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LONG CYCLES IN GRAPHS WITH LARGE DEGREE SUMS

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A number of results are established concerning long cycles in graphs with large degree sums. Let G be a graph on n vertices such that $d(x)+d(y)+d(z) \geqslant s$ for all triples of independent vertices x, y, z. Let c be the length of a longest cycle in G and α the cardinality of a maximum independent set of vertices. If G is 1-tough and $s \geqslant n$, then every longest cycle in G is a dominating cycle and $c \geqslant \min(n, n+\frac{1}{3}s-\alpha) \geqslant \min(n, \frac{1}{2}n+\frac{1}{3}s) \geqslant \frac{5}{6}n$. If G is 2-connected and $s \geqslant n+2$, then also $c \geqslant \min(n, n+\frac{1}{3}s-\alpha)$, generalizing a result of Bondy and one of Nash-Williams. Finally, if G is 2-tough and $s \geqslant n$, then G is hamiltonian.

1. Terminology

We consider only finite undirected graphs without loops or multiple edges. Our terminology is standard except as indicated. A good reference for any undefined terms is [7]. We need a few definitions and some convenient notation. Let $\omega(G)$ denote the number of components of a graph G. As introduced by Chvátal [10], a graph G is *t-tough* if $|S| \geqslant t\omega(G-S)$ for any subset S of the vertex set V of G with $\omega(G-S)>1$. The *toughness of* G, denoted $t(G)$, is the maximum value of t for which G is t-tough ($t(K_n)=\infty$ for all $n \geqslant 1$). We will denote by α the cardinality of a maximum set of independent vertices of G. A cycle C of G is a *dominating cycle* if every edge of G has at least one of its vertices on C. If C is a cycle of G we denote by $\vec{C}$ the cycle C with a given orientation. If $u, v \in V(C)$, then $u\vec{C}v$ denotes the consecutive vertices on C from u to v in the direction specified by $\vec{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We use u^+

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to denote the successor of u on $\vec{C}$ and u^- to denote its predecessor. If $v \in V$ then $N(v)$ is the set of all vertices in V adjacent to v. If $A \subseteq V(C)$, then $A^+ = \{v^+ \mid v \in A\}$. The set A^- is analogously defined.

2. Results

Our work was motivated by two recent conjectures of Ainouche and Christofides [1].

Conjecture 1. Let G be a 1-tough graph on $n \geq 3$ vertices such that $d(x) + d(y) + d(z) \geq n$ for all independent sets of vertices x, y, z. Then G is hamiltonian.

Conjecture 2. Let G be a 1-tough graph on $n \geq 3$ vertices such that $d(x) + d(y) \geq q$ for all distinct nonadjacent vertices x, y. Then G has a cycle of length at least $\min(n, q+2)$.

The following class of graphs, given in [1], shows that each conjecture, if true, would be best possible. For $n = 3r + 1 \geq 7$, construct the graph H_n from $3K_r + K_1$ by choosing one vertex from each copy of K_r, say u, v and w, and adding the edges uv, uw and vw. The graph H_n is 1-tough on $n = 3r + 1$ vertices, satisfies $d(x) + d(y) \geq 2r$ for all distinct nonadjacent vertices x, y and also satisfies $d(x) + d(y) + d(z) \geq n - 1$ for all sets of independent vertices x, y, z. Yet a longest cycle in H_n has length only $2r + 2$.

Conjecture 2 was recently proven to be true [5]. For convenience we state it as a theorem below.

Theorem 1. *Let G be a 1-tough graph on $n \geq 3$ vertices such that $d(x) + d(y) \geq q$ for all distinct nonadjacent vertices x, y. Then G has a cycle of length at least* $\min(n, q+2)$.

Conjecture 1, however, is false as indicated by the following class of graphs. For odd $n \geq 15$, construct the graph G_n from

$$\bar{K}_{\frac{1}{2}(n-1)} \cup K_m \cup K_{\frac{1}{2}(n+1)-m}, \text{ where } \tfrac{1}{3}n \leq m \leq \tfrac{1}{2}(n-5),$$

by joining every vertex in K_m to all other vertices and by adding a matching between all vertices in $K_{\frac{1}{2}(n+1)-m}$ and $\frac{1}{2}(n+1) - m$ vertices in $\bar{K}_{\frac{1}{2}(n-1)}$. Note that G_n has minimum degree m. It is easily seen that G_n is 1-tough but not hamiltonian. If $\frac{1}{2}(n+1) - m$ is odd (even) then a longest cycle in G_n has length $\frac{1}{4}(3n+1) + \frac{1}{2}m$ ($\frac{1}{4}(3n+3) + \frac{1}{2}m$). A variation of the graph G_n, with K_m replaced by $\bar{K}_m$ and $m = \frac{1}{2}(n-5)$, has already appeared in the literature [8, 13]. It can be used to show that the following theorem of Jung [11] is best possible.

Theorem 2. *Let G be a 1-tough graph on $n \geq 11$ vertices such that $d(x) + d(y) \geq n - 4$ for all distinct nonadjacent vertices x, y. Then G is hamiltonian.*

Although Conjecture 1 is false its hypothesis justifies the following conclusion, which follows immediately from Theorem 9 below.

Theorem 3. *Let G be a 1-tough graph on $n \geq 3$ vertices such that $d(x) + d(y) + d(z) \geq s \geq n$ for all independent sets of vertices x, y, z. Then G contains a cycle of length at least* $\min(n, \frac{1}{2}n + \frac{1}{3}s)$.

Corollary 4. *Let G be a 1-tough graph on $n \geq 3$ vertices with minimum degree $\delta \geq \frac{1}{3}n$. Then G contains a cycle of length at least $\frac{5}{6}n$.*

Theorem 3 is a little surprising in the following sense. If, for example, $\delta = \frac{1}{3}n$ we conclude from Theorem 1 (which is "best possible") that G has a cycle of length at least $\frac{2}{3}n + 2$. From Corollary 4 we deduce that G has a cycle of length at least $\frac{5}{6}n$. Apparently for 1-tough graphs G, as δ crosses the threshold of $\frac{1}{3}n$, the length of a longest cycle that is forced in G jumps from $\frac{2}{3}n + 2$ to at least $\frac{5}{6}n$. If Conjecture 3, mentioned in Section 4, is true then G is forced to have a cycle of length at least $\frac{1}{12}(11n + 3)$.

The proof of Theorem 3, as well as the proofs of our other results, depends on the intermediate conclusion that every longest cycle in G is a dominating cycle. This is established by our next theorem, whose proof is given in Section 3.

Theorem 5. *Let G be a 1-tough graph on n vertices such that $d(x) + d(y) + d(z) \geq n$ for all independent sets of vertices x, y, z. Then every longest cycle in G is a dominating cycle.*

Theorem 5 generalizes the following theorem of Bigalke and Jung [8].

Theorem 6. *Let G be a 1-tough graph on n vertices with $\delta \geq \frac{1}{3}n$. Then every longest cycle in G is a dominating cycle.*

The graphs H_n with $n \geq 10$ show that both Theorem 5 and Theorem 6 are best possible. We remark that for $n \geq 5$ the condition in Theorem 5 that G be 1-tough can in fact be replaced by the weaker condition that the deletion of any nonempty proper subset S of V yields a graph with at most $|S|$ nontrivial components. This weaker condition is necessary for a graph to have a dominating cycle [14]. Thus, if the condition that G be 1-tough is replaced by the above weaker condition, we obtain a result that also generalizes the following theorem of Bondy [9].

Theorem 7. *Let G be a 2-connected graph on n vertices such that $d(x) + d(y) + d(z) \geq n + 2$ for all independent sets of vertices x, y, z. Then every longest cycle in G is a dominating cycle.*

The next key lemma, proved in Section 3, is the basis for many of the results that follow.

Lemma 8. *Let G be a graph on n vertices such that* $\delta \geq 2$ *and* $d(x)+d(y)+d(z) \geq n$ *for all independent sets of vertices x, y, z. Let G contain a longest cycle C which is a dominating cycle. If* $v_0 \in V - V(C)$ *and* $A = N(v_0)$, *then* $(V - V(C)) \cup A^+$ *is an independent set of vertices.*

Lemma 8 has a number of applications. The next two theorems are obtained by combining Lemma 8 with Theorems 5 and 7, respectively. A proof of Theorem 10 and an outline proof of Theorem 9 are given in Section 3.

Theorem 9. *Let G be a 1-tough graph on* $n \geq 3$ *vertices such that* $d(x)+d(y)+d(z) \geq s \geq n$ *for all independent sets of vertices x, y, z. Then G contains a cycle of length at least* $\min(n, n+\frac{1}{3}s-\alpha)$.

Since $\alpha \leq \frac{1}{2}n$ for 1-tough graphs, Theorem 3 follows immediately from Theorem 9.

Theorem 10. *Let G be a 2-connected graph on n vertices such that* $d(x)+d(y)+d(z) \geq s \geq n+2$ *for all independent sets of vertices x, y, z. Then G contains a cycle of length at least* $\min(n, n+\frac{1}{3}s-\alpha)$.

Theorem 10 is best possible in two different ways. The graph $K_{p,q}$, with $2 \leq p \leq q \leq 2p-2$ and $q \geq 3$ has a longest cycle of length exactly $n+\frac{1}{3}s-\alpha = 2p$. The graph $H = 3K_t + 2K_1$ has $d(x)+d(y)+d(z) \geq n+1$ for all independent sets of vertices x, y, z and has a longest cycle of length $2t+2$, which is less than $\min(n, n+\frac{1}{3}s-\alpha) = \min(n, n+\frac{1}{3}(n+1)-3) = n$ $(t \geq 2)$.

It is easily seen that if $\alpha \geq 3$, the hypothesis of Theorem 10 implies $\alpha \leq n-\frac{1}{3}s$. Hence Theorem 10 generalizes the following result of Bondy [9].

Theorem 11. *Let G be a 2-connected graph on n vertices such that* $d(x)+d(y)+d(z) \geq s \geq n+2$ *for all independent sets of vertices x, y, z. Then G has a cycle of length at least* $\min(n, \frac{2}{3}s)$.

Theorem 10 also generalizes the following result of Nash-Williams [12].

Theorem 12. *Let G be a 2-connected graph on n vertices with* $\delta \geq \max(\frac{1}{3}(n+2), \alpha)$. *Then G is hamiltonian.*

Bigalke and Jung [8] also generalized Theorem 12.

Theorem 13. *Let G be a 1-tough graph on* $n \geq 3$ *vertices with* $\delta \geq \max(\frac{1}{3}n, \alpha-1)$. *Then G is hamiltonian.*

Note that Theorem 9 is only a partial generalization of Theorem 13. Theorem 9 allows us to draw conclusions concerning long, but not necessarily hamiltonian, cycles in G. However if $\delta = \alpha - 1 \geq \frac{1}{3}n$ we cannot conclude from Theorem 9 that G is hamiltonian. It is possible, however, to combine Lemma 8 with a suitably modified proof of Theorem 13 to obtain the following.

Theorem 14. *Let G be a 1-tough graph on $n \geq 3$ vertices with $\delta \geq \frac{1}{3}n$. Then G contains a cycle of length at least* $\min(n, n + \delta - \alpha + 1)$.

The proof of Theorem 14 is lengthy and will appear elsewhere [6]. Note that this result yields a slight strengthening of Corollary 4. We can actually conclude that G has a cycle of length at least $\frac{5}{6}n + 1$.

Theorem 14 completely generalizes Theorem 13 and, like Theorem 10, is best possible in two ways. If $m = \frac{1}{2}(n-5)$, the graph G_n has $n + \delta - \alpha + 1 = n - 1$ and G_n is not hamiltonian; in view of Conjecture 3 in Section 4, however, we do not believe that Theorem 14 is best possible for values of δ less than $\frac{1}{2}(n-5)$. The graph H_n has $\delta \geq \frac{1}{3}(n-1)$ and has a longest cycle of length $\frac{2}{3}(n-1)+2$, less than $\min(n, n + \delta - \alpha + 1) = \min(n, n + \frac{1}{3}(n-1) - 2) = n$.

We now turn our attention to graphs with $t(G) = \tau \geq 1$. The inequality $\alpha \leq \frac{1}{2}n$, used to prove Theorem 3 from Theorem 9, suggests that our conclusions can be strengthened if $\tau > 1$. Since obviously $\alpha \leq n/(\tau+1)$, Theorem 9 immediately implies our next result.

Corollary 15. *Let G be a graph on $n \geq 3$ vertices with $t(G) = \tau \geq 1$. If $d(x) + d(y) + d(z) \geq s \geq n$ for all independent sets of vertices x, y, z, then G has a cycle of length at least* $\min(n, n\tau/(\tau+1) + \frac{1}{3}s)$.

A special case of Corollary 15 may be a first small step toward proving the well-known conjecture that 2-tough graphs are hamiltonian [10].

Corollary 16. *Let G be a 2-tough graph on $n \geq 3$ vertices. If $d(x) + d(y) + d(z) \geq n$ for all independent sets of vertices x, y, z, then G is hamiltonian.*

3. Proofs

Proof of Theorem 5. Let C be a longest cycle of G with a fixed orientation. Assume C is not a dominating cycle of G. Then $G - V(C)$ has a nontrivial component H. Set $A = \bigcup_{v \in V(H)} N(v) - V(H)$ and let $v_1, \ldots, v_k$ be the elements of A, occurring on $\vec{C}$ in consecutive order. Since G is 1-tough, G is 2-connected in particular, so $k \geq 2$. For $i = 1, \ldots, k$, set $u_i = v_i^+$ and $w_i = v_{i+1}^-$ (indices modulo k). Since C is a longest cycle, $u_i \neq v_{i+1}$ ($i = 1, \ldots, k$). If v is a vertex in $u_i\vec{C}w_i$ such that $u_iv^+ \in E$, then v will be called an i-vertex; in particular, u_i is an

i-vertex ($i = 1, \ldots, k$). Since G is 2-connected, there exist integers r and s with $1 \leq r < s \leq k$ such that v_r and v_s are connected by a path $P_{r,s}$ of length at least 3 with all internal vertices in H. We make a number of observations.

(1) If x_r is an r-vertex and x_s an s-vertex, then there exists no (x_r, x_s)-path which is internally disjoint from C; in particular, $x_r x_s \notin E$.

Assuming the contrary to (1), let P be an (x_r, x_s)-path, internally disjoint from C. Since $x_r, x_s \notin A$, we have $V(P) \cap V(H) = \emptyset$. Now $v_r P_{r,s} v_s \overleftarrow{C} x_r^+ u_r \vec{C} x_r P x_s \overleftarrow{C} u_s x_s^+ \vec{C} v_r$, denoting the cycle having as consecutive vertices the vertices of $P_{r,s}$, $v_s \overleftarrow{C} x_r^+$, $u_r \vec{C} x_r$, P, $x_s \overleftarrow{C} u_s$ and $x_s^+ \vec{C} v_r$, respectively, has length at least $|V(C)| + 2$. This contradiction proves (1).

(2) If x_r is an r-vertex and x_s an s-vertex, then $x_r u_s^+$, $u_r^+ x_s \notin E$.

If the contrary to (2) is assumed, a cycle longer than C can be indicated as in (1). The only difference is that now this cycle has length at least $|V(C)| + 1$ instead of $|V(C)| + 2$, since it omits the vertex u_s or u_r of C.

(3) Let x_r be an r-vertex and x_s an s-vertex. If $v \in x_r \vec{C} x_s$ and $x_s v \in E$, then $x_r v^+ \notin E$. Similarly, if $v \in x_s \vec{C} x_r$ and $x_r v \in E$, then $x_s v^+ \notin E$.

To prove (3) assume, e.g. $v \in x_r \vec{C} x_s$, $x_s v \in E$ and $x_r v^+ \in E$. By (1), $v \neq x_r$ and $v^+ \neq u_s$, x_s (since u_s is also an s-vertex). If $v^+ \in x_r^{++} \vec{C} v_s$, then the cycle $v_r P_{r,s} v_s \overleftarrow{C} v^+ x_r \overleftarrow{C} u_r x_r^+ \vec{C} v x_s \overleftarrow{C} u_s x_s^+ \vec{C} v_r$ has length at least $|V(C)| + 2$, a contradiction. If $v^+ \in u_s^+ \vec{C} x_s^-$, then the cycle $v_r P_{r,s} v_s \overleftarrow{C} x_r^+ u_r \vec{C} x_r v^+ \vec{C} x_s v \overleftarrow{C} u_s x_s^+ \vec{C} v_r$ yields a similar contradiction.

(4) Let x_r be an r-vertex and x_s an s-vertex. If $v \in x_r \vec{C} x_s$ and $x_s v \in E - E(C)$, then $x_r v^{++} \notin E$. Similarly, if $v \in x_s \vec{C} x_r$ and $x_r v \in E - E(C)$, then $x_s v^{++} \notin E$.

The proof of (4) is similar to the proof of (3), except now the longer cycle has length $|V(C)| + 1$ instead of $|V(C)| + 2$.

Using observations (1) through (4) we now derive an upper bound for $d(u_0) + d(x_r) + d(x_s)$, where x_r is an r-vertex, x_s an s-vertex and u_0 an arbitrary vertex of H. Define

$$R_1(x_r) = \{v \in x_r \vec{C} x_s^- \mid x_r v^+ \in E\},$$
$$S_1(x_s) = \{v \in x_r \vec{C} x_s^- \mid x_s v \in E\},$$
$$R_2(x_r) = \{v \in x_s \vec{C} x_r^- \mid x_r v \in E\},$$
$$S_2(x_s) = \{v \in x_s \vec{C} x_r^- \mid x_s v^+ \in E\},$$
$$R_3(x_r) = \{v \in V - V(C) \mid x_r v \in E\},$$
$$S_3(x_s) = \{v \in V - V(C) \mid x_s v \in E\},$$
$$B(x_r, x_s) = R_1(x_r) \cup S_1(x_s) \cup R_2(x_r) \cup S_2(x_s).$$

By (3), $R_1(x_r) \cap S_1(x_s) = R_2(x_r) \cap S_2(x_s) = \emptyset$. By (1) and the fact that $x_r, x_s \notin A$,

$R_3(x_r) \cap S_3(x_s) = V(H) \cap (R_3(x_r) \cup S_3(x_s)) = \emptyset$. Furthermore, for $i \in \{1, \ldots, k\} - \{r, s\}$, either u_i or v_i is not in $B(x_r, x_s)$. To see this, suppose e.g. $u_i \in R_1(x_r) \cup S_1(x_s)$. Then $x_r u_i^+ \in E$, since the assumption that $x_s u_i \in E$ implies the existence of a cycle longer than C, containing the vertices of a (v_i, v_s)-path of length at least 2 with all internal vertices in H (cf. (1)). But then, by (4) with $v = v_i$, $x_s v_i \notin E$. Also, like $x_s u_i$, $x_r u_i \notin E$. It follows that $v_i \notin R_1(x_r) \cup S_1(x_s)$.

We conclude that

$$\begin{aligned}(5)\ & d(u_0) + d(x_r) + d(x_s) \\ &= d(u_0) + |R_1(x_r)| + |R_2(x_r)| + |R_3(x_r)| + |S_1(x_s)| + |S_2(x_s)| + |S_3(x_s)| \\ &\leq (k + |V(H)| - 1) + (|V(C)| - (k-2)) + |R_3(x_r)| + |S_3(x_s)| \\ &\leq (k + |V(H)| - 1) + (|V(C)| - (k-2)) + (|V| - |V(C)| - |V(H)|) = n + 1.\end{aligned}$$

On the other hand, since $\{u_0, x_r, x_s\}$ is an independent set,

$$(6)\ d(u_0) + d(x_r) + d(x_s) \geq n.$$

It follows that u_0, and hence every vertex of H, is adjacent to all but at most one of the vertices in A. This implies the existence of a (v_i, v_j)-path $P_{i,j}$ of length at least 3 with all internal vertices in H for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$. A number of conclusions now follow. We first note that (1) through (6) actually hold for arbitrary r and s with $1 \leq r < s \leq k$. Furthermore, $u_i \neq w_i$ ($i = 1, \ldots, k$). Also, it follows immediately from (2) that for $1 \leq r < s \leq k$ and $i \in \{1, \ldots, k\} - \{r, s\}$, u_i (instead of u_i or v_i) is not in $B(x_r, x_s)$, where x_r is an r-vertex and x_s an s-vertex. From (5) and (6) we also deduce the following.

(7) If x_r is an r-vertex and x_s an s-vertex, then at most one vertex of $V(C) - \{u_i \mid i \in \{1, \ldots, k\}, i \neq r, s\}$ is not in $B(x_r, x_s)$ $(1 \leq r < s \leq k)$.

The next three observations, where $s \in \{1, \ldots, k\}$, will facilitate the remainder of the proof.

(8) If $v \in u_{s+1}\vec{C}v_s$ and $u_s v \in E$, then $w_s v^- \notin E$.

Assuming the contrary, the cycle $w_s v^- \overleftarrow{C} v_{s+1} P_{s+1,s} v_s \overleftarrow{C} v u_s \vec{C} w_s$ has length at least $|V(C)| + 2$, a contradiction.

(9) If $v \in u_{s+1}\vec{C}v_s$ and $u_s v \in E$, then $w_s v^{--}$, $w_s^- v^- \notin E$.

The proof of (9) is similar to the proof of (8).

(10) If $v \in v_{s+1}\vec{C}v_s^-$ and $u_s v \in E$, then $w_s v^+$, $w_s^- v^+ \notin E$.

Assuming the contrary, the cycle $w v^+ \vec{C} v_s P_{s,s+1} v_{s+1} \vec{C} v u_s \vec{C} w$, where $w = w_s$ or $w = w_s^-$, yields a contradiction.

Using observations (1) through (10) we now derive contradictions in all cases distinguished below. If $v \in V$, then by $N'(v)$ we denote the set of vertices x such that there is a (v, x)-path of length at least 1 with all internal vertices in $V - V(C)$. In particular, $N(v) \subseteq N'(v)$.

Case 1. For all $i \in \{1, \ldots, k\}$,

$$N'(u_i) \cap V(C) \subseteq v_i\vec{C}v_{i+1} \cup A \quad \text{and} \quad N'(w_i) \cap V(C) \subseteq v_i\vec{C}v_{i+1} \cup A.$$

Suppose there exist integers r, s and vertices x, y such that $1 \leq r < s \leq k$, $x \in u_r^+\vec{C}w_r^-$, $y \in u_s^+\vec{C}w_s^-$ and $xy \in E$. Since by the hypotheses of Case 1 $u_sx, u_ry \notin E$, either u_rx^+ or u_sy^+ is in E, otherwise $x, y \notin B(u_r, u_s)$, contradicting (7). Assume, without loss of generality, that $u_rx^+ \in E$, i.e. x is an r-vertex. By (3) and (4), u_sy^+, $u_sy^{++} \notin E$ and hence $y, y^+ \notin B(u_r, u_s)$. This contradiction with (7) shows that in this case no edge, and similarly no path with all internal vertices in $V - V(C)$, joins two vertices in different sets of the collection $\{u_i\vec{C}w_i \mid 1 \leq i \leq k\}$. But then $\omega(G - A) \geq |A| + 1$, contradicting the fact that G is 1-tough.

Case 2. For some $i \in \{1, \ldots, k\}$,

$$N'(u_i) \cap V(C) \nsubseteq v_i\vec{C}v_{i+1} \cup A \quad \text{or} \quad N'(w_i) \cap V(C) \nsubseteq v_i\vec{C}v_{i+1} \cup A.$$

Assume e.g. $y_r \in N'(u_s)$, where $y_r \in u_r\vec{C}w_r$, $r < s$ and $|u_r\vec{C}y_r|$ is minimum. For convenience we also assume $u_sy_r \in E$; in case u_s and y_r are connected by a path of length at least 2 with all internal vertices in $V - V(C)$, completely analogous arguments apply, since the path must be disjoint from H. Note that by (1) and (2), $y_r \neq u_r, u_r^+$. Let x_r be the r-vertex in $u_r\vec{C}y_r^-$ that minimizes $|x_r\vec{C}y_r|$; possibly $x_r = u_r$. Either $x_r^+ = y_r^-$ or $x_r^+ = y_r$, otherwise $x_r^+, x_r^{++} \notin B(u_r, u_s)$, contradicting (7). We distinguish two subcases.

Case 2.1. $x_r^+ = y_r^-$.
Then $u_ry_r \notin E$. By (4), $u_ry_r^{++} \notin E$ and by (8) and (10), $w_sy_r^-$, $w_sy_r^+ \notin E$. Hence w_s is not an s-vertex, otherwise $y_r^-, y_r^+ \notin B(u_r, w_s)$, contradicting (7). Thus $u_sw_s^+ \notin E$. But then $u_rw_s \in E$, otherwise $y_r^-, w_s \notin B(u_r, u_s)$. Now $x_rw_s^+ \notin E$, otherwise the cycle $x_rw_s^+\vec{C}v_rP_{r,s}v_s\overleftarrow{C}y_ru_s\vec{C}w_su_r\overleftarrow{C}x_r$ is longer than C. Also, by (9), $x_rw_s \notin E$. It follows that $w_s, w_s^+ \notin B(x_r, u_s)$, a contradiction.

Case 2.2. $x_r^+ = y_r$.

Case 2.2.1. $u_sw_s^+ \notin E$.
By (8) and (9), $x_rw_s, x_rw_s^- \notin E$. Thus $u_sw_s \in E$, otherwise $w_s^-, w_s \notin B(x_r, u_s)$. In other words, w_s^- is an s-vertex. By (3) and (4), $x_ry_r^+, x_ry_r^{++} \notin E$. Recalling that $x_rw_s \notin E$ by (8), we conclude that $u_sy_r^+ \in E$, since otherwise $y_r^+, w_s \notin B(x_r, u_s)$. Now by (9) and (10), $w_s^-y_r, w_s^-y_r^+ \notin E$. It follows that $y_r, y_r^+ \notin B(x_r, w_s^-)$, a contradiction.

Case 2.2.2. $u_sw_s^+ \in E$.
Then w_s is an s-vertex. Recall that, by (3) and (4), $x_ry_r^+, x_ry_r^{++} \notin E$. By (10), $w_sy_r^+ \notin E$. Hence $w_sy_r \in E$, otherwise $y_r, y_r^+ \notin B(x_r, w_s)$. Now $x_rw_r \notin E$, otherwise

the cycle $x_r w_r \vec{C} y_r w_s \overleftarrow{C} v_{r+1} P_{r+1,s+1} v_{s+1} \vec{C} x_r$ is longer than C. Also, $w_s w_r^- \notin E$, otherwise the cycle $w_r^- w_s \overleftarrow{C} v_{r+1} P_{r+1,s+1} v_{s+1} \vec{C} w_r^-$ is longer than C. It follows that $y_r^+, w_r^- \notin B(x_r, w_s)$. Hence, by (7), $y_r^+ = w_r^-$. We now show that

(11) u_s is adjacent to all vertices in $u_s^+ \vec{C} v_{s+1}$.

Assuming the contrary, let v be the vertex in $u_s^+ \vec{C} v_{s+1}$ such that $u_s v \notin E$ and $|v \vec{C} v_{s+1}|$ is minimum. Then $v \in u_s^+ \vec{C} w_s$ and $u_s v^+ \in E$. By (4), $u_r v^- \notin E$ and by (8), $u_s y_r^+ \notin E$. Hence $v^-, y_r^+ \notin B(u_r, u_s)$. This contradiction proves (11). Similarly we have

(12) u_r is adjacent to all vertices in $u_r^+ \vec{C} y_r$.

By (9), $u_s y_r^{++} \notin E$. Recalling that $u_s y_r^+ \notin E$ we now note that for all $i \in \{1, \dots, k\} - \{r\}$ the assumption $u_i y_r^+ \in E$ or $u_i y_r^{++} \in E$ leads to a contradiction by applying the above arguments with i in place of s. Thus $u_i y_r^+, u_i y_r^{++} \notin E$ for all $i \in \{1, \dots, k\} - \{r\}$. By (3) and (4), $u_r y_r^+, u_r y_r^{++} \notin E$. Hence $u_i y_r \in E$, for otherwise $y_r, y_r^+ \notin B(u_r, u_i)$ $(i \in \{1, \dots, k\} - \{r\})$. It now follows that (11) remains true if s is replaced by i $(i \in \{1, \dots, k\} - \{r\})$. By (7), $B(u_r, u_i) = V(C) - (\{y_r^+\} \cup \{u_j \mid j \in \{1, \dots, k\} - \{r, i\}\})$, implying that

$$N(u_r) \cap V(C) = u_r^+ \vec{C} y_r \cup A \quad \text{and}$$
$$N(u_i) \cap V(C) = u_i^+ \vec{C} v_{i+1} \cup A \ (i \in \{1, \dots, k\} - \{r\}).$$

In particular, every vertex of $V(C) - (A \cup \{y_r, y_r^+\})$ is an i-vertex for some $i \in \{1, \dots, k\}$. Using (1), (3) and (4) we conclude that no edge, and similarly no path with all internal vertices in $V - V(C)$, joins two vertices in different sets of the collection $\{u_i \vec{C} w_i \mid 1 \leq i \leq k,\ i \neq r\} \cup \{u_r \vec{C} y_r^-\} \cup \{\{y_r^+, y_r^{++}\}\}$. But then $\omega(G - (A \cup \{y_r\})) \geq |A \cup \{y_r\}| + 1$, our final contradiction. □

Proof of Lemma 8. By assumption $V - V(C)$ is an independent set and a standard argument shows that A^+ is an independent set. Hence it suffices to show that no vertex in $V - V(C)$ is adjacent to a vertex in A^+. Let $A = N(v_0)$ consist of distinct vertices $x_1, x_2, \dots, x_k$ $(k \geq 2)$ on C such that $x_{i+1} \in x_i \vec{C} x_{i+2}$, $1 \leq i \leq k$ (indices mod k). Clearly v_0 is not adjacent to any vertex in A^+, i.e. $A \cap A^+ = \emptyset$. Suppose $v_1 \in V - V(C)$ and $v_1 x_1^+ \in E$. Consider the following sets of vertices.

$$A_1 = \{v \in x_1^+ \vec{C} x_k \mid v_1 v^+ \in E\},$$
$$A_2 = \{v \in x_1^+ \vec{C} x_k \mid x_k^+ v \in E\},$$
$$B_1 = \{v \in x_k^+ \vec{C} x_1 \mid v_1 v \in E\},$$
$$B_2 = \{v \in x_k^+ \vec{C} x_1 \mid x_k^+ v^+ \in E\},$$
$$D = \{v \in V - V(C) \mid x_k^+ v \in E\}.$$

Observe that for each i, $1 \leq i \leq k-1$, $x_i^+ \notin A_1$. This is clear if $i = 1$. Assuming $x_i^+ \in A_1$ for some $i \in \{2, \dots, k-1\}$, the cycle $v_1 x_i^{++} \vec{C} x_1 v_0 x_i \overleftarrow{C} x_1^+ v_1$ is a longer cycle than C, a contradiction. Since A^+ is an independent set, $x_i^+ \notin A_2$ for

$1 \leqslant i \leqslant k-1$, and $x_1 \notin B_2$. It is easy to see that $x_1 \notin B_1$ and $v_0, v_1 \notin D$. But $A_1 \cap A_2 = \emptyset$, for if $w \in A_1 \cap A_2$, then $x_1^+ v_1 w^+ \vec{C} x_k v_0 x_1 \overleftarrow{C} x_k^+ w \overleftarrow{C} x_1^+$ is a longer cycle than C, a contradiction. Similarly $B_1 \cap B_2 = \emptyset$. Thus $|D| \leqslant n - |V(C)| - 2$ and $|A_1| + |A_2| + |B_1| + |B_2| \leqslant |V(C)| - k$. Since $N(v_1) = A_1^+ \cup \{x_1^+\} \cup B_1$ and $v_1 x_k^+ \notin E$, $d(v_1) = |A_1| + |B_1| + 1$. Also $d(x_k^+) = |A_2| + |B_2| + |D|$. Thus

$$d(v_0) + d(v_1) + d(x_k^+) = k + |A_1| + |A_2| + |B_1| + |B_2| + |D| + 1 \leqslant n - 1.$$

However, v_0, v_1 and x_k^+ are independent, thus contradicting our assumption and proving the lemma. □

Outline proof of Theorem 9. Let C be a longest cycle in G. By Theorem 5, C is a dominating cycle. Assume C is chosen such that $\max\{d(v) \mid v \in V - V(C)\}$ is maximum. If $V - V(C) = \emptyset$ there is nothing to prove. Thus we assume $V - V(C) = \{v_0, v_1, \ldots, v_t\}$, $d(v_0) \geqslant d(v_1) \geqslant \cdots \geqslant d(v_t)$. Let $A = N(v_0) = \{x_1, x_2, \ldots, x_k\}$, where $k \geqslant 2$ and $x_{i+1} \in x_i \vec{C} x_{i+2}$, $1 \leqslant i \leqslant k$ (indices mod k). From Lemma 8 we have $|V - V(C)| + |A^+| \leqslant \alpha$. Hence $|V(C)| \geqslant n + |A^+| - \alpha = n + d(v_0) - \alpha$. Thus it suffices to show that $d(v_0) \geqslant \frac{1}{3}s$. This is clearly true if $t \geqslant 2$.

Suppose $t = 1$, $d(v_0) < \frac{1}{3}s$ and consider x_1^+. Suppose $x_1^+ x_j^{++} \in E$, where $2 \leqslant j \leqslant k$. Then the cycle $C' = x_1^+ x_j^{++} \vec{C} x_1 v_0 x_j \overleftarrow{C} x_1^+$ has $|V(C')| = |V(C)|$ but includes v_0 and omits x_j^+. However, v_0, v_1 and x_j^+ are independent and thus $s \leqslant d(v_0) + d(v_1) + d(x_j^+)$. This implies $d(x_j^+) > \frac{1}{3}s > d(v_0)$, contradicting the choice of C. Thus we may assume $x_1^+ x_j^{++} \notin E$ for $2 \leqslant j \leqslant k$. Since x_1^+ is not adjacent to any vertex in A^+ and $A^+ \cap A^{++} = \emptyset$, $d(x_1^+) \leqslant |V(C)| - 2d(v_0) + 1$. But then $n \leqslant d(x_1^+) + d(v_0) + d(v_1) \leqslant |V(C)| + 1 \leqslant n - 1$, a contradiction.

Finally, the proof for $t = 0$ is modelled along the lines of the proof of Theorem 5. Whenever a contradiction is obtained in the proof of Theorem 5 by finding a longer cycle, we now find a contradiction either in the same way, or by finding a cycle C' such that $|V(C')| = n - 1$ and $u_0 \in V - V(C')$ has $d(u_0) > d(v_0)$. The argument, although quite lengthy and involved, is tedious and is thus omitted here. The full proof can be found in the appendix of [4]. □

Proof of Theorem 10. If G is hamiltonian we are done. Otherwise, as in the proof of Theorem 9, let C be a longest cycle in G such that $\max\{d(v) \mid v \in V - V(C)\}$ is maximum. By Theorem 7, C is a dominating cycle. Let v_0 be a vertex in $V - V(C)$ having maximum degree among all vertices of $V - V(C)$ and set $A = N(v_0) = \{x_1, x_2, \ldots, x_k\}$, where $k \geqslant 2$ and $x_{i+1} \in x_i \vec{C} x_{i+2}$, $1 \leqslant i \leqslant k$ (indices mod k). As in the proof of Theorem 9, Lemma 8 implies that $|V(C)| \geqslant n + d(v_0) - \alpha$, so that it suffices to show that $d(v_0) \geqslant \frac{1}{3}s$. Suppose $d(v_0) < \frac{1}{3}s$ and assume, without loss of generality, that $\min\{d(x_i^+) \mid 1 \leqslant i \leqslant k\} = d(x_1^+)$. Then for $i = 2, \ldots, k$ we have $d(x_i^+) \geqslant \frac{1}{3}s$, since $d(v_0) + d(x_1^+) + d(x_i^+) \geqslant s$. It follows that $x_2^+ x_i^{++} \notin E$ for $i = 3, \ldots, k$, otherwise a cycle C' with $|V(C')| = |V(C)|$ exists that includes v_0 and omits x_i^+, contradicting the choice of C. Thus, defining $B(x_1^+, x_2^+)$

as in the proof of Theorem 5, we have that $x_3^+, \ldots, x_k^+ \notin B(x_1^+, x_2^+)$. But then

$$d(v_0) + d(x_1^+) + d(x_2^+) = k + |B(x_1^+, x_2^+)| \leq k + (|V(C)| - (k-2))$$
$$= |V(C)| + 2 \leq n + 1 < s,$$

a contradiction. □

4. Conjectures

We begin with the following conjecture.

Conjecture 3. Let G be a 1-tough graph on $n \geq 3$ vertices such that $d(x) + d(y) + d(z) \geq s \geq n$ for all independent sets of vertices x, y, z. Then G contains a cycle of length at least $\min(n, \frac{1}{4}(3n+1) + \frac{1}{6}s)$.

The graphs G_n show that Conjecture 3, if true, is best possible. Conjecture 3 would also imply the following generalization of Jung's Theorem (Theorem 2).

Conjecture 4. Let G be a 1-tough graph on $n \geq 13$ vertices such that $d(x) + d(y) + d(z) \geq \frac{1}{2}(3n - 14)$ for all independent sets of vertices x, y, z. Then G is hamiltonian.

The graph obtained from H_{13} by deleting a vertex of degree 4 shows that the requirement that $n \geq 13$ cannot be released.

We close by noting that an application of Theorem 7 and Lemma 8 leads to a simple proof of Jung's Theorem for graphs on at least 16 vertices. By applying Theorem 5 instead of Theorem 7 it is possible to obtain a new proof of Jung's entire theorem ($n \geq 11$). Details will appear elsewhere [2, 3].

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