathbf{m} = \mathbf{j}$ and $\mathbf{m}(\mathbf{r}) = 0$ outside a region containing the loop.

When two such fields interact the total energy is

$$\begin{aligned} U &= (1/2)\mu_0 \iiint H_1^2 dV + \mu_0 \iiint \mathbf{H}_1 \\ &\quad \cdot \mathbf{H}_2 dV + (1/2)\mu_0 \iiint H_2^2 dV, \end{aligned}$$

where the first and the last term at the right-hand side are equal to $(1/2) L_1 i_1^2$ and $(1/2) L_2 i_2^2$, respectively.

The cross term $\mu_0 \iiint \mathbf{H}_1 \cdot \mathbf{H}_2 dV$ can be developed as follows:

$$\begin{aligned} &\mu_0 \iiint \mathbf{H}_1 \cdot \mathbf{H}_2 dV \\ &= \mu_0 \iiint (\mathbf{m}_1 - \nabla\psi_1) \cdot (\mathbf{m}_2 - \nabla\psi_2) dV \\ &= \mu_0 \iiint \mathbf{m}_1 \cdot \mathbf{m}_2 dV \\ &\quad - \mu_0 \iiint \mathbf{m}_1 \cdot \nabla\psi_2 dV \\ &\quad - \mu_0 \iiint \mathbf{m}_2 \cdot \nabla\psi_1 dV \\ &\quad + \mu_0 \iiint \nabla\psi_1 \cdot \nabla\psi_2 dV. \end{aligned}$$

The first term is zero because $\mathbf{m}_1 \cdot \mathbf{m}_2 = 0$, the last term is analogous to a term from electrostatics, where we have

$$\mathbf{E} = -\nabla\phi(\mathbf{r}).$$

The second and third terms are typical for the situation that $\mathbf{m} \neq 0$, i.e., magnetostatics with current loops. Taking the second term, for instance, it is seen that $\mathbf{H}_2 = -\nabla\psi_2$ in the region where $\mathbf{m}_1 \neq 0$, because $\mathbf{m}_2 = 0$ there.

So we have

$$\begin{aligned} -\mu_0 \int \int \int \mathbf{m}_1 \cdot \nabla\psi_2 dV &= \mu_0 \int \int \int \mathbf{m}_1 \cdot \mathbf{H}_2 dV \\ &= \int \int \int \mathbf{m}_1 \cdot \mathbf{B}_2 dV = i_1 \phi_{21}. \end{aligned}$$

This can easily be derived, when a thin film, bounded by the current loop, is taken as the volume where $\mathbf{m}_1 \neq 0$.

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