

On the Stabilization of Bilinear Systems via Constant Feedback

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ABSTRACT

We study the problem of stabilization of a bilinear system via a constant feedback. The question reduces to an eigenvalue problem on the pencil $A + \alpha_0 B$ of two matrices. Using the idea of simultaneous triangularization of the matrices involved, some easily checkable conditions for the solvability of this question are obtained. Algorithms for checking these conditions are given and illustrated by a few examples.

1. INTRODUCTION

One of the most challenging questions in the study of nonlinear control systems is the problem of finding, whenever possible, a stabilizing (continuous) feedback control. Although the question seems of great practical interest, so far no definite answers have been obtained. Several authors have obtained various interesting partial results; see for instance Aeyels [1, 2], Banks [3], Brockett [5], Gutman [7], Jurdjevic and Quinn [8], Kalouptsidis and Tsiniias [9, 12], and Slemrod [11]. In this paper we will consider the stabilizability of the bilinear system

$$\dot{x}(t) = Ax(t) + u(t)Bx(t), \quad (1.1)$$

where $x(t)$ belongs to \mathbb{R}^n and A and B are $n \times n$ matrices. That is, we seek

a (preferably smooth) feedback $u = \alpha(x)$ such that the system

$$\dot{x} = Ax + \alpha(x)Bx \quad (1.2)$$

has the origin as an asymptotic stable equilibrium. An obvious necessary condition for $u = \alpha(x)$ to be a stabilizing control is that the linearization around $x = 0$ of (1.2),

$$\dot{x} = Ax + \alpha_0 Bx = (A + \alpha_0 B)x \quad (1.3)$$

with $\alpha_0 = \alpha(0)$, is such that all the eigenvalues of $A + \alpha_0 B$ have nonpositive real part. On the other hand, whenever an α_0 exists for which (1.3) is asymptotically stable, then the bilinear system (1.1) is itself stabilizable by choosing the constant feedback $u = \alpha_0$.

The above simple observation leads to the study of the question: *When does there exist a constant feedback $u = \alpha_0$ such that the overall system (1.3) is asymptotically stable?*

This is the problem we want to address in this paper. Notice that a positive answer to the above problem has some interesting practical advantages. First of all, the feedback which renders the asymptotic stability is extremely simple; using the “parameter” choice $u = \alpha_0$ does not require the knowledge of the state of the system and is therefore easy to implement. Formally, we should not call $u = \alpha_0$ a feedback, since it does not depend on the state of the system, but we will still use this terminology to emphasize the relation to feedback stabilization. In fact the problem we consider here can also be viewed as an eigenvalue problem for the pencil $A + \alpha_0 B$, and this note could equally well be entitled “On the stable eigenvalues of the pencil $A + \alpha_0 B$.” Secondly a solution provides an exponential rate of convergence to the origin, contrary to various other stabilizing control schemes. A final important point is that the proposed feedback is continuous, which contrasts with the approach in [3, 7].

The approach to asymptotic stabilization of a bilinear system we propose here differs considerably in two ways from the existing literature. In [8] and [11] stabilizing controllers are typically quadratic functions. To make this explicit, a Lyapunov function $V(x) = \frac{1}{2}x^T Qx$ with $Q = Q^T > 0$ is sought such that on setting $u = -x^T Q Bx$ we have $\dot{V}(x(t)) \leq 0$ along solutions. The difficulty of this technique lies in finding a suitable matrix Q . In fact this is resolved in [10, 11], by assuming that the matrix A has eigenvalues with real part nonpositive and distinct on the imaginary axis. But this means that the system is already stable when setting $u = 0$.

Note that a modification of this method, rendering an even more complex (noncontinuous) controller, has recently been proposed in [3]. The second

aspect in which our approach differs from the literature is the continuity of the stabilizing controller (see e.g. [3, 7, 11]), which of course makes it more difficult to apply the proposed controller.

So far we have argued that for asymptotic stabilization of (1.1) it is necessary that there exist a constant feedback $u = \alpha_0$ such that the matrix $A + \alpha_0 B$ appearing in (1.3) has all eigenvalues with nonpositive real parts. We emphasize that this notion of stabilizability is different from what is sometimes called "practical stability"; see e.g. [7, 8].

The method we propose for the study of the constant feedback stabilization problem is based on the analysis of the Lie algebra generated by the matrices A and B . In fact, when this Lie algebra is solvable, we obtain an easy algorithmic way of testing the solvability of our question. The basic observation is that in this case the matrices A and B can be put simultaneously into a triangular form. An extension of this idea leads to the question when the matrices A and B can be put simultaneously into block-triangular form. This of course requires a weaker condition on the Lie algebra generated by A and B . Note that, as in the solvable case, these conditions can be verified by software packages containing matrix calculations. In conclusion we stress that we obtain explicit algorithmically verifiable conditions for the aforementioned two cases. The general constant feedback stabilization problem for bilinear system needs a lot more research.

The paper is organized as follows. Section 2 contains the main results, of which the proofs are given in Section 3. In Section 4 some algorithms for practically solving the constant feedback stabilization problem are given, and they are illustrated with examples in Section 5. Section 6 contains the conclusions. Before we start with Section 2 we will give some necessary prerequisites.

Let $\mathfrak{gl}(n)$ denote the space of all $n \times n$ matrices. A matrix Lie algebra \mathcal{L} in $\mathfrak{gl}(n)$ is a subspace of $\mathfrak{gl}(n)$ which is closed under the commutator product $[P, Q] = PQ - QP$. Note that $[P, Q] \in \mathfrak{gl}(n)$ has zero trace. $P \in \mathfrak{gl}(n)$ is called nilpotent if $P^k = 0$ for some k . Note that P nilpotent iff $P^n = 0$.

Let \mathcal{L} be a matrix Lie algebra. Define a sequence $C^i(\mathcal{L})$ of subspaces of $\mathfrak{gl}(n)$ as follows: $C^0(\mathcal{L}) = \mathcal{L}$; $C^{i+1}(\mathcal{L}) = [C^i(\mathcal{L}), C^i(\mathcal{L})]$. A matrix Lie algebra \mathcal{L} is called solvable if $C^k(\mathcal{L}) = 0$ some k .

For any set $P, Q \in \mathfrak{gl}(n)$, there exists a minimal matrix Lie algebra which contains P and Q , denoted by $\mathcal{L}(P, Q)$. This Lie algebra can be determined by calculating $[P, Q]$, $[P, [P, Q]]$, $[Q, [P, Q]]$, etc. until all further commutators are contained in the span of the previous ones. Since $\mathcal{L} \subset \mathfrak{gl}(n)$ is finite dimensional, we can derive $\mathcal{L}(P, Q)$ in a finite number of calculations.

Let \mathcal{L} be a matrix Lie algebra. A linear subspace $\mathcal{I} \subset \mathcal{L}$ is called an ideal if $[\mathcal{L}, \mathcal{I}] \subset \mathcal{I}$; in other words, for all $J \in \mathcal{L}$ and $I \in \mathcal{I}$ one has $[J, I] \in \mathcal{I}$. Given $P, Q \in \mathfrak{gl}(n)$, there always exists a minimal ideal $\mathcal{I}(P, Q)$ in $\mathcal{L}(P, Q)$ which contains $[P, Q]$.

2. ASYMPTOTIC STABILIZABILITY WITH A CONSTANT FEEDBACK

Consider the single input bilinear system (1.1) with $A, B \in \text{gl}(n)$. The question is whether there exists a constant $\alpha_0 \in \mathbb{R}$ such that the system (1.3) is asymptotically stable. A basic result in systems theory is that such a system is asymptotically stable if and only if

$$\sigma(A + \alpha_0 B) \subset \mathbb{C}^-, \quad (2.1)$$

where $\sigma(\cdot)$ denotes the set of eigenvalues of a matrix. Let

$$\Omega = \{ \alpha_0 \in \mathbb{R} \mid \sigma(A + \alpha_0 B) \subset \mathbb{C}^- \}. \quad (2.2)$$

Then (1.1) is asymptotically stabilizable with constant feedback if and only if $\Omega \neq \emptyset$. Now we can state the problem of asymptotic stabilizability of (1.1) as follows: *What are the conditions on A and B such that $\Omega \neq \emptyset$?*

The eigenvalues of $A + \alpha_0 B$ are the roots of

$$\text{Det}[\lambda I_n - (A + \alpha_0 B)] = \lambda^n + p_1(\alpha_0)\lambda^{n-1} + \cdots + p_{n-1}(\alpha_0)\lambda + p_n(\alpha_0). \quad (2.3)$$

In this determinant, $p_i(\alpha_0)$ are polynomials in α_0 . The degree of the polynomial $p_i(\alpha_0)$ is generically i , $i = 1, \dots, n$.

Using the Routh-Hurwitz condition [6], we have the following result:

- I. If $n = 1$, then $\Omega = \{ \alpha \mid p_1(\alpha) > 0 \}$.
- II. If $n = 2$, then $\Omega = \{ \alpha \mid p_2(\alpha) > 0, p_1(\alpha) > 0 \}$.
- III. If $n = 3$, then $\Omega = \{ \alpha \mid p_3(\alpha) > 0, p_1(\alpha) > 0, p_1(\alpha)p_2(\alpha) - p_3(\alpha) > 0 \}$.
- IV. If $n > 3$, then Ω is described by polynomials of degree > 4 .

Hence we can solve our stabilization problem analytically in case $n = 1, 2, 3$, using the Routh-Hurwitz condition. If $n > 3$, we cannot determine Ω in this way.

When the matrices A and B satisfy certain conditions, we can also give sufficient conditions if $n > 3$. In the sequel we consider the case in which A and B are simultaneously (block-)triangularizable.

REMARK. We note that without any doubt other approaches are possible. For instance, by setting $B = PQ$ with rank B equal to the inner dimension of

QP , the problem is related to an output feedback stabilization problem for the linear system $\dot{x} = Ax + Pu$, $y = Qx$ with a scalar output gain $u = \alpha y$. Some interesting results via this method, specifically when high gain feedback ($\alpha \rightarrow \infty$) is considered, are given in Kouvaritakis and MacFarlane [14]. Application of such results in general yields a set of necessary—but not sufficient—conditions for solvability of our problem. In particular, Examples 5.1 and 5.2 show that the set of solutions Ω is a finite interval, so the gain α should be finite, in contrast to [14].

2.1. *Simultaneous Triangularization*

Suppose $P, Q \in \text{gl}(n)$ are both in (right) triangular form, i.e. $P = [p_{ij}]_{i,j=1,\dots,n}$ with $p_{ij} = 0$ if $i > j$. Q has the same properties. Then $P + \alpha_0 Q$ is also triangular, and $\sigma(P + \alpha_0 Q) = \{p_{ii} + \alpha_0 q_{ii}, i = 1, \dots, n\}$.

Let A and B in the system (1.1) be such that there exists a (complex valued) linear transformation S that simultaneously triangularizes A and B , i.e., $\tilde{A} = SAS^{-1}$ and $\tilde{B} = SBS^{-1}$ are both triangular. In this case,

$$\Omega = \left\{ \alpha_0 \in \mathbb{R} \mid \tilde{a}_{ii} + \alpha_0 \tilde{b}_{ii} \in \mathbb{C}^- \right\} = \bigcap_{i=1}^n \left\{ \alpha_0 \in \mathbb{R} \mid \tilde{a}_{ii} + \alpha_0 \tilde{b}_{ii} \in \mathbb{C}^- \right\}, \quad (2.4)$$

which is now given by n linear restrictions. The question now is: *What are the restrictions on A and B in the system (1.1) such that A and B are simultaneously triangularizable?*

THEOREM 2.1. *Two matrices A and B are simultaneously triangularizable if and only if $\mathcal{L}(A, B)$ is solvable.*

This theorem is due to Lie. A proof is given in [13]. So, given A and B , we can verify if $\mathcal{L}(A, B)$ is solvable. If it is, then we know that A and B are simultaneously triangularizable, and in principle we can derive the triangular forms and determine the set Ω .

Following this line, we still have two problems. The first is how to derive the triangularizing transformation for A and B . The second is how to check whether $\mathcal{L}(A, B)$ is solvable.

The second problem we can solve by just determining $C^i(\mathcal{L}(A, B))$, $i = 1, \dots$, and checking whether $C^k(\mathcal{L}(A, B)) = 0$ for some k . This is however a rather inefficient way. Furthermore it is difficult to derive a direct algorithm to find the triangularizing transformation from the proof of Lie.

From these considerations we conclude that it is useful to state and prove the theorem in another way, which gives conditions that are simpler to verify and for which the proof is easier to implement in an algorithm.

A proof of the theorem is given in Section 3.

THEOREM 2.2. *A and B are simultaneously triangularizable if and only if every $I \in \mathcal{I}(A, B)$ is nilpotent.*

REMARK. Of course, the situation in which A and B are simultaneously triangularizable is unusual. For instance, in this case the system (1.1) can never be accessible [4]. It can only be accessible except for a hyperplane.

2.2. Simultaneous Block Triangularization

If A and B are not simultaneously triangularizable, it may be possible to block-triangularize them simultaneously, i.e., there may exist a complex valued linear transformation S such that SAS^{-1} and SBS^{-1} are both block-triangular with blocks of dimension $n_i, i = 1, \dots, p$:

$$\tilde{A} = \begin{bmatrix} A_{11} & * & \cdots & * \\ 0 & A_{22} & & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_{pp} \end{bmatrix}, \quad \text{where } A_{ii} \text{ is an } n_i \times n_i \text{ matrix.}$$

The matrix \tilde{B} has the same configuration. If such a transformation exists, then we have

$$\Omega = \bigcap_{i=1}^p \{ \alpha_0 \in \mathbb{R} \mid \sigma(A_{ii} + \alpha_0 B_{ii}) \subset \mathbb{C}^- \}. \tag{2.5}$$

Let $\underline{n} = [n_1, \dots, n_p]$. By $\underline{n} \leq k$ we mean that $n_i \leq k$ for all $i = 1, \dots, p$. Note that the ordering of the blocks of a block-triangular matrix is not free.

If A and B are simultaneously \underline{n} -block-triangularizable with $\underline{n} \leq 3$ we can exactly determine the set Ω . So we may wonder what are the conditions on A and B such that they are simultaneously \underline{n} -block-triangularizable with $\underline{n} \leq 3$.

Note that the case $\underline{n} \leq 1$ is decided by Theorem 2.2. In what follows we only study the case $\underline{n} \leq 2$, and we leave $\underline{n} \leq 3$ for further research. Let Char_p denote the characteristic polynomial of P .

THEOREM 2.3. *Let $A, B \in \mathfrak{gl}(n)$. The following conditions are equivalent:*

- I. *A and B are simultaneously \underline{n} -block-triangularizable for some $\underline{n} \leq 2$.*

II. (1) We have

$$\text{Char}_{[A, B]} = \lambda^{n-2p} \prod_{i=1}^p (\lambda^2 - s_i). \tag{2.6}$$

- (2) There exist subspaces $\mathcal{V}_i, i = 1, \dots, n - p$, of \mathbb{C}^n such that
- (a) $\mathbb{C}^n = \bigoplus_{j=1}^{n-p} \mathcal{V}_j$;
 - (b) \mathcal{V}_i is the linear span of one (generalized) eigenvector of $\lambda = s_i^{1/2}$ and one (generalized) eigenvector of $\lambda = -s_i^{1/2}, i = 1, \dots, p$;
 - (c) \mathcal{V}_i is a (generalized) 1-dimensional eigenspace of $\lambda = 0, i = p + 1, \dots, n - p$;
 - (d) there exists a permutation π on $i = 1, \dots, n - p$ such that $\mathcal{W}_i = \bigoplus_{j=1}^i \mathcal{V}_{\pi(j)}$ is invariant under A and B for all $i = 1, \dots, n - p$.

The proof of this theorem is given in Section 3. In the case that all s_i in (2.6) are different, we propose an algorithm in Section 4 in order to find the transformation that block-triangularizes A and B . If A and B are \underline{n} -block-triangularized with $\underline{n} \leq 2$, we can determine the set Ω .

3. PROOFS OF THE THEOREMS

In this section we will give proofs of Theorems 2.2 and 2.3. In order to prove Theorem 2.2 we use the following lemmas:

LEMMA 3.1. *If $P, Q \in \text{gl}(n)$ commute, then they are simultaneously triangularizable.*

Proof. (By induction on n .) If $P, Q \in \text{gl}(1)$ then P and Q are both triangular. Next suppose two commuting $k \times k$ matrices are simultaneously triangularizable if $k < n$. Let P and Q commuting $n \times n$ matrices. Let λ be some eigenvalue of P , and \mathcal{V} the corresponding eigenspace. Then \mathcal{V} is obviously P -invariant, i.e., $x \in \mathcal{V} \Rightarrow Px \in \mathcal{V}$. Moreover \mathcal{V} is Q -invariant, since for all $v \in \mathcal{V}, P(Qv) = (PQ)v = (QP)v = Q(Pv) = \lambda(Qv)$, so $Qv \in \mathcal{V}$. Let $\{v_1, \dots, v_s\}$ be a basis for \mathcal{V} , and let $\{v_{s+1}, \dots, v_n\}$ be a completion to a basis of \mathbb{C}^n . Let $S_1^{-1} = [v_1 \ v_2 \ \dots \ v_n]$. Then

$$P_1 = S_1 P S_1^{-1} = \begin{bmatrix} \lambda I_s & * \\ 0 & \tilde{P} \end{bmatrix}, \quad Q_1 = S_1 Q S_1^{-1} = \begin{bmatrix} Q_s & * \\ 0 & \tilde{Q} \end{bmatrix}. \tag{3.1}$$

There always exists a regular $s \times s$ matrix S_s such that $\hat{Q}_s = S_s Q_s S_s^{-1}$ is triangular. Since P and Q commute, $\tilde{P}, \tilde{Q} \in \text{gl}(n-s)$ commute with $n-s < n$ and are therefore simultaneously triangularizable, suppose with regular S_{n-s} , $\hat{P} = S_{n-s} \tilde{P} S_{n-s}^{-1}$, $\hat{Q} = S_{n-s} \tilde{Q} S_{n-s}^{-1}$ triangular. Then $S_2 = \text{Diag}[S_s, S_{n-s}]$ triangularizes P_1 and Q_1 . So, P and Q are simultaneously triangularizable with transformation $S = S_2 S_1$. ■

LEMMA 3.2 (Engel’s theorem). *Suppose all elements J of the matrix Lie algebra \mathcal{J} are nilpotent. Then there exists a $v \in \mathbb{C}^n$ with $v \neq 0$ and $Jv = 0$ for all $J \in \mathcal{J}$.*

Proof. See [13]. ■

Proof of Theorem 2.2. \Rightarrow : Suppose A and B are simultaneously triangularizable, $\hat{A} = SAS^{-1}$ and $\hat{B} = SBS^{-1}$ triangular. Then $[\hat{A}, \hat{B}]$ is strictly triangular. One can easily verify that this implies that SJS^{-1} is strictly triangular for all $J \in \mathcal{J}(A, B)$. This implies that $J^n = 0$ for all $J \in \mathcal{J}(A, B)$.

\Leftarrow : Suppose $A, B \in \text{gl}(n)$. We prove the theorem by induction on n . If $A, B \in \text{gl}(1)$, then they are both triangular. Next suppose P and Q are simultaneously triangularizable if $P, Q \in \text{gl}(k)$ with $k < n$ and all elements of $\mathcal{J}(P, Q)$ nilpotent. Now suppose $A, B \in \text{gl}(n)$ and all elements of $\mathcal{J}(A, B)$ are nilpotent. Then by Lemma 3.2, the set $\mathcal{V} = \bigcap_{J \in \mathcal{J}(A, B)} \text{Ker } J \neq \{0\}$.

\mathcal{V} is invariant under A and B : Let $v \in \mathcal{V}$. Then

$$\begin{aligned} 0 &= [J, A]v = JAv - AJv = JAv \quad \text{for all } J \in \mathcal{J}(A, B), \quad \text{so } Av \in \mathcal{V}; \\ 0 &= [J, B]v = JBv - BJv = JBv \quad \text{for all } J \in \mathcal{J}(A, B), \quad \text{so } Bv \in \mathcal{V}. \end{aligned} \tag{3.2}$$

Let $\{v_1, \dots, v_s\}$ be a basis for \mathcal{V} in \mathbb{C}^n and $\{v_{s+1}, \dots, v_n\}$ a completion to a basis of \mathbb{C}^n . Let $S_1^{-1} = [v_1 \ v_2 \ \dots \ v_n]$. Then

$$A_1 = S_1 A S_1^{-1} = \begin{bmatrix} \hat{A}_s & * \\ 0 & \hat{A} \end{bmatrix}, \quad B_1 = S_1 B S_1^{-1} = \begin{bmatrix} \hat{B}_s & * \\ 0 & \hat{B} \end{bmatrix}, \tag{3.3}$$

$$J_1 = S_1 J S_1^{-1} = \begin{bmatrix} 0 & * \\ 0 & \hat{J} \end{bmatrix} \quad \text{for all } J \in \mathcal{J}(A, B). \tag{3.4}$$

Furthermore, since $[A, B] \in \mathcal{J}(A, B)$, $[\hat{A}_s, \hat{B}_s] = 0$. So \hat{A}_s are \hat{B}_s commute.

By Lemma 3.1 there exists a regular $s \times s$ matrix S_s such that $\tilde{A}_s = S_s \hat{A}_s S_s^{-1}$ and $\tilde{B}_s = S_s \hat{B}_s S_s^{-1}$ are both triangular.

Now let $\hat{\mathcal{J}}$ the set of all matrices \hat{J} we get from (3.4). Then $\hat{\mathcal{J}}$ consists of $(n - s) \times (n - s)$ matrices. Moreover, every element of $\hat{\mathcal{J}}$ is nilpotent, and $\hat{\mathcal{J}} = \mathcal{J}(\hat{A}, \hat{B})$. Since $n - s < n$, \hat{A} and \hat{B} are simultaneously triangularizable: $\tilde{A} = S_{n-s} \hat{A} S_{n-s}^{-1}$ and $\tilde{B} = S_{n-s} \hat{B} S_{n-s}^{-1}$ triangular. Then $S_2 = \text{Diag}[S_s, S_{n-s}]$ triangularizes A_1 and B_1 . So A and B are simultaneously triangularizable with transformation $S = S_2 S_1$. ■

Proof of Theorem 2.3. II \Rightarrow I: Let $n_i = \dim \mathcal{V}_{\pi(i)}$, $N_k = \sum_{i=1}^k n_i$, and $\mathcal{W}_k = \text{sp}\{v_1, \dots, v_{N_k}\}$, $k = 1, \dots, n - p$. Since all \mathcal{W}_k are invariant under A and B , with $S^{-1} = [v_1 \ v_2 \ \dots \ v_n]$, A and B must satisfy

$$\begin{aligned}
 SAS^{-1} &= \begin{bmatrix} A_1 & * & \dots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & A_{n-p} \end{bmatrix}, \\
 SBS^{-1} &= \begin{bmatrix} B_1 & * & \dots & * \\ 0 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & B_{n-p} \end{bmatrix}, \tag{3.5}
 \end{aligned}$$

where A_i and B_i are $n_i \times n_i$ matrices with $n_i \leq 2$. Therefore, A and B are simultaneously n -block-triangularizable with $\underline{n} \leq 2$.

I \Rightarrow II: Suppose A and B are simultaneously \underline{n} -block-triangularizable with $\underline{n} \leq 2$. Then for some regular $n \times n$ matrix S ,

$$\begin{aligned}
 SAS^{-1} &= \begin{bmatrix} A_1 & * & \dots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & A_{n-p} \end{bmatrix}, \\
 SBS^{-1} &= \begin{bmatrix} B_1 & * & \dots & * \\ 0 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & B_{n-p} \end{bmatrix}, \tag{3.6}
 \end{aligned}$$

where A_i and B_i are $n_i \times n_i$ matrices with $n_i \leq 2$. Then

$$S[A, B]S^{-1} = \begin{bmatrix} [A_1, B_1] & * & \cdots & * \\ 0 & [A_2, B_2] & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & [A_{n-p}, B_{n-p}] \end{bmatrix}. \tag{3.7}$$

Therefore, since $\text{Tr}[A_i, B_i] = 0$,

$$\text{Char}_{[A, B]}(\lambda) = \prod_{i=1}^{n-p} \text{Char}_{[A_i, B_i]}(\lambda) = \lambda^{n-2p} \prod_{i=1}^p (\lambda^2 - s_i). \tag{3.8}$$

With $N_i = \sum_{j=1}^i n_j$ and $\tilde{\mathcal{V}}_i = \text{sp}\{e_j, j = N_{i-1} + 1, \dots, N_i\}$, where e_j denotes the j th basis vector in \mathbb{C}^n , $\tilde{\mathcal{W}}_i = \bigoplus_{j=1}^i \tilde{\mathcal{V}}_j$ is invariant under SAS^{-1} and SBS^{-1} . Then with $\mathcal{V}_i = S\tilde{\mathcal{V}}_i$, $\mathcal{W}_i = \bigoplus_{j=1}^i \mathcal{V}_j$ is invariant under A and B for all $i = 1, \dots, n - p$. After a permutation, the subspaces are equivalent to the subspaces given in II(2). ■

4. ALGORITHMS

In this section we will give algorithms to determine the set Ω of asymptotic stabilizing constant feedbacks for a bilinear system (1.1) in the case that A and B are simultaneously \underline{n} -block-triangularizable with $\underline{n} \leq 1$ or $\underline{n} \leq 2$ ($\underline{n} \leq 1$ means triangularizability).

The main algorithms are 4.2 and 4.4. Algorithm 4.1 will be used in the other algorithms.

ALGORITHM 4.1 [Simultaneous triangularization of commuting $P, Q \in \text{gl}(n)$].

Step 1: Let $P' = P, Q' = Q, p = n, S = I_n$.

Step 2: Determine an eigenvalue of P' and $\mathcal{V} = \{x \in \mathbb{C}^p | P'x = \lambda x\} = \text{sp}\{v_1, \dots, v_s\}$. Let $\{v_{s+1}, \dots, v_p\}$ be a completion to a basis of \mathbb{C}^p , and $S_1^{-1} = [v_1 \ v_2 \ \cdots \ v_p]$,

$$P_1 = S_1 P' S_1^{-1} = \begin{bmatrix} \lambda I_s & * \\ 0 & \hat{P} \end{bmatrix}, \quad Q_1 = S_1 Q' S_1^{-1} = \begin{bmatrix} \hat{Q}_s & * \\ 0 & \hat{Q} \end{bmatrix}. \tag{4.1}$$

Step 3: Determine some S_s such that $\tilde{Q}_s = S_s \hat{Q}_s S_s^{-1}$ is triangular, and let

$$S_2 = \begin{bmatrix} S_s & 0 \\ 0 & I_{p-s} \end{bmatrix},$$

$$P_2 = S_2 P_1 S_2^{-1} = \begin{bmatrix} \lambda I_s & * \\ 0 & \hat{P} \end{bmatrix}, \quad Q_2 = S_2 Q_1 S_2^{-1} = \begin{bmatrix} \tilde{Q}_s & * \\ 0 & \hat{Q} \end{bmatrix}. \quad (4.2)$$

Let

$$S = \begin{bmatrix} I_{n-p} & 0 \\ 0 & S_2 S_1 \end{bmatrix} S. \quad (4.3)$$

We are finished if \hat{P} and \hat{Q} are triangular. Otherwise let $P' = \hat{P}$, $Q' = \hat{Q}$, $p = p - s$, and return to step 2. Since p is strictly decreasing with every loop, the algorithm ends in a finite number of steps.

Step 4: With S as given, SPS^{-1} and SQS^{-1} are triangular.

ALGORITHM 4.2 [Asymptotic stabilization of the system (1.1) if A and B are simultaneously triangularizable].

- Step 1: Given A and B , determine $\mathcal{S}(A, B)$ by calculating commutator products. Determine a basis $\{J_1, \dots, J_k\}$ of $\mathcal{S}(A, B)$.
- Step 2: Determine $J_i^n = 0$, $i = 1, \dots, k$. A and B are simultaneously triangularizable iff $J_i^n = 0$, $i = 1, \dots, k$. Stop if this is not the case. Let $p = n$, $P = A$, $Q = B$, $J'_i = J_i$, $S = I_n$.
- Step 3: Determine $\mathcal{V} = \bigcap_{i=1}^k \text{Ker } J'_i$; then $\mathcal{V} \neq \{0\}$. Determine some basis $\{v_1, \dots, v_s\}$ of \mathcal{V} and a completion $\{v_{s+1}, \dots, v_p\}$ to a basis of \mathbb{C}^p .
- Step 4: Let $S_1^{-1} = [v_1 \ v_2 \ \dots \ v_p]$ and

$$P_1 = S_1 P S_1^{-1} = \begin{bmatrix} \hat{P}_s & * \\ 0 & \hat{P} \end{bmatrix}, \quad Q_1 = S_1 Q S_1^{-1} = \begin{bmatrix} \hat{Q}_s & * \\ 0 & \hat{Q} \end{bmatrix},$$

$$J'_{i1} = S_1 J'_i S_1^{-1} = \begin{bmatrix} 0 & * \\ 0 & \hat{f}'_i \end{bmatrix}. \quad (4.4)$$

\hat{P}_s and \hat{Q}_s commute; all elements of $\mathcal{S}(\hat{P}, \hat{Q})$ are nilpotent. Let S_s

be a triangularizing transformation for \hat{P}_s and \hat{Q}_s , achieved from Algorithm 4.1, and

$$S_2 = \begin{bmatrix} S_s & 0 \\ 0 & I_{p-s} \end{bmatrix},$$

$$P_2 = S_2 P_1 S_2^{-1} = \begin{bmatrix} \tilde{P}_s & * \\ 0 & \hat{P} \end{bmatrix}, \quad Q_2 = S_2 Q_1 S_2^{-1} = \begin{bmatrix} \tilde{Q}_s & * \\ 0 & \hat{Q} \end{bmatrix}. \quad (4.5)$$

Step 5: Let

$$S = \begin{bmatrix} I_{n-p} & 0 \\ 0 & S_1 S_2 \end{bmatrix} S.$$

If \hat{P} and \hat{Q} are both triangular, then we are ready. A and B are triangularized with the transformation S . Go to step 6. Else let $P = \hat{P}$, $Q = \hat{Q}$, $p = p - s$, $J_i' = \hat{J}_i'$, and return to step 3. Since p strictly decreases with every loop, the algorithm ends in a finite number of steps.

Step 6: Let $\tilde{A} = SAS^{-1}$, $\tilde{B} = SBS^{-1}$, and determine $\Omega = \{ \alpha_0 \in \mathbb{R} \mid \text{Re}(\tilde{a}_{ii} + \alpha_0 \tilde{b}_{ii}) < 0, i = 1, \dots, n \}$. If $\Omega = \emptyset$, then the system (1.1) is not asymptotically stabilizable with constant feedback. If $\Omega \neq \emptyset$, then we can stabilize the system (1.1) asymptotically with any constant feedback $\alpha_0 \in \Omega$.

The next algorithm makes it possible to derive the set of asymptotic stabilizing constant feedbacks for the system (1.1) in the case that A and B are simultaneously \underline{n} -block-triangularizable with $\underline{n} \leq 2$. Moreover, the nonzero eigenvalues of $[A, B]$ are assumed to have algebraic multiplicity 1. In other words, if $\text{Char}_{[A, B]}(\lambda) = \lambda^{n-2p} \prod_{i=1}^p (\lambda^2 - s_i)$ then $s_i \neq s_j$ if $s_i \neq 0$ and $i \neq j$. In order to make clear step 3 of Algorithm 4.4 we give the following lemma.

LEMMA 4.3. *Let \mathcal{V}_i and $\pi(\cdot)$ be given by Theorem 2.3. Then*

$$\bigcap_{J \in \mathcal{J}(A, B)} \text{Ker } J = \{0\} \Rightarrow \dim \mathcal{V}_{\pi(1)} = 2.$$

Proof. Note that $\dim \mathcal{V}_{\pi(1)} = 1$ or 2. Suppose $\bigcap_{J \in \mathcal{J}(A, B)} \text{Ker } J = \{0\}$ and $\dim \mathcal{V}_{\pi(1)} = 1$, $\mathcal{V}_{\pi(1)} = \text{sp}\{v\}$. We show that it leads to a contradiction.

$\mathcal{V}_{\pi(1)}$ is A - and B -invariant, so $Av = \lambda v$, $Bv = \mu v$. Therefore $[A, B]v = 0$, which implies that $Jv = 0$ for all $J \in \mathcal{J}(A, B)$, so $\bigcap_{J \in \mathcal{J}(A, B)} \text{Ker } J \neq \{0\}$. Therefore $\dim \mathcal{V}_{\pi(1)} \neq 1$. ■

ALGORITHM 4.4 [Asymptotic stabilization of the system (1.1) in the case that A and B are simultaneously \underline{n} -block-triangularizable with $\underline{n} \leq 2$ and

$$\text{Char}_{[A, B]}(\lambda) = \lambda^{n-2p} \prod_{i=1}^p (\lambda^2 - s_i) \quad \text{with } s_i \neq s_j \text{ if } i \neq j]. \quad (4.6)$$

Step 1: Determine $q(\lambda) = \text{Char}_{[A, B]}(\lambda)$. Verify whether $q(\lambda)$ satisfies (4.6). If not, then the algorithm cannot be applied. Determine $\mathcal{J}(A, B) = \text{sp}\{J_1, \dots, J_k\}$. Let $P = A$, $Q = B$, $p = n$, $J'_i = J_i$, $S = I_n$.

Step 2: Determine $\tilde{\mathcal{V}} = \bigcap_{i=1}^k \text{Ker } J'_i$.

Step 3: Depending on $\tilde{\mathcal{V}}$, choose path I or II:

I: $\tilde{\mathcal{V}} \neq \{0\}$. Choose a basis $\{v_1, \dots, v_s\}$ of $\tilde{\mathcal{V}}$ and a completion to a basis of \mathbb{C}^p . Let $S_1^{-1} = [v_1 \ v_2 \ \dots \ v_p]$,

$$J'_{i1} = S_1 J'_i S_1^{-1} = \begin{bmatrix} 0 & * \\ 0 & \hat{f}_i \end{bmatrix},$$

$$P_1 = S_1 P S_1^{-1} = \begin{bmatrix} P_s & * \\ 0 & \hat{P} \end{bmatrix}, \quad Q_1 = S_1 Q S_1^{-1} = \begin{bmatrix} Q_s & * \\ 0 & \hat{Q} \end{bmatrix}. \quad (4.7)$$

P_s and Q_s commute. Use Algorithm 4.1 to derive a triangularizing transformation S_s for both P_s and Q_s . Let

$$S_2 = \begin{bmatrix} S_s & 0 \\ 0 & I_{p-s} \end{bmatrix},$$

$$P_2 = S_2 P_1 S_2^{-1} = \begin{bmatrix} \tilde{P}_s & * \\ 0 & \hat{P} \end{bmatrix}, \quad Q_2 = S_2 Q_1 S_2^{-1} = \begin{bmatrix} \tilde{Q}_s & * \\ 0 & \hat{Q} \end{bmatrix}. \quad (4.8)$$

Let

$$q(\lambda) = \frac{q(\lambda)}{\lambda^s} \quad \text{and} \quad S = \begin{bmatrix} I_{n-p} & 0 \\ 0 & S_2 S_1 \end{bmatrix} S.$$

II: $\mathcal{V}^{\sim} = \{0\}$. Then the next invariant subspace is 2-dimensional (Lemma 4.3). Determine that i for which the “eigenspace” of $\lambda^2 = s_i$ is invariant under P and Q . From condition II(2) in Theorem 2.3 we know that such an i always exists. Let $\mathcal{W} = \text{sp}\{v_1, v_2\}$ be this eigenspace. Construct a completing basis $\{v_3, \dots, v_p\}$ for \mathbb{C}^p , and let $S_1^{-1} = [v_1 \ v_2 \ \dots \ v_p]$,

$$J'_{i1} = S_1 J_i S_1^{-1} = \begin{bmatrix} \hat{J}_{i2} & * \\ 0 & J_i \end{bmatrix},$$

$$P_1 = S_1 P S_1^{-1} = \begin{bmatrix} \tilde{P}_2 & * \\ 0 & \hat{P} \end{bmatrix}, \quad Q_1 = S_1 Q S_1^{-1} = \begin{bmatrix} \tilde{Q}_2 & * \\ 0 & \hat{Q} \end{bmatrix}. \quad (4.9)$$

Let

$$q(\lambda) = \frac{q(\lambda)}{\lambda^2 - s_i} \quad \text{and} \quad S = \begin{bmatrix} I_{n-p} & 0 \\ 0 & S_1 \end{bmatrix} S.$$

Step 4: If \hat{P} and \hat{Q} are not on the desired block-triangular form yet, then let $p = p - s$, $P = \hat{P}$, $Q = \hat{Q}$, $J'_i = \hat{J}_i$, and return to step 2. Since p is strictly decreasing with every loop, the algorithm ends in a finite number of steps.

Step 5: Let $\tilde{A} = SAS^{-1}$ and $\tilde{B} = SBS^{-1}$, and determine Ω from \tilde{A} and \tilde{B} .

REMARKS.

(1) Algorithm 4.2 is contained in Algorithm 4.4, as could be expected. If A and B are simultaneously triangularizable, then in Algorithm 4.4, step 3, path I will always be chosen.

(2) Using software packages like PC-MATLAB, the calculations in the algorithms become very simple.

(3) The choice of ordering of the blocks in the block-triangular matrices is not free. So, in general, it is not possible to take first all 1×1 blocks, then the 2×2 blocks, etc. In other words, the choice of the permutation π in Theorem 2.3 is not free.

5. EXAMPLES

In this section, we will give two examples which illustrate the use of the algorithms in Section 4. The first one is an example where A and B are

simultaneously \underline{n} -block-triangularizable with $\underline{n} \leq 2$, so Algorithm 4.4 will be applied. The second one is an example from Banks [3] which shows the usefulness of asymptotic stabilization with constant feedback. The second problem is solved with Algorithm 4.2. The calculations are made using PC-MATLAB.

EXAMPLE 5.1 (Asymptotic stabilization with constant feedback control using \underline{n} -block-triangularization with $\underline{n} \leq 2$.) Consider a system (1.1) with

$$A = \begin{bmatrix} -1 & 0 & -1 & 3 \\ 3 & -3 & -3 & 3 \\ -2 & 0 & 0 & 3 \\ 3 & -4 & -3 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -4 & 4 & 3 & 1 \\ -2 & 2 & 2 & 1 \\ -5 & 4 & 4 & 1 \\ -3 & 2 & 3 & 0 \end{bmatrix}.$$

We easily verify that $1 \in \sigma(A)$, so the system is not asymptotically stable itself. Now we apply Algorithm 4.4 to investigate whether the system is asymptotically stabilizable with a constant feedback.

Step 1:

$$[A, B] = \begin{bmatrix} -13 & 14 & 13 & -11 \\ -7 & 10 & 7 & -9 \\ -13 & 14 & 13 & -11 \\ 8 & -2 & -8 & -10 \end{bmatrix},$$

so $q(\lambda) = \text{Char}_{[A, B]}(\lambda) = \lambda^2(\lambda^2 - 118)$ and is of the desired form. If we determine $\mathcal{S}(A, B) = \text{sp}\{[A, B], [A, [A, B]], [B, [A, B]], \dots\}$, then we obtain an 8-dimensional matrix space, $\mathcal{S}(A, B) = \text{sp}\{J_i, i = 1, \dots, 8\}$. Let $p = 4$, $P = A$, $Q = B$, $J'_i = J_i$, $S = I_4$.

Step 2: Determine $\tilde{\mathcal{Y}} = \bigcap_{i=1}^8 \text{Ker } J'_i = \text{sp}\{[1, 0, 1, 0]^T\}$.

Step 3: $\tilde{\mathcal{Y}} \neq \{0\}$, so go to 3-I. Let $\{[1, 0, 0, 0]^T, [0, 1, 0, 0]^T, [0, 0, 0, 1]^T\}$ be a completion to a basis of \mathbb{C}^4 . Let

$$S_1^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$P_1 = \left[\begin{array}{c|cc|c} -2 & -2 & 0 & 3 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 3 & -3 & 3 \\ 0 & 3 & -4 & 0 \end{array} \right], \quad Q_1 = \left[\begin{array}{c|cc|c} -1 & -5 & 4 & 1 \\ \hline 0 & 1 & 0 & 0 \\ 0 & -2 & 2 & 1 \\ 0 & -3 & 2 & 0 \end{array} \right].$$

$\hat{P}_s = [-2]$ and $\hat{Q}_s = [-1]$ are triangular. We can choose $S_2 = I_4$. Further,

$$\hat{P} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -3 & 3 \\ 3 & -4 & 0 \end{bmatrix} \quad \text{and} \quad \hat{Q} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 1 \\ -3 & 2 & 0 \end{bmatrix}.$$

If we compute \hat{J}_i , then we obtain three zero matrices; hence $\hat{\mathcal{J}}$ is 5-dimensional.

Step 4: \hat{P} and \hat{Q} are not of the desired form yet. Return to step 2 with $q(\lambda) = \lambda(\lambda^2 - 118)$, $P = \hat{P}$, $Q = \hat{Q}$, $p = 3$, $J'_i = \hat{J}_i$, and $S = S_2 S_1 I_n$.

Step 2: $\tilde{\mathcal{Y}} = \bigcap_{i=1}^8 \text{Ker } J'_i = \{0\}$.

Step 3: $\tilde{\mathcal{Y}} = \{0\}$, so go to 3-II. There is only one 2-dimensional ‘‘eigenspace,’’ the one belonging to $\lambda^2 - 118$, given by $\mathcal{W} = \text{sp}\{[0, 1, 0]^T, [0, 0, 1]^T\}$. \mathcal{W} is invariant under P and Q . $\{[1, 0, 0]^T\}$ is a completion to a basis of \mathbb{C}^3 . Let

$$S_1^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then

$$P_1 = \left[\begin{array}{ccc|c} -3 & 3 & 3 & 3 \\ -4 & 0 & 3 & 3 \\ \hline 0 & 0 & 1 & 1 \end{array} \right], \quad Q_1 = \left[\begin{array}{ccc|c} 2 & 1 & -2 & -2 \\ 2 & 0 & -3 & -3 \\ \hline 0 & 0 & 1 & 1 \end{array} \right].$$

When we compute \hat{J}_i , $i = 1, \dots, 5$, we find two of them are zero; hence $\hat{\mathcal{J}}$ is 3-dimensional.

Step 4: Now $\hat{P} = [1]$ and $\hat{Q} = [1]$ are of the desired form, and we go to step 5 with

$$S = \begin{bmatrix} 1 & 0 \\ 0 & S_1 \end{bmatrix} S.$$

Step 5:

$$\tilde{A} = \left[\begin{array}{ccc|c} -2 & 0 & 3 & -2 \\ \hline 0 & -3 & 3 & 3 \\ 0 & -4 & 0 & 3 \\ \hline 0 & 0 & 0 & 1 \end{array} \right], \quad \tilde{B} = \left[\begin{array}{ccc|c} -1 & 4 & 1 & -5 \\ \hline 0 & 2 & 1 & -2 \\ 0 & 2 & 0 & -3 \\ \hline 0 & 0 & 0 & 1 \end{array} \right].$$

Then $\Omega = \{-2 < \alpha_0 < -1\} \neq \emptyset$. Therefore, the system (2.1) is asymptotically stabilizable with any constant feedback control $-2 < \alpha_0 < -1$.

EXAMPLE 5.2 (Banks [3]). In [3] Banks gives an example where he asymptotically stabilizes a system (1.1) with

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 5 & 3 \end{bmatrix}$$

by means of a feedback

$$u(x) = \begin{cases} -\frac{-x_1^2 + x_2^2 + 1/2}{[x_2 - g(x_1)]x_2} & \text{if } [x_2 - g(x_1)]x_2 \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

for some specific function $g(\cdot)$ and $x = (x_1 \ x_2)^T$. Application of this feedback control leads to a system which is neither linear nor bilinear.

We easily verify that $[A, B] = 0$, so A and B commute and are therefore simultaneously triangularizable. Application of Algorithm 5.2 gives that $\Omega = \{-1 < \alpha_0 < -\frac{1}{4}\} \neq \emptyset$, so the given system is also asymptotically stabilizable with a constant feedback.

6. CONCLUSIONS

In this paper we have considered the stabilizability of a bilinear system via a constant feedback. When the matrices involved are simultaneously (block-)triangularizable this leads to easily checkable conditions on these matrices. For the simultaneous triangularization we use a modified version of Lie's theorem. In this way we obtain an algorithm for checking the solvability of the constant feedback stabilization problem. A similar approach is used for block triangularization of the matrices involved, where it is assumed that each block is at most 2×2 . Obviously further research is required for the general solution of the constant feedback stabilization problem for bilinear systems.

REFERENCES

- 1 D. Aeyels, Stabilization of a class of nonlinear systems by a smooth feedback control, *Systems Control Lett.* 5:289-294 (1985).

- 2 D. Aeyels, Local and global stabilizability for nonlinear systems, in *Theory and Applications of Nonlinear Control Systems* (C. J. Byrnes and A. Lindquist, Eds.), North Holland, 1986.
- 3 S. P. Banks, Stabilizability of finite- and infinite-dimensional bilinear systems, *IMA J. Math. Control Inform.* 3:255–271 (1986).
- 4 R. W. Brockett, System theory on group manifolds and coset spaces, *SIAM J. Control Optim.* 10:265–284 (1972).
- 5 R. W. Brockett, Asymptotic stability and feedback stabilization, in *Differential Geometric Control Theory* (R. W. Brockett, R. S. Millman, and H. J. Sussmann, Eds.), Birkhäuser, Boston, 1983.
- 6 F. R. Gantmacher, *The Theory of Matrices*, Chelsea, New York, 1959.
- 7 P. O. Gutman, Stabilizing controllers for bilinear systems, *IEEE Trans. Automat. Control* 26:917–922 (1981).
- 8 V. Jurdjevic and J. P. Quinn, Controlability and stability, *J. Differential Equations* 28:381–389 (1978).
- 9 N. Kalouptsidis, and J. Tsinias, Stability improvement of nonlinear systems by feedback, *IEEE Trans. Automat. Control* 29:364–367 (1984).
- 10 E. P. Ryan, Global asymptotic stabilization of a class of bilinear systems, *Internat. J. Control* 38:359–367 (1983).
- 11 M. Slemrod, Stabilization of bilinear control systems with applications to nonconservative problems in elasticity, *SIAM J. Control Optim.* 16:131–141 (1978).
- 12 J. Tsinias, N. Kalouptsidis, and A. Bacciotti, Lyapunov functions and stability of dynamical systems, *Math. Systems Theory* 19:333–354 (1987).
- 13 V. S. Varadarajan, *Lie Groups, Lie Algebras and Their Representations*, Prentice-Hall, Englewood Cliffs, N.J., 1974.
- 14 B. Kouvaritakis and A. G. J. MacFarlane, Geometric approach to analysis and synthesis of system zeros, Parts I, II, *Internat. J. Control* 23:149–181 (1976).

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