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An Application of the N-Strip Method of Integral Relations for Analyzing the Flow around a Circular Cone

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SUMMARY

The solution of sets of non-linear partial differential equations using the method of integral relations is considered. Emphasis is laid on the derivation of a general N-strip approximation algorithm. In order to check the applicability of this algorithm a program has been written to obtain the solution of the flow field around a circular cone at incidence in supersonic flow. Using the method of Stone, the angle of attack has been taken into account up to the second order. Thus a comparison can be made with the results given by Kopal.

The results show that the N-strip algorithm in the case studied is a very attractive method which leads straightforward to results of high accuracy.

1. Introduction

In many instances interesting fysical phenomena are governed by sets of partial differential equations, which are non linear and have complicated boundary conditions, often involving an in advance unknown geometrical boundary of the region of interest. In most cases an analytical solution is out of the question, but even a numerical solution by means of finite difference methods is hard to achieve in the case of an unknown boundary and would normally require a rather complicated and unwieldy iteration scheme to find this boundary as part of the solution.

By Dorodnitsyn a method was proposed for solving such problems approximately for those cases in which the partial differential equations can be written as a system of divergence expressions [1]. This so-called method of integral relations e.g. transforms a two-dimensional problem into a set of ordinary differential equations, by dividing the region of interest into a number of strips and by assuming a certain behaviour of the vector and scalar fields involved. The size of this set depends on the number of strips. An essential feature is that a priori unknown boundaries can be included quite naturally into the analysis. Since a number of methods is available for solving systems of ordinary differential equations, an approximation can be found in this way.

In many problems on fluid flow the flow quantities such as pressure and velocity may show strong variations throughout the flow field, whereas the components of the vector fields involved in the divergence expressions are much more smooth. This then probably is the reason for some rather sensational successes when applying the method with only one or two strips to problems that otherwise are much harder to solve. References [1–4] give a number of examples on the various applications. The results can be improved when using more strips, say three or four as has been shown by Belotserkovskii and Chuskin [3] and [4]. However, it appears that this improvement is obtained at the cost of solving a system of equations rapidly increasing in size and complexity. This complexity, which is mainly due to the vast number of algebraic manipulations, already becomes practically prohibitive for five strips.

Nevertheless, the method is one of the very few known today for the direct solution of non-linear problems. Naturally the search therefore is to present it in a form which eliminates or reduces its complexity, so that it would become possible to compute general N-strip approximations. Although of course, once such a scheme would be achieved the necessity arises to consider the convergence and the consistence of the method for N going to infinity, these questions will be left aside for the time being, at least their theoretical aspects.

The model problem which will be analyzed here, namely the flow around a circular cone at an angle of attack, has been the subject of a large number of papers (see e.g. [5–9]). As such it gives a fine base for comparison.

It will be shown that it is possible in this case to present an algorithm for the general N-strip approximation by using matrix calculus. The resulting system of ordinary differential equations will be simplified further, by reducing it to a system of algebraic equations, by using a method due to Stone [6]. In this way the results can be compared directly to the tables calculated by Kopal [7], [10] and [11]. The peculiar aspects involved in this approach i.e. the neglect of the vortical layer will be discussed in the paper.

The first part of the paper is devoted to a brief description of the method of integral relations and its application to the equations for conical flow. The emphasis lies on the formulation of the general N-strip scheme.

The second part is devoted to the evaluation of the systems of algebraic equations which result when applying the method of Stone. Three terms in the Stone series will be considered.

The third part gives a thorough discussion of the results. Four examples are calculated with up to ten strips.

2. Description of the Method of Integral Relations

In many cases the partial differential equations of physics and mechanics can be written as a system of generalized continuity equations, i.e. given a number of vector fields V_i and scalar fields S_i, each V_i can be considered as the velocity field of an incompressible medium with unit density and S_i as a corresponding source distribution. Hence

$$\nabla \cdot V_i = S_i \qquad i = 1 \dots n \tag{2.1}$$

The components of the vector fields and the scalar fields will usually depend on the independent and dependent variables.

In the two-dimensional case, which will be considered henceforth, the system (2.1) becomes

$$\frac{\partial P_i}{\partial x} + \frac{\partial Q_i}{\partial y} = S_i \tag{2.2}$$

where P_i and Q_i are the components of V_i.

Let it be necessary to find for a given set of boundary conditions a solution of this system in a region bounded by $x = a$, $x = b$, $y = \delta_o(x)$ and $y = \delta_N(x)$ where the quantities $\delta_o(x)$ and $\delta_N(x)$ may be unknown a priori, but are part of the solution.

Assuming that such a solution exists, this is constructed approximately by using the method of integral relations, as follows. The region is divided in N strips as indicated in Fig. 1 and it is assumed that the behaviour of $\partial P_i/\partial x$ and S_i as a function of y and for a certain x can be represented by a set of $(N+1)$ functions, which means for example

$$\frac{\partial P_i}{\partial x} = \sum_{j=0}^{N} g_{ij}(y)\, \frac{\partial P_i^j}{\partial x} \tag{2.3}$$

where the only condition on g_{ij} is: $g_{ij}(\delta_k) = \delta_{jk}$ (Kronecker delta) and where

$$\frac{\partial P_i^j}{\partial x} = \left(\frac{\partial P_i}{\partial x}\right)_{y=\delta_j}.$$

The essence of the method is the requirement that eq. (2.2) is satisfied "in the mean" for each of the regions $y = \delta_o(x) - y = \delta_k(x)$. This can be expressed as

$$\int_{\delta_o}^{\delta_k} \frac{\partial P_i}{\partial x}\, dy + \int_{\delta_o}^{\delta_k} \frac{\partial Q_i}{\partial y}\, dy = \int_{\delta_o}^{\delta_k} S_i dy\,. \tag{2.4}$$

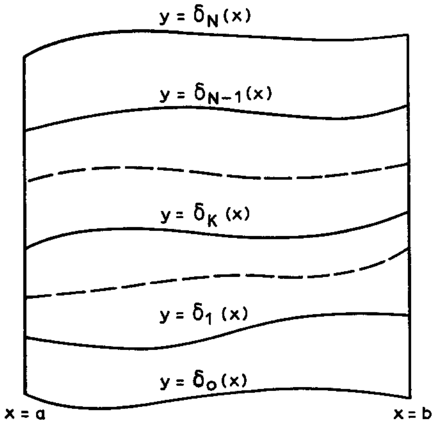


Figure 1. The region of interest divided in N strips.

When using the representation (2.3) for each of the functions $\partial P_i/\partial x$ and S_i and assuming g_{ij} to be independent of i there is found when positing

$$\int_{\delta_o}^{\delta_k} g_j(y)\,dy = a_{kj} \qquad k = 1 \ldots N \tag{2.5}$$

that

$$\sum_{j=0}^{N} a_{kj}\frac{\partial P_i^j}{\partial x} + Q_i(\delta_k) - Q_i(\delta_o) = \sum_{j=0}^{N} a_{kj} S_i^j\,. \tag{2.6}$$

It should be remarked that since g_j in general is dependent on the set $\{\delta_j(x)\}$, the values a_{kj} are dependent on x.

The system of equations (2.6) can be further transformed by using the directional derivative dP_i^j/dx given by

$$\frac{dP_i^j}{dx} = \frac{\partial P_i^j}{\partial x} + \frac{d\delta_j(x)}{dx}\frac{\partial P_i^j}{\partial y} \tag{2.7}$$

and eliminating $\partial P_i^j/\partial y$ by using a representation analogous to eq. (2.3) for P_i itself, hence

$$\frac{\partial P_i}{\partial y} = \sum_{l=0}^{N} h_{il}(y)\,P_i^l\,. \tag{2.8}$$

Substituting eqs (2.7) and (2.8) into eq. (2.6) there finally is obtained (assuming h_{il} to be independent on i)

$$\sum_{j=0}^{N} a_{kj}\left\{\frac{dP_i^j}{dx} - \frac{d\delta_j(x)}{dx}\sum_{l=0}^{N} h_l(\delta_j)P_i^l\right\} + Q_i(\delta_k) - Q_i(\delta_o) = \sum_{j=0}^{N} a_{kj} S_i^j\,. \tag{2.9}$$

In this way a system of nN differential equations is obtained, which usually will be non-linear and quite complex in character since P_i, Q_i and S_i are itself expressions in the unknown quantities. Especially for large values of N the system can become practically unmanageable, when it is tried to write down the equations.

However, in the sequel it will be shown that a large part of the algebraic manipulations can be performed by the computer.

3. An N-Strip Scheme for Conical Flow

In this chapter it will be shown first that the differential equations governing conical flow can be brought into the form of eq. (2.2). Then the boundary conditions will be analyzed to gain an insight into the properties of the region where the solution has to be found and finally using this knowledge an algorithm will be derived for the N-strip method in this case.

3.1. *The equations governing conical flow*

When using a spherical coordinate system (r, θ, ϕ) and accordingly defined velocities (u, v, w) the differential equations governing the flow of an isentropic ideal gas are

$$ru\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial \theta} + \frac{w}{\sin\theta}\frac{\partial u}{\partial \phi} - (v^2+w^2) = -\frac{r}{\rho}\frac{\partial p}{\partial r} \tag{3.1a}$$

$$ru\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial \theta} + \frac{w}{\sin\theta}\frac{\partial v}{\partial \phi} + (uv - w^2\cot\theta) = -\frac{1}{\rho}\frac{\partial p}{\partial \theta} \tag{3.1b}$$

$$ru\frac{\partial w}{\partial r} + v\frac{\partial w}{\partial \theta} + \frac{w}{\sin\theta}\frac{\partial w}{\partial \phi} + (uw + vw\cot\theta) = -\frac{1}{\rho\sin\theta}\frac{\partial p}{\partial \phi} \tag{3.1c}$$

$$\frac{\partial}{\partial r}(r^2\rho u\sin\theta) + \frac{\partial}{\partial \theta}(r\rho v\sin\theta) + \frac{\partial}{\partial \phi}(r\rho w) = 0\,. \tag{3.1d}$$

The relation between the pressure p and the density ρ is given by the velocity of sound a as follows

$$\gamma\frac{p}{\rho} = a^2 = \frac{1}{M_\infty^2} + \frac{\gamma-1}{2}(1-u^2-v^2-w^2) \tag{3.2}$$

where γ is the ratio of the specific heats and M_∞ is the Mach number of the free stream. It is assumed that u, v and w are nondimensionalized with respect to the free stream velocity U_∞, ρ with ρ_∞ and p with $\rho_\infty U_\infty^2$.

The system (3.1) can be brought into the form

$$\nabla\cdot V_i = S_i \qquad i = 1 \ldots 4 \tag{3.3}$$

where $\nabla = \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right)$ and

$$V_1 = ([\rho u^2+p]r^2\sin\theta,\ \rho uvr\sin\theta,\ \rho uwr) \qquad S_1 = (\rho v^2+\rho w^2+2p)r\sin\theta$$
$$V_2 = (\rho uvr^3\sin\theta,\ [\rho v^2+p]r^2\sin\theta,\ \rho vwr^2) \qquad S_2 = (\rho w^2+p)r^2\cos\theta$$
$$V_3 = (\rho ur^2\sin\theta,\ \rho vr\sin\theta,\ \rho wr) \qquad S_3 = 0$$
$$V_4 = (\rho uwr^3\sin^2\theta,\ \rho vwr^2\sin^2\theta,\ [\rho w^2+p]r^2\sin\theta) \qquad S_4 = 0$$

In the case of conical flow, i.e. supersonic flow with attached shock wave around a conical body, the flow quantities are independent on the radial coordinate r. Hence the system (3.3) transforms into

$$\nabla\cdot F_i = T_i \qquad i = 1 \ldots 4 \tag{3.4}$$

where $\nabla = \left(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta}\right)$ and

$$F_1 = (\rho uw,\ \rho uv\sin\theta) \qquad T_1 = \rho(-2u^2+v^2+w^2)\sin\theta$$
$$F_2 = (\rho vw,\ [\rho v^2+p]\sin\theta) \qquad T_2 = (\rho w^2+p)\cos\theta - 3\rho uv\sin\theta$$
$$F_3 = (\rho w,\ \rho v\sin\theta) \qquad T_3 = -2\rho u\sin\theta$$
$$F_4 = (\rho w^2+p,\ \rho vw\sin\theta) \qquad T_4 = -\rho w(3u\sin\theta + v\cos\theta)\,.$$

This clearly demonstrates that eq. (3.4) has the form of eq. (2.2) and hence the method of integral relations can be applied.

3.2. *The boundary conditions*

In order to solve the system given by eq. (3.4) the appropriate boundary conditions have to be formulated. At the shock wave, which as should be emphasized has an a priori unknown shape, the usual conditions of conservation of mass, momentum and energy should apply. At the surface of the conical body the normal velocity has to be zero.

First the conditions at the shock wave will be derived. If it is assumed that the spherical coordinate system is defined by an axis lying in the $x-z$ plane and having an inclination α with respect to the x-axis (see Fig. 2) the general equations of a conical shock read.

$$x = r\cos\theta_w(\phi)\cos\alpha + r\sin\theta_w(\phi)\cos\phi\sin\alpha \tag{3.5a}$$

$$y = r\sin\theta_w(\phi)\sin\phi \tag{3.5b}$$

$$z = -r\cos\theta_w(\phi)\sin\alpha + r\sin\theta_w(\phi)\cos\phi\cos\alpha \tag{3.5c}$$

where w indicates the shock wave.

It is possible now to find at every point of the shock surface the normal vector $\boldsymbol{n}$ and two tangent vectors $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$. The components of the velocity $\boldsymbol{q}$ in the direction of these vectors will be denoted by u_n, u_{t_1} and u_{t_2} respectively.

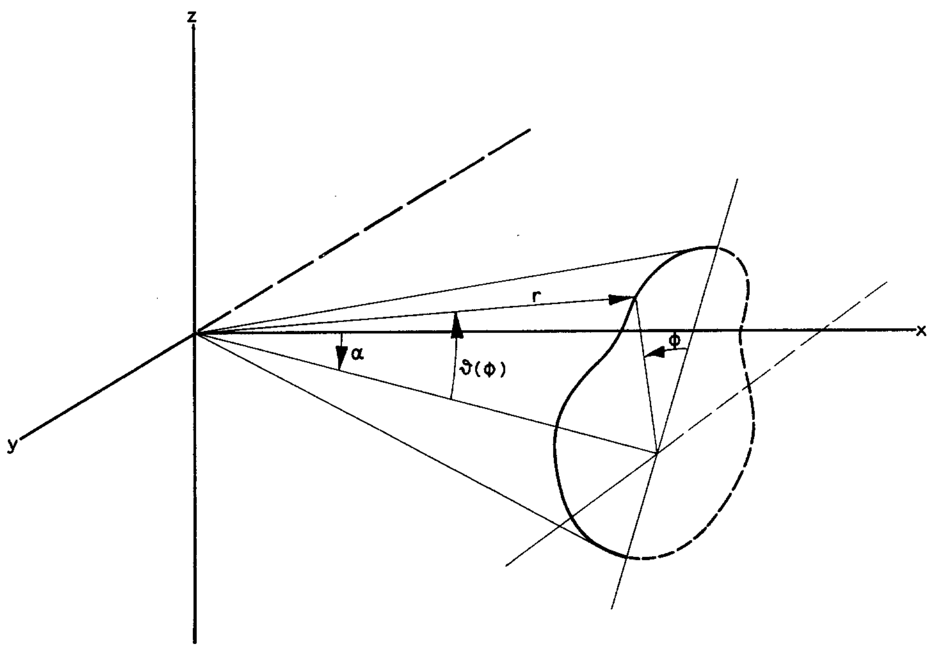


Figure 2. Geometrical representation of a conical shock.

If the quantities in front of the shock are indicated by the suffix f and those aft of the shock by the suffix a, the following three conditions must be fulfilled (see e.g. [12]).

$$u_{f_{t_1}} = u_{a_{t_1}} \tag{3.6a}$$

$$u_{f_{t_2}} = u_{a_{t_2}} \tag{3.6b}$$

$$\frac{u_{a_n}}{u_{f_n}} = \frac{1}{\gamma+1}\,\frac{(\gamma-1)M_n^2+2}{M_n^2} \tag{3.6c}$$

where $M_n = u_{f_n}/a_f$.

Assuming the free stream velocity U_∞ to be directed along the x-axis and using eqs. (3.5) and (3.6) the following expressions for the spherical components of the velocity aft of the shock can be found.

$$u_a = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos\phi \tag{3.7a}$$

$$v_a = \frac{\frac{\gamma-1}{\gamma+1} A\sin\theta_w + B\frac{d\theta_w}{d\phi}}{\left(\frac{d\theta_w}{d\phi}\right)^2 + \sin^2\theta_w} + 2\frac{\sin\theta_w}{(\gamma+1)M_\infty^2 A} \tag{3.7b}$$

$$w_a = \frac{-\frac{\gamma-1}{\gamma+1} A\frac{d\theta_w}{d\phi} + B\sin\theta_w}{\left(\frac{d\theta_w}{d\phi}\right)^2 + \sin^2\theta_w} - 2\frac{\frac{d\theta_w}{d\phi}}{(\gamma+1)M_\infty^2 A}\,. \tag{3.7c}$$

The quantities A and B are given by

$$A = \sin\alpha\sin\phi\frac{d\theta_w}{d\phi} - \sin^2\theta_w\cos\alpha + \sin\theta_w\cos\theta_w\cos\phi\sin\alpha \tag{3.8a}$$

$$B = (-\sin\theta_w\cos\alpha + \cos\theta_w\cos\phi\sin\alpha)\frac{d\theta_w}{d\phi} - \sin\theta_w\sin\phi\sin\alpha\,. \tag{3.8b}$$

Using eq. (3.6c) together with eq. (3.2) the quantities ρ and p can readily be derived and are as follows

$$\rho_a = \frac{(\gamma+1)M_\infty^2 A^2}{(\gamma-1)M_\infty^2 A^2 + 2\left\{\left(\frac{d\theta_w}{d\phi}\right)^2 + \sin^2\theta_w\right\}} \tag{3.9a}$$

$$p_a = \frac{\left[-\frac{\gamma-1}{\gamma+1}\left\{\left(\frac{d\theta_w}{d\phi}\right)^2 + \sin^2\theta_w\right\} + \frac{2\gamma}{\gamma+1}M_\infty^2 A^2\right]}{\gamma M_\infty^2\cdot\left\{\left(\frac{d\theta_w}{d\phi}\right)^2 + \sin^2\theta_w\right\}}\,. \tag{3.9b}$$

As is clear from eqs. (3.7), (3.8) and (3.9) all the quantities at the shock depend on the unknown quantity θ_w only.

Since the surface of the conical body can be described by a system similar to eq. (3.5), it is easily derived that the condition of zero normal velocity reads

$$v\sin\theta_s - w\frac{d\theta_s}{d\phi} = 0 \tag{3.10}$$

where the suffix s refers to the surface of the body.

3.3 *An algorithm for the N-strip method*

So far the actual form of the divergence expressions for conical flow have been derived together with the appropriate boundary conditions. From the discussion given in section 3.2 it is clear that the flow quantities have to be determined in a region which is indeed as indicated in Fig. 1. The known boundaries of this region are apparently $\phi = 0$ and $\phi = 2\pi$, and the surface of the conical body under consideration i.e.

$$\theta = \delta_o(\phi) = \theta_s(\phi)\,. \tag{3.11}$$

The upper boundary $\theta = \delta_N(\phi) = \theta_w(\phi)$ being the shock wave is an a priori unknown quantity.

Hence the next problem is to transform eq. (2.9) into a form which can be used for practical computations, using the information just discussed. For this purpose it is necessary to establish the actual form of the coefficients a_{kj} and $h_l(\delta_j)$ occurring in eq. (2.9). First the geometry of the strips should be chosen. In terms of the still unknown shock wave $\delta_N(\phi) = \theta_w$ it is assumed to be given by

$$\theta = \delta_j(\phi) = \theta_s(\phi) + j\,\Delta\theta(\phi) \qquad j = 0 \ldots N \tag{3.12}$$

where $\Delta\theta = (\theta_w - \theta_s)/N$.

It is further assumed that the quantities of interest say F_i, can be written as polynomials of degree N in θ, hence

$$F_i = \sum_{l=0}^{N} c_{li}\theta^l \,. \tag{3.13}$$

Thus it follows that the value of F_i^j is

$$F_i^j = \sum_{l=0}^{N} c_{li}(\theta_s + j\Delta\theta)^l$$

or in matrix notation $\tilde{F}_i = Kc_i$.

Eliminating c_{li} from this set of equations there follows

$$c_i = K^{-1}\tilde{F}_i \,. \tag{3.14}$$

Combining eq. (3.14) and eq. (2.3) there is obtained

$$g_j = \sum_{l=0}^{N} K_{lj}^{-1}\theta^l$$

where K_{lj}^{-1} is an element of the matrix K^{-1}.

The coefficients a_{kj} can now be found from eq. (2.5) and are given by

$$a_{kj} = \int_{\delta_o}^{\delta_k} g_j d\theta = \sum_{l=0}^{N} \frac{1}{l+1} K_{lj}^{-1}\{(\theta_s + k\Delta\theta)^{l+1} - \theta_s^{l+1}\} \,. \tag{3.15}$$

In order to find the coefficients $h_j(\delta_m)$, eq. (3.13) has to be differentiated with respect to θ, hence

$$\frac{\partial F_i}{\partial\theta} = \sum_{l=0}^{N} l c_{li}\theta^{l-1} \,.$$

Using eq. (3.14) together with eq. (2.8) there follows immediately

$$h_j(\delta_m) = \sum_{l=0}^{N} l K_{lj}^{-1}(\theta_s + m\Delta\theta)^{l-1} \,. \tag{3.16}$$

It should be remarked that the quantities a_{kj} and $h_j(\delta_m)$ have to be independent on θ_s, due to their invariance for a translation in θ. Furthermore it is easily seen that a_{kj} is homogeneous in $\Delta\theta$, while $h_j(\delta_m)$ should be homogeneous in $(\Delta\theta)^{-1}$. These facts can be used to simplify the calculation of the matrices involved. Taking then in this calculation $\theta_s = 1$ and $\Delta\theta = 1$, it is readely verified that eqs. (3.15) and (3.16) can be written as

$$[a_{kj}] = \bar{b}\bar{K}^{-1}\Delta\theta = \bar{M}\Delta\theta \tag{3.17}$$

where

$$[\bar{K}_{jl}] = [(1+j)^l], \quad \text{and} \quad [\bar{b}_{kl}] = \left[\frac{(1+k)^{l+1}-1}{l+1}\right]$$

while further

$$[h_j(\delta_m)] = \bar{d}\bar{K}^{-1}(\Delta\theta)^{-1} = \bar{N}(\Delta\theta)^{-1} \tag{3.18}$$

where

$$[\bar{d}_{ml}] = [l(m+1)^{l-1}] \,.$$

Inserting eqs. (3.17) and (3.18) into eq. (2.9) and taking into account eq. (3.12) there is found

$$\sum_{j=0}^{N} \bar{M}_{kj}\left\{\frac{dP_i^j}{d\phi}\Delta\theta - \left(\frac{d\theta_s}{d\phi} + j\frac{d\Delta\theta}{d\phi}\right)\sum_{l=0}^{N} \bar{N}_{jl}P_i^l\right\} + Q_i(\delta_k) - Q_i(\delta_o) = \sum_{j=0}^{N} \bar{M}_{kj}S_i^j\Delta\theta \,.$$

This equation can be further simplified by introducing the following matrices

$$[\bar{e}_{ml}] = [l(1+m)^{l-1}m], \ \bar{P} = \bar{e}\bar{K}^{-1}, \ \bar{U} = \bar{M}\bar{N}, \ \bar{V} = \bar{M}\bar{P}.$$

The resulting equation proves to be

$$\bar{M}\frac{d\tilde{P}_i}{d\phi}\Delta\theta - \bar{U}\tilde{P}_i\frac{d\theta_s}{d\phi} - \bar{V}\tilde{P}_i\frac{d\Delta\theta}{d\phi} + '\tilde{Q}_i - Q_i^0 = \bar{M}\tilde{S}_i\Delta\theta \tag{3.19}$$

where e.g. $\tilde{P}_i$ is the vector (P_i^j), and $'\tilde{Q}_i$ is the vector $\tilde{Q}_i$ with the first element omitted.

When instead of P_i, Q_i and S_i the components of the vectors F_i and the scalar fields T_i are used as defined in eq. (3.4), eq. (3.19) gives the algorithm for the N-strip approximation for conical flow. The next problem is to obtain a solution of eq. (3.19) for a specific configuration. This point will be considered in the following chapter.

4. The Solution of the Algorithm for a Circular Cone by the Method of Stone.

The system of ordinary differential equations as given by eq. (3.19) is certainly not in a form already which makes it amenable to a solution. In the case considered here the unknown quantity is the set $\{u, v, w, \rho\}$ and by using eq. (3.4) it is possible to transform eq. (3.19) into a system of equations for the elements of this set. This system can be solved in principle by any of the known methods. It should be remarked that the compatibility conditions for $\phi = 0$ and $\phi = 2\pi$ lead to a boundary value problem which seems, for a number of reasons, not easy to solve.

Here it has not been tried to follow this way, since the purpose of the paper is to show the convergence of the method by comparing the results with those already available in literature.

Therefore the configuration to be considered is a circular cone at an angle of attack α and the method of representing the unknown quantities will be the Stone method [6].

In the Stone method it is assumed that the quantities u, v, w, ρ and θ_w can be expanded in a power series in α as follows

$$u = u_0 + \alpha u'' \cos\phi + \alpha^2(u_1^* + u_2^* \cos 2\phi) \tag{4.1a}$$

$$v = v_0 + \alpha v'' \cos\phi + \alpha^2(v_1^* + v_2^* \cos 2\phi) \tag{4.1b}$$

$$\rho = \rho_0 + \alpha\rho'' \cos\phi + \alpha^2(\rho_1^* + \rho_2^* \cos 2\phi) \tag{4.1c}$$

$$\theta_w = \theta_{w_0} + \alpha(1-\varepsilon)\cos\phi + \alpha^2(\lambda_2 + \lambda_3 \cos 2\phi) \tag{4.1d}$$

$$w = \alpha w'' \sin\phi + \alpha^2 w_2^* \sin 2\phi \tag{4.1e}$$

while $\theta_s = \theta_{s_0}$. (4.1f)

As is well known, this approach has the deficiency that it is not able to represent the so-called vortical layer in the vicinity of the body [8]. As has been shown by a number of investigators among them Willett [9] and Munson [13] it is possible to remedy this situation by a careful analysis, using the method of the matching asymptotic expansions. It then turns out that the pressure as found by the Stone method is valid throughout the field, whereas u, v, w and ρ need a correction in a small layer near the body. Since in many cases the pressure will be the most important quantity, these corrections can be neglected then.

The real reason, however, to use the Stone approach is that eq. (3.19) can be reduced to a system of sets of algebraic equations which can be solved consecutively, while abundant numerical results are available [7], [10] and [11].

In the sequel the zero-, first- and second order terms will be treated in a number of sections.

4.1 *The solution for axially symmetric flow*

In the case of axially symmetric flow the system (3.19) can be strongly simplified. Denoting the components of F_i by F_{i1} and F_{i2}, the fact that w has to be zero, while all quantities have to be independent of ϕ lead to

$$'\tilde{F}_{i2} - F^0_{i2} = \overline{M}\,\tilde{T}_i \Delta\theta \qquad i = 1 \dots 3 . \tag{4.2}$$

The boundary conditions are readily verified from eqs. (3.7), (3.9) and (3.10) to be

$$u^N_o = \cos\theta_{w_0} \tag{4.3a}$$

$$\rho^N_o = (\gamma+1) M^2_\infty \frac{\sin^2\theta_{w_0}}{(\gamma-1) M^2_\infty \sin^2\theta_{w_0} + 2} \tag{4.3b}$$

$$v^N_o = -\frac{\sin\theta_{w_0}}{\rho} \tag{4.3c}$$

$$v^0_o = 0 . \tag{4.3d}$$

The system (4.2) is a set of transcendental algebraic equations which cannot be solved directly. However, for a given value of $\Delta\theta$ the system reduces to a set of algebraic equations of third degree. This latter set can be solved rather easily in an iterative way by applying the Newton–Raphson procedure. Hence the following method was used for obtaining the solution of eq. (4.2).

- A set of estimated values is chosen for θ_{w_0} and hence for $\Delta\theta$
- For each value of θ_{w_0} the system minus one omitted equation is solved by applying the Newton–Raphson procedure.
- The omitted equation is used to determine a new estimate of θ_{w_0} and so on until the required accuracy has been reached.

The actual form of the system of equations to be solved becomes after $(p-1)$ iteration steps of the Newton–Raphson procedure

$$R_p x_p = B_p \tag{4.4}$$

where R_p is a matrix with $(3N-1)\times(3N-1)$ elements and where x_p and B_p are vectors with $(3N-1)$ elements.

The elements of the unknown vector x_p are given by the following expression

$$x^T_p = \{(u^q_{p+1} - u^q_p), (v^r_{p+1} - v^r_p), (\rho^q_{p+1} - \rho^q_p)\} \qquad q = 0 \dots (N \dots 1); r = 1 \dots (N \dots 1)$$

The subscript zero, referring to the zero angle of attack case has been deleted here and will be deleted in the rest of this sections. The elements of the known vector B_p are

$$B^T_p = \{\overline{M}\tilde{T}_1 \Delta\theta - '\tilde{F}_{12} + F^0_{12}, \overline{M}\tilde{T}_2 \Delta\theta - '\tilde{F}_{22} + F^0_{22}, \overline{M}'\tilde{T}_3 - '\tilde{F}_{32} + F^0_{32}\}$$

where the prime in the last expression means the deletion of the last row of $\overline{M}$ and the last element of $\tilde{F}_{32}$. It will be clear that the values of the elements are computed from the known values of u_p, v_p and ρ_p. Solving the system a set of values for u_{p+1}, v_{p+1} and ρ_{p+1} is obtained. After completion of this iteration, a new value for θ_w can be found by solving as a function of θ_w the omitted equation

$$F^N_{32} - F^0_{32} - \sum_{j=0}^{N} \overline{M}_{Nj} T^j_3 \Delta\theta = 0 . \tag{4.5}$$

Since this equation is a transcendental expression in θ_w, this leads to another iteration with respect to the value of θ_w. Experience has shown that the process as described here is rapid convergent to the true solution of the original system (4.2).

The attention is once more drawn to the fact that eqs. (4.3) and (4.4) indeed present an algorithm since they lead to a program whose size is virtually independent of the number of strips N.

4.2. *The first order lift case*

Once the solution of the flow quantities for the zero angle of attack case has been obtained, it is possible to proceed to the solution of the first order quantities u'', v'', w'', ρ'' and ε. The governing system of equations and the appropriate boundary conditions can be derived from eqs.

(3.7), (3.9) and (3.19) by inserting eq. (4.1) and equating to zero all terms in $\alpha \cos \phi$ and $\alpha \sin \phi$.

The boundary conditions give rise to the following expressions for the shock wave quantities

$$u_N'' = \varepsilon \sin \theta_{w_0} \tag{4.6a}$$

$$v_N'' = \varepsilon \cot\theta_{w_0} \left\{ \frac{\gamma-1}{\gamma+1} \sin\theta_{w_0} - \frac{2}{(\gamma+1) M_\infty^2 \sin \theta_{w_0}} \right\} \tag{4.6b}$$

$$w_N'' = \varepsilon \frac{1-\rho_0^N}{\rho_0^N} - \frac{1}{\rho_0^N} \tag{4.6c}$$

$$\rho_N'' = -\frac{4\varepsilon (\rho_0^N)^2}{(\gamma+1) M_\infty^2 \sin^2 \theta_{w_0}} \cot \theta_{w_0} \tag{4.6d}$$

$$v_0'' = 0 \tag{4.6e}$$

where for reasons of readibility the superscript N in the first order quantities has been changed into a subscript.

The system of governing equations, when expanded to first order in α gives rise to the following set of equations.

$$'\tilde{A}_i - A_i^0 + \overline{M}\tilde{B}_i \Delta\theta - \overline{M}\tilde{C}_i \Delta\theta - \overline{M}\tilde{\tilde{C}}_i \frac{1-\varepsilon}{N} = 0 \qquad i = 1 \dots 3$$

$$'\tilde{A}_4 - A_4^0 + \overline{M}\tilde{B}_4 \Delta\theta - \overline{M}\tilde{C}_4 \Delta\theta + \overline{V}\tilde{\tilde{B}}_4 \frac{1-\varepsilon}{N} = 0 \tag{4.7}$$

It follows that the eqs. (4.7) lead to a linear algebraic system of $4N$ equations for the $4N$ unknowns ε, u_0'', w_0'', ρ_0'', u_r'', v_r'', w_r'' and ρ_r'' $(r = 1 \dots (N \dots 1))$. It will be clear that the shock wave quantities are all expressed in the unknown quantity ε by using eqs. (4.6a)–(4.6d).

Here too the total amount of programming is independent of the number of strips, while no difficulties are encountered in solving the system.

4.3. *The second order lift case*

Having obtained the zero-order and first order terms in the expansion given by eq. (4.1), the results can be used to calculate the second order terms. It is easily seen that the governing equations and the boundary conditions can be split into two independent systems.

The boundary conditions for the first system read

$$u_{N1}^* = -\tfrac{1}{4}(1+\varepsilon^2) \cos \theta_{w_0} - \lambda_2 \sin \theta_{w_0} \tag{4.8a}$$

$$v_{N1}^* = \frac{\gamma-1}{4(\gamma+1) \sin \theta_{w_0}} \{(1-2\varepsilon+3\varepsilon^2) - (1+\varepsilon^2) \cos^2 \theta_{w_0} - 2\lambda_2 \sin 2\theta_{w_0}\} + $$
$$+ \frac{\varepsilon(1-\varepsilon)}{2 \sin \theta_{w_0}} + \frac{1}{2(\gamma+1) M_\infty^2 \sin^3 \theta_{w_0}} \{(1-2\varepsilon-\varepsilon^2) + (1-\varepsilon^2) \cos^2 \theta_{w_0} + 2\lambda_2 \sin 2\theta_{w_0}\} \tag{4.8b}$$

$$\rho_{N1}^* = \tfrac{1}{2} \frac{(\rho_N'')^2}{\rho_{N_0}} - \frac{2(\rho_{N_0})^2}{(\gamma+1) M_\infty^2 \sin^4 \theta_{w_0}} \{(\varepsilon^2 - \tfrac{1}{2}) \cos^2 \theta_{w_0} + \varepsilon^2 - \lambda_2 \sin 2\theta_{w_0}\} \tag{4.8c}$$

$$v_{01}^* = 0$$

while the set of linear equations is given by

$$'\tilde{A}_{i1}^* - A_{i1}^{*0} + \overline{M}\tilde{B}_{i1}^* - \overline{V}\tilde{\tilde{B}}_{i1}^* - \overline{M}\tilde{C}_{i1}^* = 0 \qquad i = 1 \dots 3 \tag{4.9}$$

As is clear these terms in fact give a correction to the axially symmetric field due to the angle of attack. Although the various expressions become more elaborate, it is not difficult to write a program for the system (4.9), thereby taking into account of the eqs. (4.8).

The boundary conditions for the second system read

$$u^*_{N2} = \tfrac{1}{4}(1-\varepsilon^2)\cos\theta_{w_0} - \lambda_3 \sin\theta_{w_0} \tag{4.10a}$$

$$v^*_{N2} = -\frac{\gamma-1}{4(\gamma+1)\sin\theta_{w_0}}\{(1-\varepsilon)^2-(1-\varepsilon)^2\cos^2\theta_{w_0}+2\lambda_3\sin 2\theta_{w_0}\} + \\ -\frac{\varepsilon(1-\varepsilon)}{2\sin\theta_{w_0}} + \frac{1}{2(\gamma+1)M_\infty^2\sin^3\theta_{w_0}}\{(1-\varepsilon)^2+(1+\varepsilon^2)\cos^2\theta_{w_0}-2\lambda_3\sin 2\theta_{w_0}\} \tag{4.10b}$$

$$w^*_{N2} = -\frac{1-\varepsilon}{\gamma+1}\cot\theta_{w_0}\left\{1-\frac{1-2\varepsilon}{M_\infty^2\sin^2\theta_{w_0}}\right\} + \frac{4}{\gamma+1}\lambda_3\left\{1-\frac{1}{M_\infty^2\sin^2\theta_{w_0}}\right\} \tag{4.10c}$$

$$\rho^*_{N2} = \frac{1}{2}\frac{(\rho''_N)^2}{\rho_{N_0}} - \frac{2(\rho_{N_0})^2}{(\gamma+1)M_\infty^2\sin^4\theta_{w_0}}\{(\varepsilon^2+\tfrac{1}{2})\cos^2\theta_{w_0}-\lambda_3\sin 2\theta_{w_0}\} \tag{4.10d}$$

$$v^*_{02} = 0 \tag{4.10e}$$

The set of equations proves to be

$$\tilde{A}^*_{i2} - A^{*0}_{i2} + \overline{M}\tilde{B}^*_{i2} - \overline{V}\tilde{\overline{B}}^*_{i2} - \overline{M}C^*_{i2} = 0 \qquad i = 1\ldots 4\,. \tag{4.11}$$

While the system (4.9) is a set of $3N$ equations for the $3N$ unknowns λ_2, u^*_{01}, ρ^*_{01}, u^*_{j1}, v^*_{j1}, ρ^*_{j1}, the system (4.11) is a set of $4N$ equations in the $4N$ unknown quantities λ_3, u^*_{02}, w^*_{02}, ρ^*_{02}, u^*_{r2}, v^*_{r2}, w^*_{r2}, ρ^*_{r2}. In this case too, the programming itself is independent of the number of strips and presents no difficulties.

A more detailed derivation of the coefficients in eqs. (4.4), (4.7), (4.9) and (4.11) is contained in [15].

5. Discussion of Results

In the preceding chapter the general formulae are given from which the actual flow quantities in a certain case can be computed. In order to do so four separate programs have been written which can be linked together by using discs or magnetic tapes.

The first program uses as input estimated values of u_{s_0}, ρ_{s_0}, two values of θ_{w_0}, the quantities γ, M_∞, θ_s and N, the number of strips to be used. The program computes the values in the axially-symmetric case according to the formulae given in section 4.1. Experience has shown that the choice of the values of u_{s_0} and ρ_{s_0} is not very critical for the convergence of the Newton–Raphson method. The output of the first program can be used as input for the second program which calculates the double primed quantities according to the expressions given in section 4.2. The output of the second program consists of the flow quantities of the axially-symmetric case and the first order lift case. It can be used as input for either the third or the fourth program. The third program calculates the second order corrections to the axially-symmetric flow field, while the fourth program computes the second order corrections due to harmonic distortions in the boundary conditions. To show the capability of the algorithm four different cases of conical flow have been considered, *viz.*

$\theta_s = 10^\circ$ $\quad M_\infty = 2.3869$ and $M_\infty = 5.4223$

$\theta_s = 20^\circ$ $\quad M_\infty = 2.4331$ and $M_\infty = 5.5457$.

In this way the behaviour of the flow can be shown in low and high Mach number cases and for small and large semi-top angles. In Table 1 results are given for the axially-symmetric case. The largest number of strips in each case is such as to give results correct up to the fourth decimal place.

In Table 2 results are given for the first order lift case. Here too the number of strips has been chosen as to fulfill the same requirement as for the axially symmetric case.

In Tables 3 and 4 the results of the two second order cases are given. Here in both cases the results are calculated for up to ten strips.

TABLE 1 *Results for axially symmetric flow*

M_∞	θ_s	θ_w	u_s	ρ_s	u_w	v_w	ρ_w	number of strips
2.3869	10°	0.46012	0.95641	1.26044	0.89601	−0.40347	1.10056	Kopal
		0.51632	0.98898	1.05797	0.86964	−0.37880	1.30330	1
		0.45903	0.95203	1.24135	0.89648	−0.40404	1.09661	2
		0.46117	0.95559	1.26052	0.89553	−0.40295	1.10435	3
		0.46065	0.95645	1.26059	0.89576	−0.40321	1.10248	4
		0.46043	0.95643	1.26076	0.89586	−0.40333	1.10168	5
		0.46032	0.95640	1.26046	0.89591	−0.40338	1.10128	6
		0.46013	0.95643	1.26046	0.89599	−0.40348	1.10059	7
5.4223	10°	0.26203	0.97265	1.90404	0.96586	−0.15281	1.69528	Kopal
		0.26648	0.97319	1.87008	0.96470	−0.15175	1.73531	1
		0.26204	0.97249	1.90078	0.96586	−0.15281	1.69527	2
		0.26204	0.97266	1.90418	0.96586	−0.15281	1.69527	3
2.4431	20°	0.57674	0.87867	1.77523	0.83824	−0.34733	1.57001	Kopal
		0.60133	0.87852	1.75755	0.82458	−0.34154	1.65643	1
		0.57649	0.87814	1.77143	0.83838	−0.34740	1.56904	2
		0.57689	0.87848	1.77518	0.83816	−0.34730	1.57047	3
		0.57677	0.87867	1.77515	0.83823	−0.34733	1.57006	4
		0.57676	0.87868	1.77523	0.83824	−0.34733	1.57000	5
5.5457	20°	0.42644	0.91570	3.27586	0.91045	−0.13503	3.06331	Kopal
		0.42826	0.91544	3.26556	0.90970	−0.13504	3.07516	1
		0.42643	0.91567	3.27484	0.91045	−0.13503	3.06325	2
		0.42644	0.91570	3.27584	0.91045	−0.13503	3.06332	3
		0.42643	0.91570	3.27581	0.91045	−0.13503	3.06329	4

TABLE 2 *Results for first-order lift case*

M_∞	θ_s	θ_w	ε	u_s''	w_s''	ρ_s''	u_w''	v_w''	w_w''	ρ_w''	number of strips
2.3869	10°	0.46012	0.25663	0.26657	−1.52926	−1.65852	0.11420	−0.14535	−0.93255	−0.96357	Kopal
		0.45588	0.20308	0.92784	−1.41959	−1.50947	0.08941	−0.10662	−0.93744	−0.73466	1
		0.45903	0.30171	0.17822	−1.43725	−2.04005	0.13368	−0.15555	−0.93848	−1.09160	2
		0.46117	0.24893	0.24611	−1.48986	−1.73077	0.11077	−0.12678	−0.92903	−0.90071	3
		0.46065	0.25865	0.26515	−1.51513	−1.67005	0.11498	−0.13212	−0.93109	−0.93586	4
		0.46043	0.25843	0.26576	−1.52430	−1.66257	0.11483	−0.13217	−0.93155	−0.93504	5
5.4223	10°	0.26203	0.69010	0.20890	−1.02262	−6.79773	0.17873	−0.16900	−0.87262	−6.23430	Kopal
		0.26203	0.67779	0.26099	−1.08509	−6.72482	0.17557	−0.16571	−0.86785	−6.12325	1
		0.26204	0.69029	0.20527	−1.05032	−6.87205	0.17882	0.16873	−0.87298	−6.23511	2
		0.26204	0.68968	0.20866	−1.04793	−6.79845	0.17866	0.16859	−0.87273	−6.22957	3
2.4431	20°	0.57674	0.66461	0.43555	−1.13362	−2.68804	0.36237	−0.16638	−0.87821	−2.35732	Kopal
		0.57243	0.67647	0.57157	−1.07924	−2.66310	0.36643	−0.17426	−0.88458	−2.40948	1
		0.57649	0.66762	0.42491	−1.14568	−2.74146	0.36390	−0.16821	−0.87945	−2.37092	2
		0.57689	0.66300	0.43310	−1.15095	−2.69351	0.36161	−0.16668	−0.87759	−2.35383	3
		0.57677	0.66384	0.43541	−1.14873	−2.68869	0.36201	−0.16700	−0.87795	−2.35702	4
		0.57676	0.66406	0.43557	−1.14713	−2.68767	0.36212	−0.16707	−0.87803	−2.35782	5
5.5457	20°	0.42644	0.93746	0.39181	−0.62017	−6.40733	0.38772	0.00882	−0.95771	−6.11905	Kopal
		0.42616	0.94170	0.39645	−0.78782	−6.25440	0.38928	0.00871	−0.96074	−6.15197	1
		0.42643	0.93722	0.39171	−0.72227	−6.40606	0.38766	0.00883	−0.95772	−6.11915	2
		0.42644	0.93747	0.39186	−0.69771	−6.40066	0.38776	0.00884	−0.95788	−6.12056	3
		0.42643	0.93741	0.39179	−0.68357	−6.40517	0.38774	0.00883	−0.95784	−6.12027	4

TABLE 3 *Results for second-order lift case (zero-harmonic)*

M_∞	θ_s	θ_w	λ_2	u^*_{s1}	ρ^*_{s1}	p^*_{s1}	u^*_{w1}	v^*_{w1}	ρ^*_{w1}	p^*_{w1}	number of strips
2.3869	10°	0.46012	0.28452	−0.49683	−0.65600	−0.07540	−0.36865	0.82992	2.72124	0.51476	Kopal
		0.45588	0.74790	−0.01940	−6.27958	−0.76898	−0.56299	1.13165	4.45784	0.59538	1
		0.45903	0.19246	−0.67895	−1.41639	−0.10251	−0.32980	0.75193	2.32200	0.46430	2
		0.46117	0.30410	−0.53238	−0.80658	−0.07447	−0.37308	0.84327	2.78108	0.53839	3
		0.46065	0.28484	−0.49936	−0.70453	−0.07700	−0.36554	0.82856	2.70274	0.52420	4
		0.46043	0.29537	−0.49700	−0.67657	−0.07669	−0.37016	0.83509	2.74216	0.53131	5
		0.46032	0.29291	−0.49481	−0.66220	−0.07641	−0.36926	0.83299	2.73171	0.51652	6
		0.46025	0.29385	−0.49519	−0.65949	−0.07637	−0.36974	0.83329	2.73469	0.51704	7
		0.46021	0.29311	−0.49485	−0.65785	−0.07633	−0.36948	0.83260	2.73146	0.51656	8
		0.46019	0.29273	−0.49543	−0.65900	−0.07632	−0.36935	0.83222	2.72978	0.51628	9
		0.46012	0.29258	−0.49452	−0.65000	−0.07650	−0.36936	0.83190	2.72872	0.51612	10
5.4223	10°	0.26198	0.84105	−0.41160	−6.18993	0.14245	−0.57316	0.42396	10.91468	0.69661	1
		0.26204	0.75321	−0.58612	−2.78853	0.26383	−0.55165	0.39339	10.06437	0.67475	2
		0.26204	0.76050	−0.58696	−2.83286	0.26501	−0.55333	0.39670	10.13934	0.67783	3
		0.26203	0.75938	−0.60093	−3.10651	0.26509	−0.55312	0.39576	10.12569	0.67734	4
		0.26202	0.75883	−0.61281	−3.32875	0.26509	−0.55303	0.39527	10.11873	0.67710	5
		0.26202	0.75859	−0.62305	−3.51763	0.26514	−0.55297	0.39514	10.11625	0.67699	6
		0.26202	0.75852	−0.63148	−3.67236	0.26512	−0.55297	0.39505	10.11520	0.67696	7
		0.26202	0.75844	−0.63905	−3.80873	0.26516	−0.55294	0.39504	10.11446	0.67693	8
		0.26202	0.75846	−0.64559	−3.92802	0.26514	−0.55295	0.39503	10.11455	0.67693	9
		0.26202	0.75860	−0.65138	−4.01991	0.26514	−0.55299	0.39504	10.11568	0.67699	10
2.4431	20°		0.49610			+0.07143	−0.057160	+0.23742	1.95148	0.55425	Kopal
		0.57243	0.55697	−0.40939	−1.87989	−0.05717	−0.60801	0.25106	2.17832	0.60393	1
		0.57649	0.47980	−0.59215	−1.05116	0.06233	−0.56455	0.23917	1.91166	0.54682	2
		0.57689	0.49327	−0.58568	−0.98452	0.06773	−0.57069	0.24757	1.97319	0.55842	3
		0.57677	0.49175	−0.59552	−1.02180	0.06754	−0.57006	0.24631	1.96554	0.55702	4
		0.57676	0.49159	−0.60676	−1.05814	0.06753	−0.57004	0.24603	1.96432	0.55684	5
		0.57675	0.49131	−0.61634	−1.08910	0.06762	−0.56990	0.24590	1.96318	0.55661	6
		0.57675	0.49132	−0.62432	−1.11494	0.06760	−0.56992	0.24586	1.96308	0.55661	7
		0.57674	0.49121	−0.63147	−1.13760	0.06765	−0.56986	0.24582	1.96266	0.55652	8
		0.57675	0.49125	−0.63767	−1.15771	0.06763	−0.56988	0.24583	1.96278	0.55655	9
		0.57675	0.49131	−0.64305	−1.17191	0.06763	−0.56992	0.24582	1.96294	0.55659	10
5.5457	20°	0.42616	0.32951	−0.57485	−11.51054	0.13865	−0.56572	−0.07541	−2.14428	0.42502	1
		0.42643	0.34339	−0.70283	−13.01751	0.17498	−0.56958	−0.06920	−1.97980	0.43477	2
		0.42644	0.34345	−0.74422	−13.98075	0.17441	−0.56971	−0.06952	−1.98354	0.43473	3
		0.42643	0.34355	−0.77791	−14.69048	0.17540	−0.56972	−0.06945	−1.98189	0.43481	4
		0.42643	0.34362	−0.80236	−15.21122	0.17522	−0.56977	−0.06950	−1.98198	0.43485	5
		0.42643	0.34359	−0.82333	−15.63977	0.17548	−0.56975	−0.06947	−1.98188	0.43484	6
		0.42643	0.34364	−0.84033	−15.99002	0.17540	−0.56977	−0.06949	−1.98173	0.43487	7
		0.42643	0.34362	−0.85549	−16.29499	0.17551	−0.56976	−0.06948	−1.98178	0.43485	8
		0.42643	0.34365	−0.86849	−16.55797	0.17546	−0.56977	−0.06948	−1.98165	0.43487	9
		0.42644	0.34366	−0.88034	−16.79249	0.17552	−0.56978	−0.06948	−1.98150	0.43489	10

TABLE 4 *Results for sec nd-order lift case (second-harmonic)*

M_∞	θ_s	θ_w	λ_3	u^*_{s2}	W_{s2}	ρ^*_{s2}	p^*_{s2}	u^*_{w2}	v^*_{w2}	W_{w2}	ρ^*_{w2}	p^*_{w2}	number of strips
			0.68548	−0.07137	−0.84826	3.62518	0.73946	−0.09304	−0.43298	−0.23352	0.79730	0.16287	Kopal
2.3869	10°	0.45588	0.62457	2.98424	−1.73534	14.32094	0.62407	−0.05974	−0.47331	−0.21431	0.51147	0.10306	1
		0.45903	0.52580	−0.40735	−0.75743	2.25611	0.68290	−0.02925	−0.51597	−0.28652	0.28242	0.08161	2
		0.46117	0.72135	−0.04879	−0.68595	3.35460	0.73280	−0.11099	−0.41118	−0.21243	0.93134	0.18598	3
		0.46065	0.68901	0.06511	−0.73430	3.61893	0.73711	−0.09733	−0.42889	−0.22674	0.82358	0.16732	4
		0.46043	0.69825	0.06387	−0.75472	3.60838	0.73850	−0.10125	−0.42455	−0.22576	0.85592	0.17315	5
		0.46032	0.69091	0.06795	−0.76622	3.61615	0.73910	−0.09811	−0.42856	−0.22870	0.83098	0.16879	6
		0.46025	0.69035	0.06911	−0.77248	3.61815	0.73940	−0.09789	−0.42898	−0.22944	0.82046	0.16857	7
		0.46021	0.68774	0.06978	−0.77371	3.61953	0.73943	−0.09677	−0.43041	−0.23047	0.82061	0.16702	8
		0.46019	0.68640	0.07076	−0.77525	3.62239	0.73952	−0.09621	−0.43113	−0.23099	0.81611	0.16624	9
		0.46012	0.68298	0.07044	−0.79017	3.61573	0.73981	−0.09472	−0.43304	−0.23236	0.80431	0.16416	10
5.4223	10°	0.26197	0.56595	0.40225	−1.28995	12.59695	0.54223	−0.01897	−0.81148	−0.11384	0.09948	0.21339	1
		0.26204	0.51314	0.10872	−1.23748	9.79929	0.59898	−0.00652	−0.82270	−0.15185	−0.34193	0.19531	2
		0.26204	0.51791	0.14928	−1.26804	10.59878	0.60286	−0.00755	−0.82176	−0.14878	−0.30496	0.19664	3
		0.26203	0.51539	0.16694	−1.25172	10.91091	0.60240	−0.00699	−0.82236	−0.15054	−0.32496	0.19588	4
		0.26202	0.51358	0.17927	−1.24355	11.14934	0.60266	−0.00657	−0.82277	−0.15184	−0.33976	0.19528	5
		0.26202	0.53159	0.18929	−1.23111	11.32918	0.60245	−0.00657	−0.82277	−0.15185	−0.33945	0.19531	6
		0.26202	0.51306	0.19785	−1.22291	11.48766	0.60251	−0.00645	−0.82289	−0.15221	−0.34389	0.19513	7
		0.26202	0.51321	0.20532	−1.21483	11.62071	0.60243	−0.00649	−0.82286	−0.15209	−0.34252	0.19519	8
		0.26202	0.51303	0.21191	−1.20802	11.74128	0.60246	−0.00644	−0.82289	−0.15223	−0.34419	0.19511	9
		0.26202	0.51277	0.21768	−1.20627	11.83370	0.60249	−0.00638	−0.82296	−0.15242	−0.34638	0.19502	10
2.4431	20°		0.22390				0.60519	−0.00567	−0.36186	−0.08953	+0.01923	0.14010	Kopal
		0.57243	0.17445	0.42960	−0.59626	3.28043	0.55870	0.01949	−0.38409	−0.12636	−0.15798	0.10720	1
		0.57649	0.20415	0.10387	−0.58140	2.23994	0.58636	0.00490	−0.37287	−0.10468	−0.06653	0.12344	2
		0.57689	0.21496	0.14532	−0.59878	2.39539	0.59230	0.00019	−0.37064	−0.09870	−0.03592	0.12814	3
		0.57677	0.21268	0.16699	−0.58989	2.45853	0.59166	0.00123	−0.37119	−0.10008	−0.04264	0.12703	4
		0.57676	0.21126	0.17893	−0.58645	2.49931	0.59200	0.00195	−0.37152	−0.10102	−0.04732	0.12612	5
		0.57675	0.21133	0.18821	−0.57973	2.52802	0.59172	0.00100	−0.37150	−0.10095	−0.04697	0.12621	6
		0.57675	0.21087	0.19641	−0.57558	2.55500	0.59181	0.00214	−0.37161	−0.10126	−0.04853	0.12590	7
		0.57674	0.21101	0.20340	−0.57123	2.57670	0.59169	0.00206	−0.37157	−0.10116	−0.04802	0.12601	8
		0.57675	0.21083	0.20969	−0.56769	2.59721	0.59173	0.00215	−0.37161	−0.10128	−0.04866	0.12587	9
		0.57675	0.21070	0.21504	−0.56679	2.61153	0.59176	0.00222	−0.37164	−0.10137	−0.04909	0.12579	10
5.5457	20°	0.42616	−0.01003	0.11508	−0.45781	5.09624	0.56772	0.02992	−0.20713	−0.07588	−3.86217	0.26042	1
		0.42643	−0.01051	0.19601	−0.37803	6.70778	0.56006	0.03203	−0.21028	−0.08116	−3.86158	0.25425	3
		0.42644	−0.01226	0.23987	−0.32041	7.72827	0.56101	0.03265	−0.21006	−0.08326	−3.87302	0.25347	3
		0.42643	−0.01183	0.27339	−0.28186	8.41636	0.55954	0.03249	−0.21012	−0.08274	−3.87021	0.25366	4
		0.42643	−0.01220	0.29781	−0.25399	8.93836	0.55978	0.03263	−0.21009	−0.08321	−3.87268	0.25347	5
		0.42643	−0.01204	0.31873	−0.23356	9.36231	0.55941	0.03257	−0.21010	−0.08301	−3.87159	0.25356	6
		0.42643	−0.01217	0.33573	−0.21704	9.71332	0.55952	0.03262	−0.21010	−0.08318	−3.87246	0.25348	7
		0.42643	−0.01209	0.35087	−0.20393	10.01675	0.55937	0.03259	−0.21010	−0.08308	−3.87195	0.25353	8
		0.42643	−0.01216	0.36386	−0.19277	10.28017	0.55943	0.03262	−0.21010	−0.08316	−3.87237	0.25349	9
		0.42644	−0.01216	0.37572	−0.18314	10.51415	0.55935	0.03262	−0.21010	−0.08316	−3.87235	0.25349	10

As is clear from Tables 1 and 2 convergence is reached more rapidly (with a fewer number of strips) in the high Mach number cases than in the low Mach number cases. An other tendency which is visible is the fact that the small semi-top angle cases require more strips than the large semi-top angle cases. This behaviour can be explained by observing that the gradients in the flow field become stronger at low Mach number-small semi-top angle cases.

As to the Tables 3 and 4 it is seen that apart from the quantities u_{s1}^*, ρ_{s1}^*, u_{s2}^*, w_{s2}^* and ρ_{s2}^* the same tendencies are present. However, the convergence is now only in about three decimal places. This behaviour must be due to the fact that the second order approximation according to Stone will give rise to infinite values for the quantities referred to above, which in itself is a consequence of the neglection of the vortical layer in the Stone theory. By using the method of integral relations the singular behaviour of the quantities at the cone surface is suppressed, but of course one may not expect convergence of these quantities when increasing the number of strips. It is very remarkable and a severe test on the method that notwithstanding this the other quantities show the desired convergence, albeit in less digits.

Combining the various results it is possible to calculate the pressure distribution along the cone surface for different angles of attack.

In Figs. 3a–b this pressure distribution in two of the cases considered is given for two

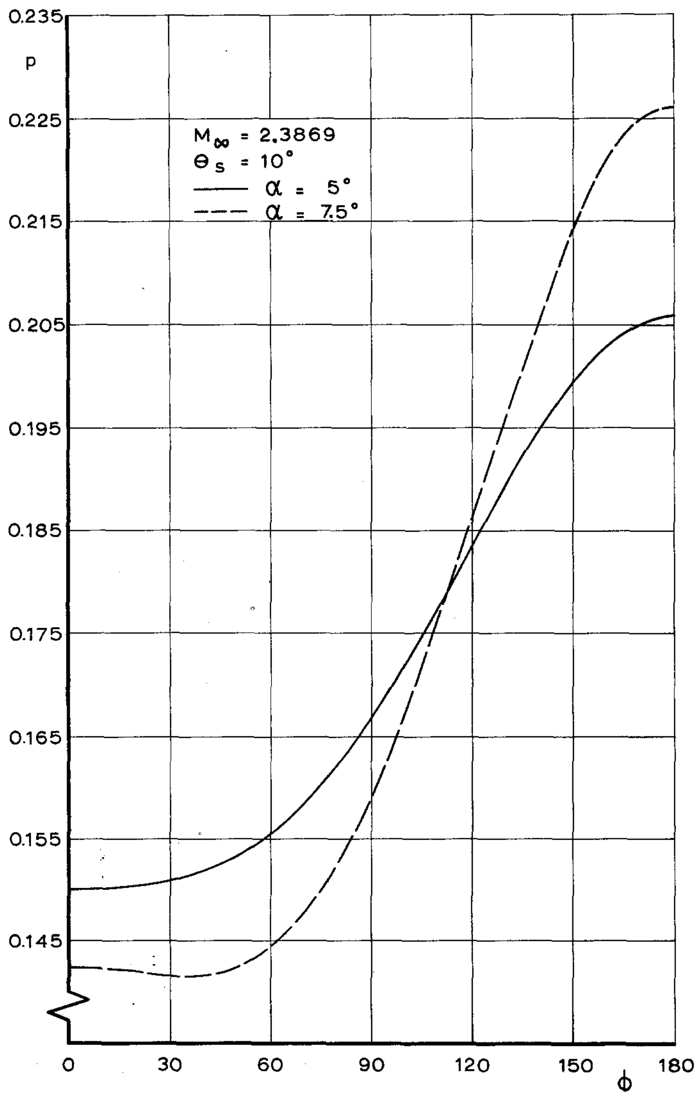


Figure 3a. Pressure distribution at the cone surface.

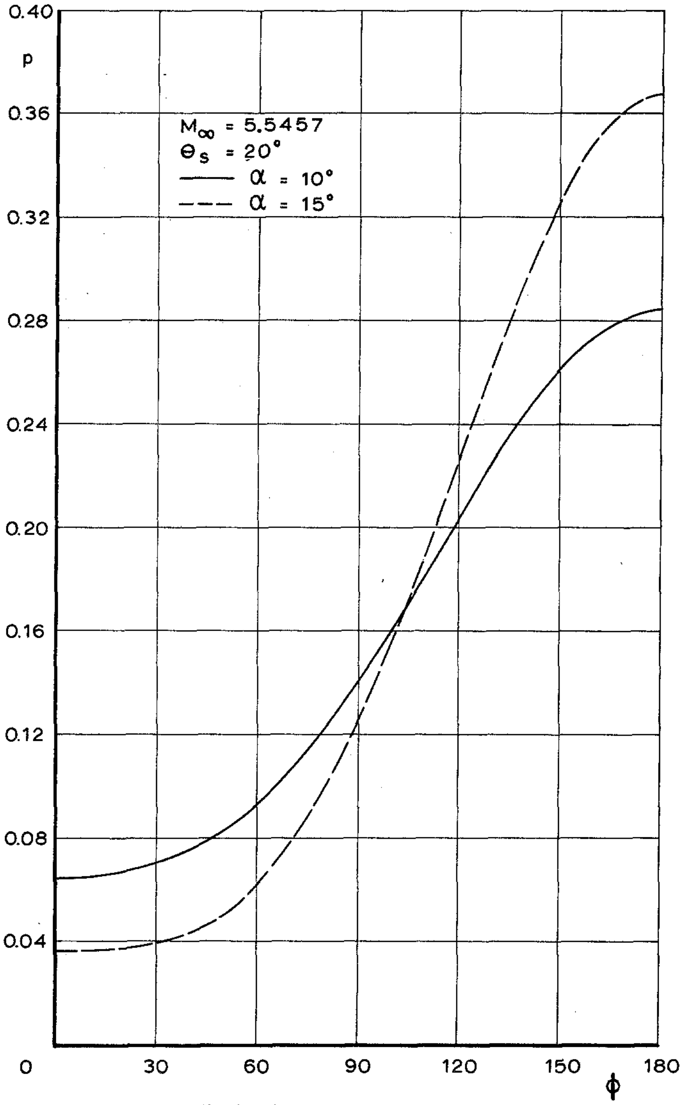
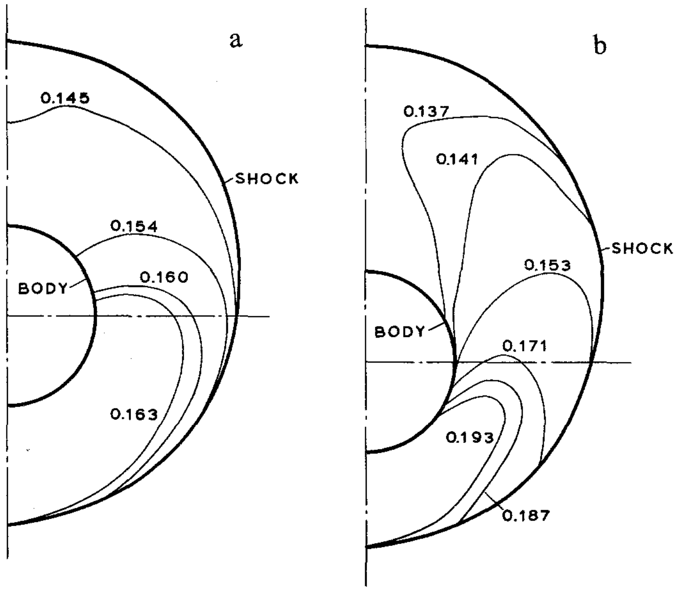


Figure 3b. Pressure distribution at the cone surface.

different angles of attack in each case. It can be concluded that the influence of the second order terms is quite large. The behaviour of the pressure throughout the field has been given in Figs. 4a–e by depicting the isobar pattern. Although there is no sufficient experimental evidence it seems that the lower angle of attack cases look more realistic than the larger angle of attack cases.

Of course it is interesting to compare the results obtained by the N-strip algorithm to those obtained by Kopal with a different method of approach [7], [10] and [11]. Therefore the four cases considered here are identical to some of those considered by Kopal. However, since Kopal works in a coordinate system where the direction of the undisturbed flow is the direction of the x-axis, whereas in this analysis a body-axis system is used, the results are not comparable directly, but have to be transformed according to [14] for the first- and second order cases.

In Tables 1 and 2 the Kopal values for the various quantities are also given and it is evident that the agreement is very well indeed. To check this more closely in one of the cases, the Kopal pressure distribution as a function of θ for $\phi=0$ and $\phi=\pi$ has been compared to results of a ten-strip approximation (see Fig. 5). This comparison in which second-order terms have been omitted shows a remarkable agreement. Due to the singular behaviour of the second order terms in the vicinity of the surface, Kopal ran into serious difficulties where trying to obtain



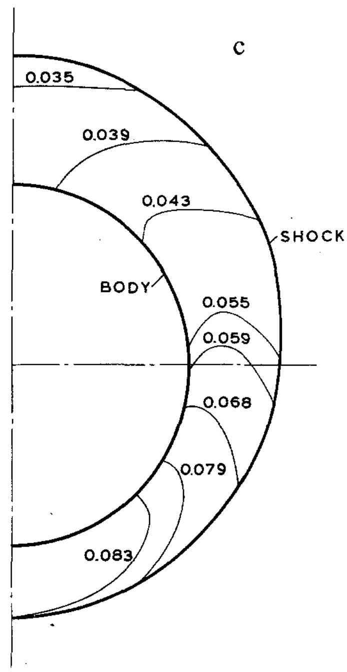


Figure 4a. Isobars at 5° incidence ($\theta_s = 10°$, $M_\infty = 2.3869$).
Figure 4b. Isobars at 10° incidence ($\theta_s = 10°$, $M_\infty = 2.3869$).
Figure 4c. Isobars at 5° incidence ($\theta_s = 10°$, $M_\infty = 5.4223$).

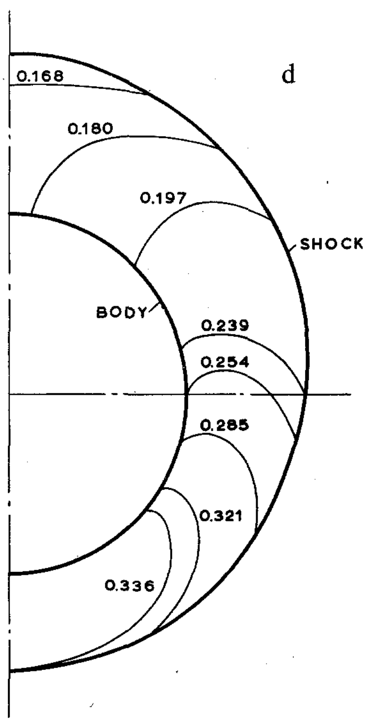


Figure 4d. Isobars at 10° incidence ($\theta_s = 20°$, $M_\infty = 2.4432$).

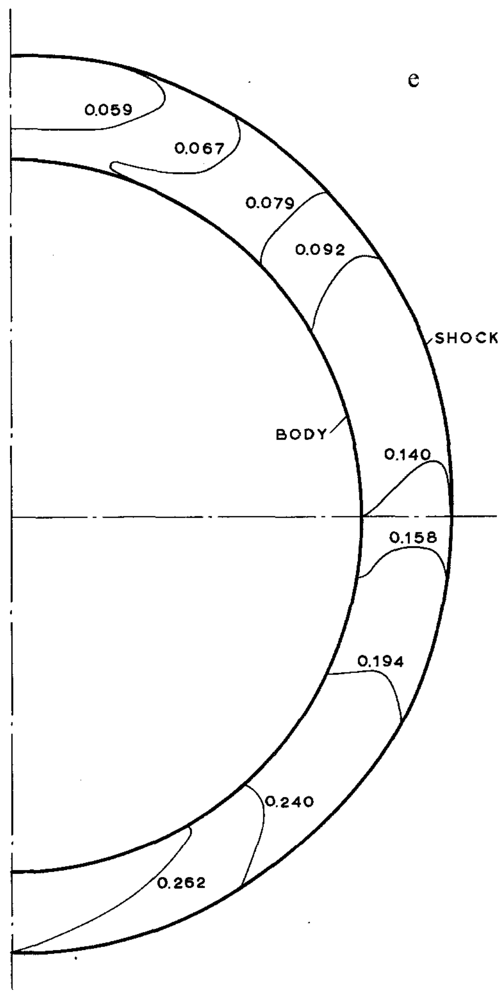


Figure 4e. Isobars at 10° incidence ($\theta_s = 20°$, $M_\infty = 5.5457$).

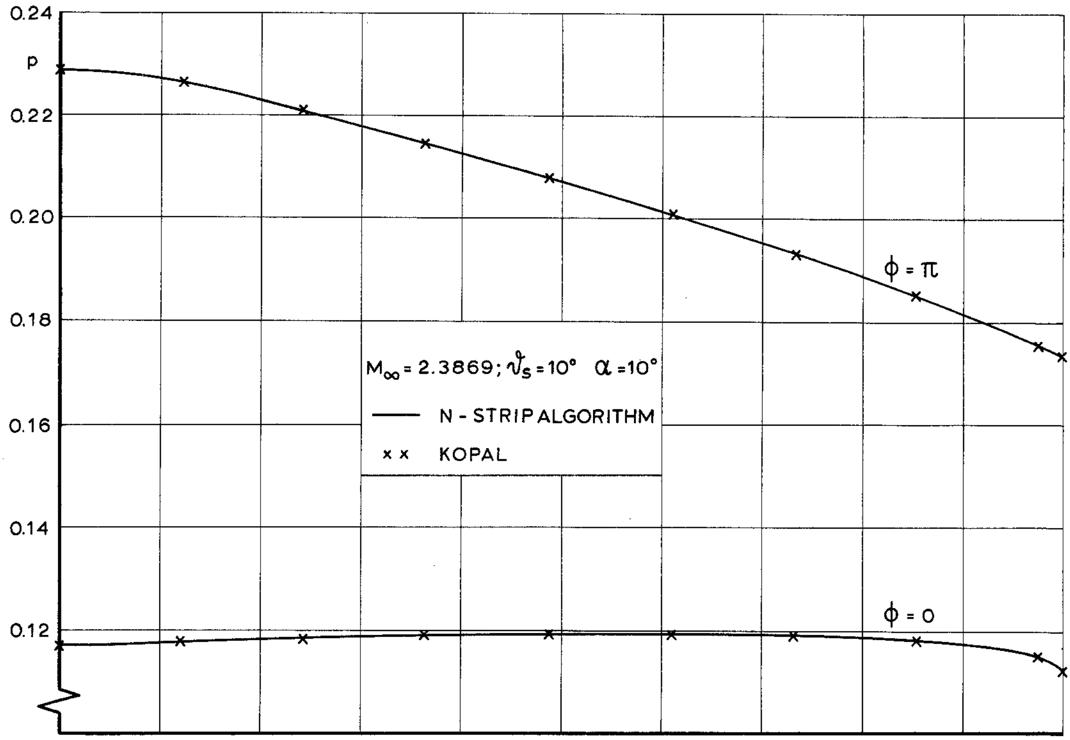


Figure 5. Comparison of Kopal values and the N-strip algorithm in a first-order lift case.

feasible values for the surface-quantities. In the cases when it seemed acceptable he computed these surface-quantities by extrapolating field values. This is the reason why in only two cases Kopal-values for second-order terms are available.

In these cases the agreement is quite well again. When considering the convergence in Tables 2, 3 and 4 it is seen that in all the cases 4 to 6 strips are sufficient to obtain accurate results.

CONCLUSIONS

In this paper using matrix calculus an algorithm is derived for an approximate method of solution of certain types of partial differential equations, known as the method of integral relations. To show its usefullness the algorithm which enables an approximate solution for an in general arbitrary number of strips, is applied to the problem of the calculation of supersonic conical flow. By using the series expansion in powers of the angle of attack, a method due to Stone, it is possible to check the results with those calculated by Kopal. In spite of the complications involved—which are related to the occurrence of the vortical layer—the numerical results give great confidence in the algorithm.

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