ELSEVIER Journal of Computational and Applied Mathematics 101 (1999) 203–229

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Monotonicity preserving interpolatory subdivision schemes

Frans Kuijt *, [1], Ruud van Damme

Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, NL-7500 AE Enschede, The Netherlands

Received 30 April 1998; received in revised form 28 September 1998

Abstract

A class of local nonlinear stationary subdivision schemes that interpolate equidistant data and that preserve monotonicity in the data is examined. The limit function obtained after repeated application of these schemes exists and is monotone for arbitrary monotone initial data. Next a class of rational subdivision schemes is investigated. These schemes generate limit functions that are continuously differentiable for any strictly monotone data. The approximation order of the schemes is four. Some generalisations, such as preservation of piecewise monotonicity and application to homogeneous grid refinement, are briefly discussed. © 1999 Elsevier Science B.V. All rights reserved.

AMS classification: 41A05; 41A29; 65D05; 65D17

Keywords: Subdivision; Interpolation; Monotonicity preservation; Shape preservation; Computer aided geometric design

1. Introduction

In this article, we examine four-point interpolatory monotonicity preserving subdivision schemes. These schemes are used for interpolation of univariate data that are uniform and monotone increasing (or decreasing).

Interpolatory subdivision schemes are based on iterative refinement of a data set. The usual subdivision schemes roughly double the number of data points every iteration. Overviews on subdivision schemes can be found in, e.g., [3, 6, 7]. Many subdivision schemes however fail to preserve monotonicity in the data. Linear monotonicity preserving subdivision schemes are discussed in [19], but the schemes discussed there are not interpolatory. Monotonicity preservation of the interpolatory linear four-point scheme [5] is discussed in [2]. The author determines ranges on the tension parameter such that the scheme is monotonicity preserving. Since the tension parameter depends on the initial data, the resulting subdivision scheme is stationary but data dependent.

* Corresponding author. E-mail: f.kuijt@math.utwente.nl.

[1] Financially supported by the Dutch Technology Foundation STW.

PII: S0377-0427(98)00220-9

We restrict ourselves to subdivision schemes that guarantee the preservation of monotonicity in the data. It is stressed that there is an analogy with [16] (and [15]) in which convexity preserving interpolatory subdivision schemes are examined. The same constructive approach is used, and so there is a similarity in some of the proofs. The remark in [16] that convexity preserving interpolatory subdivision schemes generating C^1 limit functions must necessarily be nonlinear also holds for monotonicity preserving schemes. The schemes examined in this article are stationary and they do not contain any data-dependent tension parameter.

The overview of this article is as follows. Section 2 states the problem and the class of schemes under investigation. The condition for preservation of monotonicity is derived in Section 3, and in Section 4 convergence to a limit function is examined. The analysis for convexity preservation in [16] led to a scheme that is unique in some sense, if convergence to a C^1 function is required. Requiring convergence to a C^1-smooth *monotone* function leads to a larger class of subdivision schemes however. Sufficient conditions for convergence to a C^1 limit function are given in Section 5, and since these conditions are too complex for constructing explicit subdivision schemes, we restrict ourselves to a specific class of schemes, namely rational subdivision schemes, see Section 6. Section 7 shows that these schemes have the property that ratios of adjacent first order differences tend to 1 as k tends to infinity. For any initial monotone data these schemes converge to continuously differentiable limit functions (see Section 8) and have approximation order four (Section 9). Some generalisations are briefly discussed in Section 10: e.g., piecewise monotonicity and application to homogeneous grid refinement which is a useful property for subdivision schemes for functional nonuniform data.

2. Problem definition

First, we state the problem that is examined in this article.

Definition 1 (*Problem definition*). Given is a finite bounded data set $\{(t_i^{(0)}, x_i^{(0)}) \in \mathbb{R}^2\}_{i=0}^N$, where the data are uniform, i.e., $t_i^{(0)} = ih$, where $h > 0$ is the mesh size. The data are assumed to be monotone, i.e., $x_i^{(0)} \leqslant x_{i+1}^{(0)}$, $\forall i$, or $x_i^{(0)} \geqslant x_{i+1}^{(0)}$, $\forall i$. Subdivision in t is defined as $t_i^{(k)} = 2^{-k} ih$, $i = 0, \ldots, 2^k N$. The aim is to characterise a class of subdivision schemes that are interpolatory and monotonicity preserving if the data are monotone. The second goal is to restrict this class of subdivision schemes to schemes that generate continuously differentiable limit functions and are fourth order accurate.

First we make a remark how to treat the boundaries, see [16]. Every initial monotone data set $\{(t_i^{(0)}, x_i^{(0)})\}_{i=0}^N$ can be extended in an arbitrary but monotonicity preserving way to $\{(t_i^{(0)}, x_i^{(0)})\}_{i=-2}^{N+2}$, such that the limit function is defined in the whole interval $I = [t_0^{(0)}, t_N^{(0)}]$. This means that all relevant properties on the kth iterate are easily shown to hold for the index set $\{0, \ldots, 2^k N\}$. Thus, all relevant properties are consistently proved on the original domain $I = [t_0^{(0)}, t_N^{(0)}] = [t_0^{(k)}, t_{2^k N}^{(k)}]$.

A constructive approach is used to derive monotonicity preserving subdivision schemes. Without loss of generality, we consider monotone increasing data, briefly denoted as *monotone* data.

We restrict ourselves to the following class of schemes:

$$
\begin{aligned}
x_{2i}^{(k+1)} &= x_i^{(k)}, \\
x_{2i+1}^{(k+1)} &= \tfrac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)}) + G_1(x_{i-1}^{(k)}, x_i^{(k)}, x_{i+1}^{(k)}, x_{i+2}^{(k)}),
\end{aligned}
\tag{1}
$$

for some function G_1. This implies that
1. the subdivision schemes are *interpolatory*,
2. the subdivision schemes are *local, using four points*.

First order differences $s_i^{(k)}$ are defined by

$$s_i^{(k)} := x_{i+1}^{(k)} - x_i^{(k)}. \tag{2}$$

Subdivision scheme (1) can then be rewritten in the following form:

$$\begin{aligned} x_{2i}^{(k+1)} &= x_i^{(k)}, \\ x_{2i+1}^{(k+1)} &= \tfrac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)}) + G_2(\tfrac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)}), s_{i-1}^{(k)}, s_i^{(k)}, s_{i+1}^{(k)}), \end{aligned} \tag{3}$$

where G_2 is another function representing the same class of subdivision schemes.

The third condition on the subdivision schemes deals with invariance under addition of constants:

3. The subdivision scheme is *invariant under addition of constant functions*, i.e., if the data $(t_i^{(0)}, x_i^{(0)})$ generate subdivision points $(t_i^{(k)}, x_i^{(k)})$, then the data $(t_i^{(0)}, x_i^{(0)} + \mu)$, with $\mu \in \mathbb{R}$ yield subdivision points $(t_i^{(k)}, x_i^{(k)} + \mu)$.

Imposing this condition yields

$$x_{2i+1}^{(k+1)} + \mu = \tfrac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)}) + \mu + G_2(\tfrac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)}) + \mu, s_{i-1}^{(k)}, s_i^{(k)}, s_{i+1}^{(k)})$$

It follows that G_2 cannot depend on its first argument. Condition 3 therefore yields that subdivision scheme (3) must be of the following form:

$$\begin{aligned} x_{2i}^{(k+1)} &= x_i^{(k)}, \\ x_{2i+1}^{(k+1)} &= \tfrac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)}) + G_3(s_{i-1}^{(k)}, s_i^{(k)}, s_{i+1}^{(k)}). \end{aligned} \tag{4}$$

Next we add a natural requirement on the subdivision schemes:

4. The subdivision scheme is *homogeneous*, i.e., if initial data $(t_i^{(0)}, x_i^{(0)})$ give subdivision points $(t_i^{(k)}, x_i^{(k)})$, then initial data $(t_i^{(0)}, \lambda x_i^{(0)})$ yield points $(t_i^{(k)}, \lambda x_i^{(k)})$.

A direct consequence of homogeneity of the subdivision schemes is that the function G_3 is homogeneous:

$$G_3(\lambda a, \lambda b, \lambda c) = \lambda G_3(a, b, c), \quad \forall \lambda. \tag{5}$$

Subdivision scheme (4) then necessarily reproduces constant functions, i.e., if $x_i^{(0)} = \mu_0$, then $x_i^{(k)} = \mu_0$ (take $\lambda = 0$ in (5)).

Further simplification of the representation of subdivision scheme (4) is obtained by using the homogeneity of G_3 in (5) as follows:

$$G_3(s_{i-1}^{(k)}, s_i^{(k)}, s_{i+1}^{(k)}) = s_i^{(k)} G_3(r_i^{(k)}, 1, R_{i+1}^{(k)}) = \tfrac{1}{2} s_i^{(k)} G(r_i^{(k)}, R_{i+1}^{(k)}),$$

where ratios of adjacent first differences are defined by

$$r_i^{(k)} := s_{i-1}^{(k)}/s_i^{(k)} \quad \text{and} \quad R_i^{(k)} = 1/r_i^{(k)}, \tag{6}$$

and the function G is defined by $G(r,R) := 2G_3(r,1,R)$.

Since it follows in the next sections that G is a bounded function, this reformulation of G does not cause problems in case $s_i^{(k)} = 0$.

The class of subdivision schemes (1) is rewritten in the form:

$$\begin{aligned} x_{2i}^{(k+1)} &= x_i^{(k)}, \\ x_{2i+1}^{(k+1)} &= \tfrac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)}) + \tfrac{1}{2}s_i^{(k)}G(r_i^{(k)}, R_{i+1}^{(k)}), \end{aligned} \tag{7}$$

and it is this subdivision scheme that is examined in this article.

Remark 2. Note that the class of subdivision schemes (7) automatically satisfies conditions 1–4.

The next general assumption on the subdivision scheme concerns with invariance under affine transformations of the variable t:

5. The subdivision scheme is *invariant under affine transformations of the variable t*, i.e., if the initial data $(t_i^{(0)}, x_i^{(0)})$ yield subdivision points $(t_i^{(k)}, x_i^{(k)})$, then the data $(\lambda t_i^{(0)} + \mu_0, x_i^{(0)})$, with $\mu_0 \in \mathbb{R}$ yield subdivision points $(\lambda t_i^{(k)} + \mu_0, x_i^{(k)})$ for $\lambda > 0$, and $(\lambda t_i^{(k)} + \mu_0, -x_i^{(k)})$ for $\lambda < 0$.

By taking $\lambda = -1$ and $\mu_0 = 0$ in Condition 5, it follows that G is anti-symmetric under interchanging its arguments, i.e., it is obtained that

$$G(r,R) = -G(R,r), \quad \forall r, R, \tag{8}$$

which directly implies $G(r,r) = 0$, $\forall r$. Under this condition on G, subdivision scheme (7) necessarily reproduces linear functions, i.e., if $x_i^{(0)} = \lambda_1 t_i^{(0)} + \mu_1$, then $x_i^{(k)} = \lambda_1 t_i^{(k)} + \mu_1$ (as $s_i^{(k)} = \lambda_1 h$, $r_i^{(k)} = 1$, $\forall i$).

Remark 3. Invariance under addition of linear functions is not a natural condition in case of monotonicity preservation, in contrast with convexity preservation (see [16]).

In the following sections, we discuss conditions for monotonicity preservation and smoothness properties of subdivision scheme (7) satisfying (8), which requires additional conditions on the function G.

3. Monotonicity preservation

In this section, we examine monotonicity preservation of the class of four-point interpolatory subdivision schemes (7) satisfying (8).

Theorem 4 (Monotonicity preservation). *Subdivision scheme* (7) *satisfying condition* (8) *preserves monotonicity if and only if the subdivision function G satisfies*

$$|G(r,R)| \leqslant 1, \quad \forall r,R \geqslant 0. \tag{9}$$

Proof. Monotonicity preservation is achieved if and only if the scheme generates differences that satisfy

$$s_i^{(k)} \geqslant 0, \quad \forall i, \ \forall k.$$

Therefore assume that for some k, the data $x_i^{(k)}$ satisfy $s_i^{(k)} \geqslant 0$, $\forall i$. Necessary and sufficient for monotonicity preservation is that the differences in the data at level $(k+1)$ are also nonnegative, i.e., $s_i^{(k+1)} \geqslant 0$, $\forall i$.

Two differences, $s_{2i}^{(k+1)}$ and $s_{2i+1}^{(k+1)}$, have to be analysed:

$$\begin{aligned} s_{2i}^{(k+1)} &= x_{2i+1}^{(k+1)} - x_{2i}^{(k+1)} = \tfrac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)}) + \tfrac{1}{2}s_i^{(k)}G(r_i^{(k)},R_{i+1}^{(k)}) - x_i^{(k)} \\ &= \tfrac{1}{2}(x_{i+1}^{(k)} - x_i^{(k)}) + \tfrac{1}{2}s_i^{(k)}G(r_i^{(k)},R_{i+1}^{(k)}) \\ &= \tfrac{1}{2}s_i^{(k)}(1 + G(r_i^{(k)},R_{i+1}^{(k)})), \end{aligned} \tag{10}$$

$$\begin{aligned} s_{2i+1}^{(k+1)} &= x_{2i+2}^{(k+1)} - x_{2i+1}^{(k+1)} = x_{i+1}^{(k)} - \tfrac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)}) - \tfrac{1}{2}s_i^{(k)}G(r_i^{(k)},R_{i+1}^{(k)}) \\ &= \tfrac{1}{2}(x_{i+1}^{(k)} - x_i^{(k)}) - \tfrac{1}{2}s_i^{(k)}G(r_i^{(k)},R_{i+1}^{(k)}) \\ &= \tfrac{1}{2}s_i^{(k)}(1 - G(r_i^{(k)},R_{i+1}^{(k)})). \end{aligned} \tag{11}$$

Since $s_i^{(k)} \geqslant 0$, it is necessary for monotonicity preservation that

$$1 + G(r_i^{(k)},R_{i+1}^{(k)}) \geqslant 0 \text{ and } 1 - G(r_i^{(k)},R_{i+1}^{(k)}) \geqslant 0,$$

which yields that condition (9) is sufficient for monotonicity preservation of subdivision scheme (7).

Since we consider arbitrary monotone data, this condition is also necessary. □

Remark 5 (*Preservation of strict monotonicity*). A sufficient condition for preservation of strict monotonicity is:

$$\exists \mu < 1 \text{ such that } |G(r,R)| \leqslant \mu, \quad \forall r,R > 0.$$

Remark 6 (*Nonlinearity*). Observe that the function G must be necessarily *nonlinear* for monotonicity preservation and C^1-smoothness of subdivision scheme (7): the only scheme that is polynomial in its arguments and that satisfies (9) is given by $G \equiv 0$. The resulting two-point subdivision scheme however, generates the piecewise linear interpolant as limit function, which is obviously only C^0.

Remark 7 (*The linear four-point scheme*). The well-known linear four-point scheme of Dyn, Gregory and Levin [5] with $w = 1/16$ is given by the function

$$\bar{G}(r,R) = \tfrac{1}{8}(r - R). \tag{12}$$

Since this function $\bar{G}$ is linear, it obviously cannot satisfy the monotonicity condition (9).

4. Convergence to a continuous function

In this section convergence of subdivision scheme (7) to continuous limit functions is investigated. The proof follows the lines of the proof of existence and continuity of the limit curve generated by the linear four-point scheme in [5].

Theorem 8 (C^0-convergence). *Given is a monotone data set $\{(t_i^{(0)}, x_i^{(0)}) \in \mathbb{R}^2\}_i$, where $t_i^{(0)} = ih$. The kth stage data $\{(t_i^{(k)}, x_i^{(k)})\}_i$ are defined at values $t_i^{(k)} = 2^{-k}ih$.*

Repeated application of subdivision scheme (7) satisfying

$$\exists \mu < 1 \text{ such that } |G(r,R)| \leqslant \mu, \quad \forall r, R \geqslant 0. \tag{13}$$

leads to a continuous function which is monotone and interpolates the initial data points $(t_i^{(0)}, x_i^{(0)})$.

Proof. The interpolatory property of the subdivision scheme is a direct consequence of the definition $x_{2i}^{(k+1)} = x_i^{(k)}$. Preservation of monotonicity of the subdivision scheme was shown in the previous section. Therefore, the continuous function $x^{(k)}$, defined as the linear interpolant to the data $\{(t_i^{(k)}, x_i^{(k)})\}_i$, is monotone.

Remains to prove that the sequence of functions $x^{(k)}$ converges, i.e., the limit function

$$x^{(\infty)} := \lim_{k \to \infty} x^{(k)}$$

exists and is continuous. Sufficient for convergence is that $x^{(k)}$ is a Cauchy sequence in k with limit 0, i.e., it suffices to show that

$$\|x^{(k+1)} - x^{(k)}\|_\infty \leqslant C_0 \lambda_0^k \quad \text{where } C_0 < \infty \text{ and } \lambda_0 < 1.$$

The distance $\|x^{(k+1)} - x^{(k)}\|_\infty$ is calculated by

$$\|x^{(k+1)} - x^{(k)}\|_\infty = \max_i \max\left\{\left|x_{2i}^{(k+1)} - x_i^{(k)}\right|, \left|x_{2i+1}^{(k+1)} - \tfrac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)})\right|\right\}.$$

The maximal distance between the functions $x^{(k)}$ and $x^{(k+1)}$ occurs at a point $t_{2i+1}^{(k+1)} = 2^{-k}(i + \frac{1}{2})h$ for some i, which thus gives, using condition (9)

$$\begin{aligned}
\|x^{(k+1)} - x^{(k)}\|_\infty &= \max_i |x_{2i+1}^{(k+1)} - \tfrac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)})| \\
&= \tfrac{1}{2} \max_i |s_i^{(k)} G(r_i^{(k)}, R_{i+1}^{(k)})| \\
&\leqslant \tfrac{1}{2} \max_i |G(r_i^{(k)}, R_{i+1}^{(k)})| \max_i |s_i^{(k)}| \leqslant \tfrac{1}{2}\mu \max_i s_i^{(k)}.
\end{aligned} \tag{14}$$

We now prove that $\max_i s_i^{(k)}$ converges to 0. Using (10), (11) and monotonicity condition (9), it is obtained that

$$\begin{aligned}
\max_i s_i^{(k+1)} &= \max_i \max\{s_{2i}^{(k+1)}, s_{2i+1}^{(k+1)}\} = \tfrac{1}{2} \max_i \{s_i^{(k)}(1 \pm G(r_i^{(k)}, R_{i+1}^{(k)}))\} \\
&\leqslant \tfrac{1}{2}(1+\mu) \max_i s_i^{(k)}.
\end{aligned}$$

Combining this result with (14) yields

$$\|x^{(k+1)} - x^{(k)}\|_\infty \leqslant \tfrac{1}{2}\mu \max_i s_i^{(0)} \left(\frac{1+\mu}{2}\right)^k .$$

As $\mu < 1$, this proves convergence of $x^{(k)}$, and since all functions $x^{(k)}$ are continuous by construction, the limit function $x^{(\infty)}$ is continuous. □

5. Convergence to a continuously differentiable function

In the previous sections, we derived sufficient conditions on the function G such that the subdivision scheme preserves monotonicity and that a continuous limit function exists. An additional sufficient condition on the scheme such that it generates *continuously differentiable* functions is presented in this section.

Theorem 9 (C^1-convergence). *Given is a strictly monotone data set* $\{(t_i^{(0)}, x_i^{(0)}) \in \mathbb{R}^2\}_i$, *where* $t_i^{(0)} = ih$, *with* $h > 0$. *The kth stage data* $\{(t_i^{(k)}, x_i^{(k)})\}_i$ *are defined at values* $t_i^{(k)} = 2^{-k} ih$.

Let the function G satisfy (13) *and the Lipschitz condition*

$$\exists \alpha > 0,\ \forall x, y:\quad |G(x+\varepsilon_1, y+\varepsilon_2) - G(x,y)| \leqslant B_1 \|\varepsilon\|^\alpha, \quad B_1 < \infty. \tag{15}$$

Moreover, subdivision scheme (7) *has the property that the ratios of adjacent first order differences, defined in* (6), *obey*

$$\exists \rho < 1:\quad \max_i \left| \max\left\{ r_i^{(k)}, \frac{1}{r_i^{(k)}} \right\} - 1 \right| \leqslant B_2 \rho^k, \quad B_2 < \infty. \tag{16}$$

Repeated application of such a subdivision scheme generates a continuously differentiable function which is monotone and interpolates the initial data points $(t_i^{(0)}, x_i^{(0)})$.

Proof. Starting from a strictly monotone data set $\{(t_i^{(k)}, x_i^{(k)})\}_i$, first order divided differences $y_i^{(k)}$ in the data are defined by

$$y_i^{(k)} := \frac{x_{i+1}^{(k)} - x_i^{(k)}}{t_{i+1}^{(k)} - t_i^{(k)}} = \frac{2^k}{h}(x_{i+1}^{(k)} - x_i^{(k)}) = \frac{2^k}{h} s_i^{(k)}. \tag{17}$$

The function $y^{(k)}$ is defined as the linear interpolant of the data points $(t_{2i+1}^{(k+1)}, y_i^{(k)})$. All functions $y^{(k)}$ are therefore continuous by construction.

It has to be proved that the functions $y^{(k)}$ converge to a function $y^{(\infty)}$, and secondly this $y^{(\infty)}$ must be the derivative of $x^{(\infty)}$, defined in the previous section.

Sufficient for convergence of the sequence of functions $y^{(k)}$ is that they form a Cauchy sequence in k with limit 0, i.e., there must exist a $\lambda_1 < 1$ and $C_1 \in \mathbb{R}$ such that

$$\|y^{(k+1)} - y^{(k)}\|_\infty \leqslant C_1 \lambda_1{}^k. \tag{18}$$

By construction, the maximal distance between the functions $y^{(k+1)}$ and $y^{(k)}$ occurs at a point $t_{4i+j}^{(k+2)} = 2^{-k}(i + j/4)h$, for some i and j, and these distances $d_{4i+j}^{(k+1)}$ satisfy

$$d_{4i+j}^{(k+1)} = |y^{(k+1)}(t_{4i+j}^{(k+2)}) - y^{(k)}(t_{4i+j}^{(k+2)})|. \tag{19}$$

Subsequent application of (19), (17), and the subdivision relations (10) and (11), for example yields for the distance $d_{4i+1}^{(k+1)}$:

$$\begin{aligned} d_{4i+1}^{(k+1)} &= \left| y_{2i}^{(k+1)} - \left(\frac{1}{4} y_{i-1}^{(k)} + \frac{3}{4} y_i^{(k)} \right) \right| = \frac{2^k}{h} \left| 2 s_{2i}^{(k+1)} - \frac{1}{4} s_{i-1}^{(k)} - \frac{3}{4} s_i^{(k)} \right| \\ &= \frac{2^k}{h} \left| \frac{1}{4}(s_i^{(k)} - s_{i-1}^{(k)}) + s_i^{(k)} G(r_i^{(k)}, R_{i+1}^{(k)}) \right| \\ &= \frac{2^k}{h} s_i^{(k)} \left| \frac{1}{4}(1 - r_i^{(k)}) + G(r_i^{(k)}, R_{i+1}^{(k)}) \right|, \end{aligned}$$

and the other distances are determined similarly:

$$\begin{aligned} d_{4i}^{(k+1)} &= \frac{2^k}{h} \left| \frac{1}{2} s_i^{(k)} G(r_i^{(k)}, R_{i+1}^{(k)}) - \frac{1}{2} s_{i-1}^{(k)} G(r_{i-1}^{(k)}, R_i^{(k)}) \right|, \\ d_{4i+1}^{(k+1)} &= \frac{2^k}{h} s_i^{(k)} \left| \frac{1}{4}(1 - r_i^{(k)}) + G(r_i^{(k)}, R_{i+1}^{(k)}) \right|, \\ d_{4i+2}^{(k+1)} &= 0, \\ d_{4i+3}^{(k+1)} &= \frac{2^k}{h} s_i^{(k)} \left| \frac{1}{4}(1 - R_{i+1}^{(k)}) - G(r_i^{(k)}, R_{i+1}^{(k)}) \right|. \end{aligned}$$

The distance $d_{4i}^{(k+1)}$ is easily estimated with

$$d_{4i}^{(k+1)} \leqslant \frac{2^k}{h} \max_i s_i^{(k)} \max_i |G(r_i^{(k)}, R_{i+1}^{(k)})|,$$

and thus, it is obtained that

$$\begin{aligned} \max_j d_j^{(k+1)} \leqslant \frac{2^k}{h} \max_i s_i^{(k)} \max_i \max \Bigg\{ & |G(r_i^{(k)}, R_{i+1}^{(k)})|, \left| \frac{1}{4}(1 - r_i^{(k)}) + G(r_i^{(k)}, R_{i+1}^{(k)}) \right|, \\ & \left| \frac{1}{4}(1 - R_{i+1}^{(k)}) - G(r_i^{(k)}, R_{i+1}^{(k)}) \right| \Bigg\}. \end{aligned} \tag{20}$$

Subsequently, we apply assumptions (15) and (16), and also use the fact that $G(1,1) = 0$, see (8). The function G can now be estimated as follows:

$$\begin{aligned} |G(r_i^{(k)}, R_{i+1}^{(k)})| &= |G(r_i^{(k)}, R_{i+1}^{(k)}) - G(1,1)| \leqslant B_1 \|(r_i^{(k)} - 1, R_{i+1}^{(k)} - 1)\|^{\alpha} \\ &\leqslant B_3 \rho^{\alpha k}, \end{aligned}$$

where $B_3 < \infty$. According to (10) and (11), the first part of (20) can be estimated as

$$\max_i s_i^{(k+1)} \leqslant \tfrac{1}{2} \left(1 + \max_i |G(r_i^{(k)}, R_{i+1}^{(k)})| \right) \max_i s_i^{(k)} \leqslant \tfrac{1}{2}(1 + B_3 \rho^{\alpha k}) \max_i s_i^{(k)},$$

which yields

$$\max_i s_i^{(k)} \leqslant \left(\frac{1}{2}\right)^k \max_i s_i^{(0)} \prod_{\ell=0}^{k-1}(1 + B_3\rho^{\alpha\ell}).$$

Since $1 + x \leqslant \mathrm{e}^x$, we obtain

$$\begin{aligned}\prod_{\ell=0}^{k-1}(1 + B_3\rho^{\alpha\ell}) \leqslant \prod_{\ell=0}^{k-1} \exp(B_3\rho^{\alpha\ell}) &= \exp\left(B_3 \sum_{\ell=0}^{k-1} \rho^{\alpha\ell}\right) \\ &= \exp\left(B_3 \frac{1-\rho^{\alpha k}}{1-\rho^{\alpha}}\right) \leqslant \exp\left(\frac{B_3}{1-\rho^{\alpha}}\right) =: B_4 < \infty,\end{aligned}$$

and hence

$$\max_i s_i^{(k)} \leqslant B_4 \left(\frac{1}{2}\right)^k \max_i s_i^{(0)}.$$

The second part of (20) is estimated as

$$\max_i \max\{|G(r_i^{(k)}, R_{i+1}^{(k)})|, |\tfrac{1}{4}(1 - r_i^{(k)}) + G(r_i^{(k)}, R_{i+1}^{(k)})|, |\tfrac{1}{4}(1 - R_{i+1}^{(k)}) - G(r_i^{(k)}, R_{i+1}^{(k)})|\} \leqslant B_5 \hat{\rho}^k,$$

where

$$\hat{\rho} := \max\{\rho, \rho^{\alpha}\} < 1 \quad \text{and} \quad B_5 := \max\{2B_1, \tfrac{1}{2}B_2, B_3\}.$$

We complete the proof of (18) with

$$\|y^{(k+1)} - y^{(k)}\|_{\infty} = \frac{2^k}{h}\left(\frac{1}{2}\right)^k B_4 B_5 \max_i s_i^{(0)} \hat{\rho}^k = B_6 \hat{\rho}^k$$

with

$$B_6 := \frac{B_4 B_5}{h} \max_i s_i^{(0)} < \infty,$$

as $\hat{\rho} < 1$. The remaining part of the proof is to show that also $y^{(\infty)} = x^{(\infty)\prime}$, i.e., $y^{(k)}$ converges to the derivative of the limit function $x^{(\infty)}$. This can be done by the standard approach using the uniform convergence of Bernstein polynomials: the derivative of the Bernstein polynomial determined by the data $\{x_i^{(k)}\}_i$ on the interval $[x_0^{(0)}, x_N^{(0)}]$ is the Bernstein polynomial of the data $\{y_i^{(k)}\}_i$ on the same interval. This is described in [5]. The limit derivative $y^{(\infty)}$ is C^0, so the limit function $x^{(\infty)}$ is C^1, which completes the proof. □

Remark 10 (*The linear four-point scheme*). Note that minimisation of the second part of (20) yields the linear four-point scheme [5] given in (12), i.e., the linear four-point scheme is "as smooth as possible".

6. Construction of rational subdivision schemes

In this section we restrict the class of subdivision schemes to schemes that generate continuously differentiable limit functions.

Since G cannot be polynomial, see Remark 6, a relatively simple restriction is achieved by choosing the function G of a specific nonlinear form: a rational function. Observe that the convexity preserving subdivision scheme in [16] is also rational.

Theorem 11. *Let the data $x_i^{(0)}$ be drawn from a strictly monotone and four times continuously differentiable function g, as follows*:

$$x_i^{(0)} = g(ih).$$

Assume that the function G is rational and bilinear in the numerator and the denominator, and additionally require that G satisfies monotonicity condition (9).

Then, subdivision scheme (7) *can only have approximation order four if G is of the form*

$$G(r,R) = \frac{r-R}{\ell_1 + (1+\ell_2)(r+R) + \ell_3 rR} \quad (\ell_1,\ell_2,\ell_3) \in \Omega, \tag{21}$$

where the triangular domain Ω is defined as

$$\Omega = \{(\ell_1,\ell_2,\ell_3), \text{ such that } \ell_1,\ell_2,\ell_3 \geqslant 0 \text{ and } \ell_1 + 2\ell_2 + \ell_3 = 6\}. \tag{22}$$

Proof. The class of rational functions G where the numerator and the denominator are bilinear functions in r and R is denoted by

$$G(r,R) = \frac{b_1 + b_2 r + b_3 R + b_4 rR}{b_5 + b_6 r + b_7 R + b_8 rR} \quad \text{where } b_j \in \mathbb{R}. \tag{23}$$

First, we impose conditions on the parameters b_j that are necessary for fourth order accuracy of the subdivision scheme. Necessary conditions are achieved by application to initial data that by definition satisfy $x_i^{(0)} = \bar{x}_i^{(0)}$ and therefore $r_i^{(0)} = \bar{r}_i^{(0)}$, using the rational function G as in (23) compared with the linear function $\bar{G}$ as in (12). It is easily checked that the following condition is necessary for fourth-order accuracy (see also Section 9):

$$|x_{2i+1}^{(1)} - \bar{x}_{2i+1}^{(1)}| = \tfrac{1}{2} s_i^{(0)} |G(r_i^{(0)}, R_{i+1}^{(0)}) - \bar{G}(r_i^{(0)}, R_{i+1}^{(0)})| = \mathrm{O}(h^4),$$

where

$$r_i^{(0)} = \frac{g(ih) - g((i-1)h)}{g((i+1)h) - g(ih)} \quad \text{and} \quad R_{i+1}^{(0)} = \frac{g((i+2)h) - g((i+1)h)}{g((i+1)h) - g(ih)}.$$

Since $s_i^{(0)} = \mathrm{O}(h)$, it must hold that

$$G(r_i^{(0)}, R_{i+1}^{(0)}) - \bar{G}(r_i^{(0)}, R_{i+1}^{(0)}) = \mathrm{O}(h^3).$$

Combining these constraints on the parameters b_j with condition (8), one easily obtains that the function G can be written in the form

$$G(r,R) = \frac{r-R}{\ell_1 + \ell_0(r+R) + rR(8 - 2\ell_0 - \ell_1)}. \tag{24}$$

Additional necessary conditions on the values of ℓ_0 and ℓ_1 in (24) are determined by condition (9) and the requirement that G may not contain poles for positive values of r and R. A simple calculation shows that necessarily

$$\ell_0 \geqslant 1, \quad \ell_1 \geqslant 0, \quad 8 - 2\ell_0 - \ell_1 \geqslant 0.$$

Defining $\ell_2 = \ell_0 - 1$ and $\ell_3 = 6 - \ell_1 - 2\ell_2$ yields that the function G can be written as (21). □

Remark 12. A simple calculation shows that the subdivision scheme with G in (21) reproduces quadratic polynomials if $\ell_3 = 0$, i.e., $\ell_1 + 2\ell_2 = 6, \ell_1, \ell_2 \geqslant 0$.

Remark 13. It is easily checked that G in (21) satisfies condition 5 from Section 2.

In addition, note that G in (21) automatically satisfies the following natural property:

- $G(r, R^*)$ is strict monotone increasing in r, at fixed $R^* \geqslant 0$.
- $G(r^*, R)$ is strict monotone decreasing in R, at fixed $r^* \geqslant 0$.

Applying (8) yields that such a function G also satisfies the natural condition

$$G(r, R) > 0, \quad \forall r > R > 0.$$

Remark 14. In the special case $\ell_1 = 2$, $\ell_2 = 1$ and $\ell_3 = 2$, the function G in (21) reduces to

$$G_C(r, R) = \frac{1}{2}\left(\frac{1}{1+R} - \frac{1}{1+r}\right). \tag{25}$$

In this case, the subdivision function (21) can be factorised as a difference of two univariate functions in r and in R respectively. This factorisation is only possible for this specific choice of the parameters ℓ_1, ℓ_2 and ℓ_3.

In this section, we showed that the class of subdivision schemes (7) with (21) satisfies necessary conditions for approximation order four. It is proved in Section 9 that this class of rational monotonicity preserving interpolatory subdivision schemes has indeed approximation order four.

To be able to prove the smoothness properties and approximation order four, we use some additional properties on subdivision scheme (7) with (21). These properties are discussed in the next section.

7. Ratios of first-order differences

In this section we investigate the behaviour of ratios of first-order differences obtained after application of subdivision scheme (7) with G as in (21).

7.1. Boundedness of difference ratios

In this section we prove that the ratios of adjacent differences in iteration step $k+1$ are bounded by the maximum of the ratios of differences in iteration k.

Theorem 15. *Define numbers* $q_i^{(k)}$ *and* $q^{(k)}$ *by*

$$q_i^{(k)} := \max\left\{ r_i^{(k)}, \frac{1}{r_i^{(k)}} \right\} \quad \textit{and} \quad q^{(k)} := \max_i q_i^{(k)}. \tag{26}$$

Then, application of subdivision scheme (7) *with* (21) *yields*

$$q^{(k+1)} \leqslant q^{(k)}. \tag{27}$$

Proof. Let the data $x_i^{(k)}$ be given and its ratios of differences $r_i^{(k)}$ as defined in (6). Since $r_{2i+1}^{(k+1)}$ can be written as a function of $r_i^{(k)}$ and $r_{i+1}^{(k)}$, and $r_{2i}^{(k+1)}$ as a function of $r_{i-1}^{(k)}$, $r_i^{(k)}$ and $r_{i+1}^{(k)}$, we prove here that ratios of first-order differences at level $k+1$ are bounded as follows:

$$\max\left\{ r_{2i+1}^{(k+1)}, \frac{1}{r_{2i+1}^{(k+1)}} \right\} \leqslant \max\left\{ r_i^{(k)}, \frac{1}{r_i^{(k)}}, r_{i+1}^{(k)}, \frac{1}{r_{i+1}^{(k)}} \right\}$$

and

$$\max\left\{ r_{2i}^{(k+1)}, \frac{1}{r_{2i}^{(k+1)}} \right\} \leqslant \max\left\{ r_{i-1}^{(k)}, \frac{1}{r_{i-1}^{(k)}}, r_i^{(k)}, \frac{1}{r_i^{(k)}}, r_{i+1}^{(k)}, \frac{1}{r_{i+1}^{(k)}} \right\}.$$

We illustrate the proof by the treatment of $r_{2i+1}^{(k+1)}$. Since the properties that must be proved contain maximum functions, the proof has to enumerate several situations depending on the size of the $r_j^{(k)}$. We, therefore, order the ratios $r_j^{(k)}$ in size. The proof is based on treating all partitions separately.

Consider the case that $1/r_{i+1}^{(k)}$ is maximal, i.e., one of the following two partitions is valid:

$$r_{i+1}^{(k)} \leqslant r_i^{(k)} \leqslant 1 \leqslant \frac{1}{r_i^k} \leqslant \frac{1}{r_{i+1}^{(k)}} \tag{28}$$

or

$$r_{i+1}^{(k)} \leqslant \frac{1}{r_i^{(k)}} \leqslant 1 \leqslant r_i^{(k)} \leqslant \frac{1}{r_{i+1}^{(k)}}. \tag{29}$$

Then it must be proved that

$$\frac{1}{r_{i+1}^{(k)}} - r_{2i+1}^{(k+1)} \geqslant 0 \quad \text{and} \quad \frac{1}{r_{i+1}^{(k)}} - \frac{1}{r_{2i+1}^{(k+1)}} \geqslant 0. \tag{30}$$

As an example we give the construction for the second partition (29). A convenient transformation of variables,

$$r_{i+1}^{(k)} = \frac{1}{1+x} \quad \text{and} \quad r_i^{(k)} = 1 + x\frac{1}{1+y} \qquad \text{where } x, y \geqslant 0,$$

is substituted in (30). Inequalities (30) then result in rational expressions that must hold for all $x, y \geqslant 0$.

By requiring that both the numerator and the denominator of such an expression is positive (or negative), it is sufficient for positiveness of the rational expression that the coefficients in both the numerator and the denominator have the same sign.

The construction for the even ratios $r_{2i}^{(k+1)}$ requires a more sophisticated substitution for three variables ($r_{i-1}^{(k)}$, $r_i^{(k)}$ and $r_{i+1}^{(k)}$) in each partition. As an example for the partition

$$r_{i+1}^{(k)} \leqslant \frac{1}{r_{i-1}^{(k)}} \leqslant r_i^{(k)} \leqslant 1 \leqslant \frac{1}{r_i^{(k)}} \leqslant r_{i-1}^{(k)} \leqslant \frac{1}{r_{i+1}^{(k)}},$$

the substitution

$$r_{i+1}^{(k)} = \frac{1}{1+x}, \quad r_{i-1}^{(k)} = 1 + x\frac{1}{1+y}, \quad r_i^{(k)} = \frac{1}{1+x\frac{1}{1+z}} \quad \text{where } x,y,z \geqslant 0,$$

has been used. Again it is required that the coefficient in the numerator and the denominator have the same sign.

The coefficients however depend on the parameters ℓ_1 and ℓ_2 (note that $\ell_3 = 6 - \ell_1 - 2\ell_2$). By enumeration over all different expressions to be proved and all different partitions, a large set of constraints $c_j(\ell_1, \ell_2) \geqslant 0$ is constructed in this way. Since we have to prove the validity of many constraints, the calculations are performed using algebraic manipulation software. We used Maple [4] to generate all equations and to solve the constraints. It is algebraically checked that all constraints lie outside the domain Ω defined in (22) or on its boundary. As an illustration, all constraints $c_j(\ell_1,\ell_2)=0$ are shown in Fig. 1. Since in addition $c_j(2,1)\geqslant 0$, $\forall j$, it is thus proved that $c_j(\ell_1,\ell_2) \geqslant 0$, $\forall(\ell_1,\ell_2)\in\Omega\backslash\partial\Omega$. The conclusion also holds for Ω including its boundaries, since all rational expressions are continuous.

It is thus shown in this proof that

$$q_{2i+1}^{(k+1)} \leqslant \max\{q_i^{(k)}, q_{i+1}^{(k)}\}, \quad \forall i,$$

$$q_{2i}^{(k+1)} \leqslant \max\{q_{i-1}^{(k)}, q_i^{(k)}, q_{i+1}^{(k)}\}, \quad \forall i$$

which completes the proof. □

This result is also maximal in a single step relation between two subsequent subdivision iterations:

Theorem 16. *Let the numbers $q^{(k)}$ be defined as in Theorem 15. Then, in general, there does not exist a $\rho<1$ such that*

$$q^{(k+1)} - 1 \leqslant \rho(q^{(k)} - 1).$$

Proof. Consider the data $x_i^{(0)}$ with differences $s_i^{(0)}$ satisfying

$$s_{-1}^{(0)} = a, \quad s_0^{(0)} = b, \quad s_1^{(0)} = a \quad \text{and} \quad s_2^{(0)} = b \quad \text{where } a > b > 0.$$

The maximum ratio $q^{(0)}$ is equal to

$$q^{(0)} = \frac{s_1^{(0)}}{s_0^{(0)}} = \frac{a}{b} > 1,$$

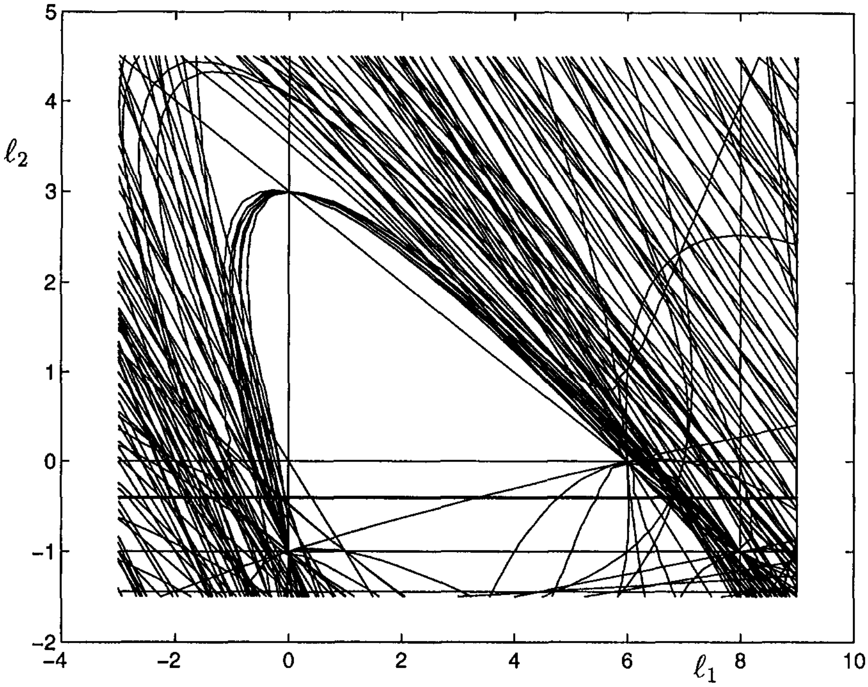

Fig. 1. The constraints $c_j(\ell_1, \ell_2) = 0$.

and because of the symmetry of the scheme, one subdivision yields

$$s_1^{(1)} = \frac{b}{2}, \quad s_2^{(1)} = \frac{a}{2} \Rightarrow q^{(1)} = \frac{s_2^{(1)}}{s_1^{(1)}} = \frac{a}{b} = q^{(0)}.$$

This counterexample shows that the maximum ratio in general does not become smaller in a single subdivision iteration. □

Next it is proved, using a double step strategy, that the ratios $r_i^{(k)}$ converge to 1.

7.2. Strict convergence of difference ratios

In the previous subsection, Theorem 16 indicates that we cannot establish convergence of the difference ratios to 1 using a single step strategy. In this section, we prove that ratios converge to 1 using a double step strategy.

Theorem 17. *Let the numbers $q^{(k)}$ be defined as in Theorem* 15. *Then*

$$q^{(k+2)} - 1 \leqslant \tfrac{3}{4}(q^{(k)} - 1). \tag{31}$$

Proof. It is proved that the $q_j^{(k+2)}$ satisfy

$$q_{4i-1}^{(k+2)} - 1 \leqslant \tfrac{5}{16}(\max\{q_{i-1}^{(k)}, q_i^{(k)}, q_{i+1}^{(k)}\} - 1),$$

$$q_{4i}^{(k+2)} - 1 \leqslant \tfrac{3}{4}(\max\{q_{i-1}^{(k)}, q_i^{(k)}, q_{i+1}^{(k)}\} - 1),$$

$$q_{4i+1}^{(k+2)} - 1 \leqslant \tfrac{5}{16}(\max\{q_{i-1}^{(k)}, q_i^{(k)}, q_{i+1}^{(k)}\} - 1),$$

$$q_{4i+2}^{(k+2)} - 1 \leqslant \tfrac{1}{4}(\max\{q_{i-1}^{(k)}, q_i^{(k)}, q_{i+1}^{(k)}, q_{i+2}^{(k)}\} - 1).$$

To illustrate the proof, we examine the first pair of inequalities:

$$\frac{5}{16} - \frac{r_{4i-1}^{(k+2)} - 1}{\max\{q_{i-1}^{(k)}, q_i^{(k)}, q_{i+1}^{(k)}\} - 1} \geqslant 0 \quad \text{and} \quad \frac{5}{16} + \frac{r_{4i-1}^{(k+2)} - 1}{\max\{q_{i-1}^{(k)}, q_i^{(k)}, q_{i+1}^{(k)}\} - 1} \geqslant 0.$$

Using the same approach as in the proof of Theorem 15, constraints on ℓ_1 and ℓ_2 have been generated. Again, it is algebraically checked that all constraints lie outside the domain Ω, see (22), or on its boundary. This proves that the theorem holds for all $(\ell_1, \ell_2) \in \Omega$. The result can be written as

$$\max_j \left\{ r_j^{(k+2)}, \frac{1}{r_j^{(k+2)}} \right\} - 1 \leqslant \frac{3}{4} \left(\max_j \left\{ r_j^{(k)}, \frac{1}{r_j^{(k)}} \right\} - 1 \right)$$

and this completes the proof. □

The factors 5/16, 3/4 and 1/4 have been conjectured by an asymptotic analysis on arbitrarily chosen data $x_i^{(0)}$ with $r_i^{(0)} = 1 + \delta_i \varepsilon$, where $0 < \varepsilon \ll 1$ and $\delta_i \in [-1, 1]$. The proofs however, are given for general data.

Remark 18. Numerical experiments show that the factor $\frac{3}{4}$ cannot be improved by optimising the parameters ℓ_1 and ℓ_2: all parameter choices within the triangle Ω give the same contraction factor.

8. Convergence of rational subdivision schemes

It is proved in this section that subdivision scheme (7) with (21) preserves monotonicity and generates continuously differentiable limit functions.

Theorem 19 (Monotonicity preservation). *Subdivision scheme* (7) *with G given in* (21) *preserves monotonicity.*

Proof. The function G satisfies (9), which is a direct result from the construction in Section 6. □

With respect to convergence, the following theorem holds:

Theorem 20 (C^0-convergence). *Let the same conditions hold as in Theorem* 8.

Then, repeated application of subdivision scheme (7) *with* (21) *leads to a continuous function which is monotone and interpolates the initial data points* $(t_i^{(0)},x_i^{(0)})$ *if* $\ell_2>0$ *or the initial data are strictly monotone.*

Proof. First, if $\ell_2>0$, it is shown that G satisfies

$$\begin{aligned}|G(r,R)| &= \left|\frac{r-R}{\ell_1+(1+\ell_2)(r+R)+\ell_3 rR}\right| \leqslant \frac{r+R}{\ell_1+(1+\ell_2)(r+R)+\ell_3 rR}\\ &\leqslant \frac{r+R}{(1+\ell_2)(r+R)} = \frac{1}{1+\ell_2} = \mu<1, \end{aligned} \tag{32}$$

i.e., G satisfies condition (13). If the initial data are strictly monotone, we remark that the ratios of first order differences $r_i^{(k)}$ can be estimated using Theorem 15:

$$1 \leqslant \max_i \max\left\{ r_i^{(k)}, \frac{1}{r_i^{(k)}} \right\} \leqslant q^{(0)} < \infty.$$

The function G defined in (21) is monotone in both arguments, see Remark 13, and hence is maximal in the case

$$r_i^{(k)} = q^{(0)} \quad \text{and} \quad R_{i+1}^{(k)} = 1/q^{(0)},$$

which yields that G can be estimated as

$$|G(r_i^{(k)},R_{i+1}^{(k)})| \leqslant \frac{(q^{(0)})^2-1}{(\ell_1+\ell_3)q^{(0)}+(1+\ell_2)((q^{(0)})^2+1)} = \mu<1, \tag{33}$$

which proves (13) for all $(\ell_1,\ell_2,\ell_3)\in\Omega$, as $q^{(0)}<\infty$. □

Concerning C^1-convergence, we can formulate the following result:

Theorem 21 (C^1-convergence). *Let the same conditions hold as in Theorem* 9, *and let the data be strictly monotone.*

Then repeated application of subdivision scheme (7) *with* (21) *leads to a continuously differentiable function which is monotone and interpolates the initial data points* $(t_i^{(0)},x_i^{(0)})$.

Proof. As the function G is continuously differentiable in r and R, for all $r,R\geqslant 0$, it satisfies Lipschitz condition (15). It is shown in Section 7 that this function G yields that ratios of adjacent first-order differences converge to 1, i.e., the C^1-requirement (16) is satisfied. □

The analysis of the subdivision scheme for monotone, but not strictly monotone data is more difficult to examine. At any diadic point the left and right derivative can be proved to be equal, but this is not sufficient for convergence to a continuously differentiable limit function. Numerical experiments however show that the subdivision scheme yields limit functions that are continuously differentiable even in such cases:

Conjecture 22 (*Always* C^1). Let the same conditions hold as in Theorem 9, but let the data be monotone but not necessarily strictly monotone.

Table 1
The data set used in the numerical example

$t_i^{(0)}$	−2	−1	0	1	2	3	4	5	6	7	8
$x_i^{(0)}$	−2	−1	0	1/2	1	6	6	7	8	9	10

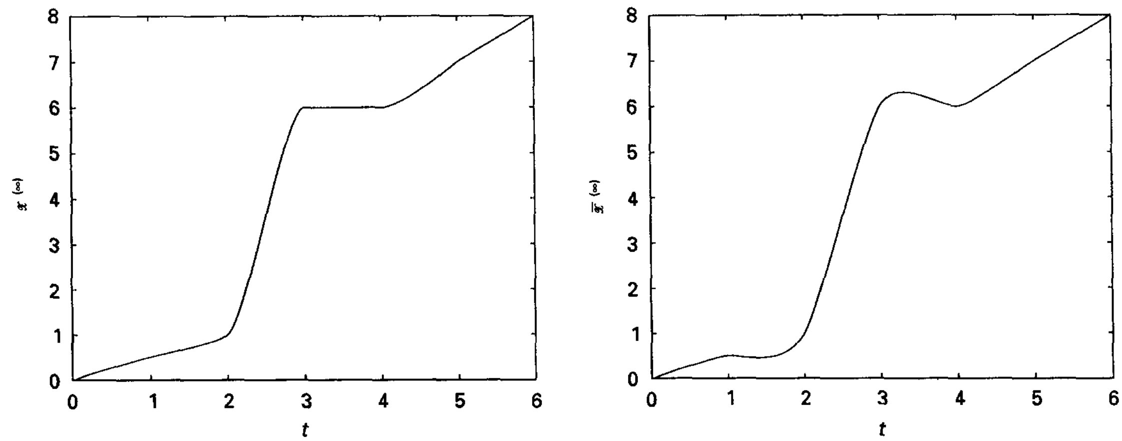

Fig. 2. The limit function obtained by the monotonicity preserving scheme (with $\ell_1 = 2$, $\ell_2 = 1$, $\ell_3 = 2$) and the linear four-point scheme.

Repeated application of subdivision scheme (7) with (21) leads to a continuously differentiable function which is monotone and interpolates the initial data points $(t_i^{(0)}, x_i^{(0)})$.

In the following example we show the graphical capabilities of the subdivision scheme and illustrate tension control that is provided by the parameters ℓ_1, ℓ_2 and ℓ_3.

Example 23 (*Numerical example*). Consider the uniform data set defined in Table 23. Some visual results are shown after repeated application of subdivision scheme (7) with G in (21).

First, the limit function is shown in the interval $[t_0^{(0)}, t_6^{(0)}]$ for the monotonicity preserving scheme with the parameter choice $\ell_1 = 2$, $\ell_2 = 1$ and $\ell_3 = 2$, see Remark 14, which has proved to generate results that are visually pleasing. This result is compared with the graphical performance of the linear four-point scheme [5], see (12), in Fig. 2, which clearly does not preserve monotonicity for this data set.

In the next plots, see Fig. 3, we again display the limit function of the monotonicity preserving subdivision scheme with $\ell_1 = 2$, $\ell_2 = 1$ and $\ell_3 = 2$, together with its derivative, which is nonnegative in the whole interval.

Finally, it is shown in Fig. 4 that the parameters ℓ_j act as tension parameters. Two extreme choices are compared: $\ell_1 = 0$, $\ell_2 = 0$, $\ell_3 = 6$ and $\ell_1 = 3$, $\ell_2 = 3/2$, $\ell_3 = 0$ respectively. The first case leads to a limit function that is almost piecewise constant in difficult areas, whereas the second choice of the tension parameters leads to an almost piecewise linear function.

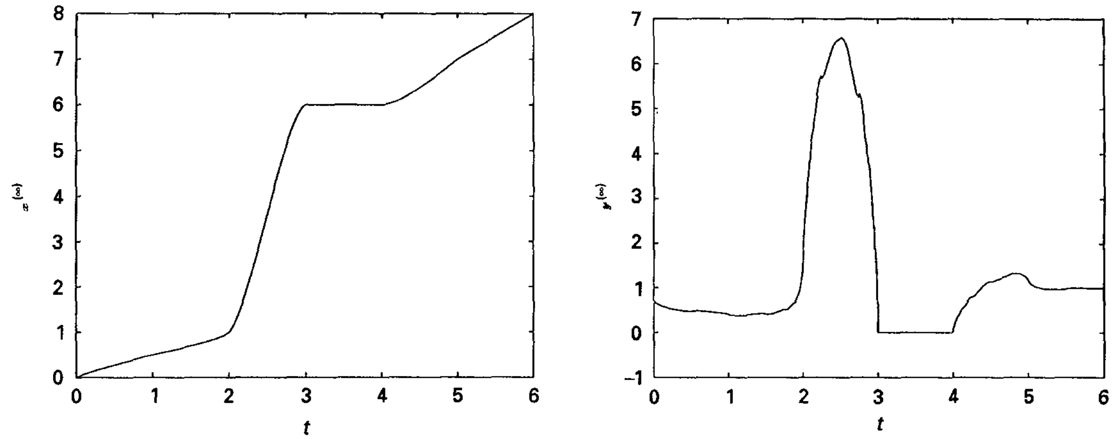

Fig. 3. The limit function $x^{(\infty)}$ and its derivative $y^{(\infty)}$ obtained by the monotonicity preserving scheme (with $\ell_1=2$, $\ell_2=1$, $\ell_3=2$).

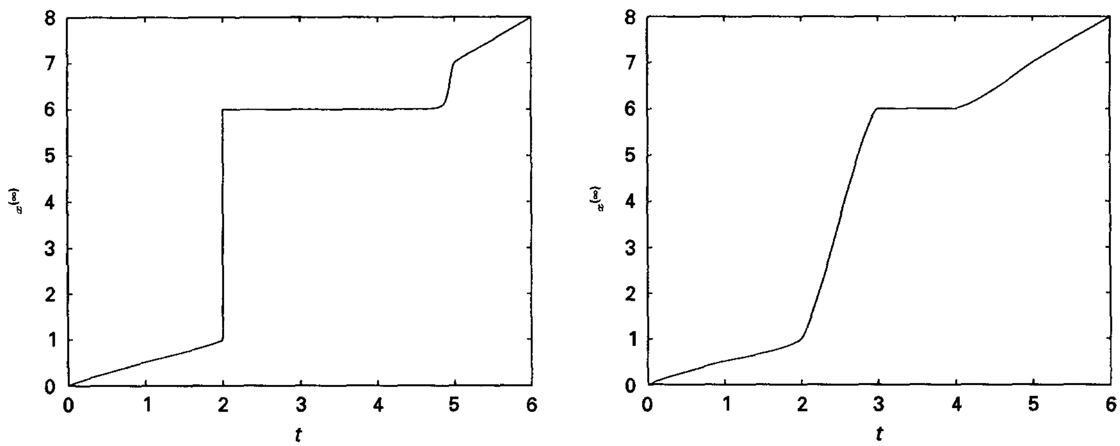

Fig. 4. Tension control illustrated by the limit function obtained by the monotonicity preserving scheme with $\ell_1=0$, $\ell_2=0$, $\ell_3=6$ and $\ell_1=3$, $\ell_2=3/2$, $\ell_3=0$ respectively.

9. Approximation order

In this section, the approximation properties of the monotonicity preserving subdivision scheme from the previous sections are examined. Although a simple calculation shows that the scheme only reproduces linear functions (and quadratics if $\ell_3=0$), it can be proved that the scheme has approximation order four.

Theorem 24 (Approximation order). *Consider the data set* $\{(t_i^{(0)}, x_i^{(0)})\}_{i=0}^{N}$ *drawn from a strictly monotone function* $g \in C^4(I)$, *where* $I = [t_0^{(0)}, t_N^{(0)}] = [0,1]$, *such that*

$$x_i^{(0)} = g(t_i^{(0)}) \quad \text{where } t_i^{(0)} = ih \quad \text{and} \quad Nh = 1.$$

Define $x_h^{(\infty)}$ *as the limit function obtained by repeated application of the monotonicity preserving subdivision scheme* (7) *with* (21) *on the data* $x_i^{(0)}$.

Then, subdivision scheme (7) *with* (21) *is fourth-order accurate, i.e. there exists a constant* C, *not depending on* h, *such that*

$$\|x_h^{(\infty)} - g\|_{I,\infty} \leqslant Ch^4,$$

provided the boundaries are treated properly.

Proof. The boundaries are treated first. Note that, using Taylor series of g in $t_0^{(0)}$ and $t_N^{(0)}$, any strictly monotone function $g \in C^4(I)$ can be extended to a strictly monotone function $\tilde{g} \in C^4(\tilde{I})$, where $\tilde{I} = [-\varepsilon, 1+\varepsilon])$, such that $\tilde{g}(x) = g(x)$, $\forall x \in I$. Provided h is small enough, such an $\varepsilon > 0$ always exists. The boundary data points $x_{-2}^{(0)}$, $x_{-1}^{(0)}$, $x_{N+1}^{(0)}$, and $x_{N+2}^{(0)}$, which are necessary to determine the limit function $x^{(\infty)}$ in I, are now drawn from this extended function $\tilde{g}$.

To prove fourth-order accuracy, we compare the monotonicity preserving scheme with the linear four-point scheme [5] with $w = 1/16$, see (12). This scheme reproduces cubic polynomials. The data generated by the linear scheme are denoted by $\bar{x}_i^{(k)}$, and therefore

$$\begin{aligned} \bar{x}_{2i}^{(k+1)} &= \bar{x}_i^{(k)}, \\ \bar{x}_{2i+1}^{(k+1)} &= \tfrac{9}{16}(\bar{x}_i^{(k)} + \bar{x}_{i+1}^{(k)}) - \tfrac{1}{16}(\bar{x}_{i-1}^{(k)} + \bar{x}_{i+2}^{(k)}). \end{aligned} \tag{34}$$

The linear four-point scheme with $w = \frac{1}{16}$ has proved to be fourth-order accurate in [5]: there exists a $B \in \mathbb{R}$ such that

$$\|\bar{x}_h^{(\infty)} - g\|_{I,\infty} \leqslant Bh^4. \tag{35}$$

Starting from the initial data $x_i^{(0)} = \bar{x}_i^{(0)}, \forall i$, it will be proved that

$$\forall k \quad \exists\, C_k < \infty \quad \text{such that } \max_i |x_i^{(k)} - \bar{x}_i^{(k)}| \leqslant C_k h^4, \tag{36}$$

and

$$\lim_{k\to\infty} C_k < \infty.$$

The proof of (36) is based on induction with respect to k. Clearly, $C_0 = 0$, since the initial data satisfy $x_i^{(0)} = \bar{x}_i^{(0)}, \forall i$. Suppose that (36) is valid for some k, it will be proved that

$$\max_i |x_i^{(k+1)} - \bar{x}_i^{(k+1)}| = \max_i \max\{|x_{2i}^{(k+1)} - \bar{x}_{2i}^{(k+1)}|, |x_{2i+1}^{(k+1)} - \bar{x}_{2i+1}^{(k+1)}|\} \leqslant C_{k+1} h^4,$$

where a relation between C_{k+1} and C_k will be given. The first term gives

$$\max_i |x_{2i}^{(k+1)} - \bar{x}_{2i}^{(k+1)}| = \max_i |x_i^{(k)} - \bar{x}_i^{(k)}| \leqslant C_k h^4,$$

so it remains to estimate

$$
\begin{aligned}
&|x_{2i+1}^{(k+1)} - \bar{x}_{2i+1}^{(k+1)}| \\
&\quad = |\{\tfrac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)}) + \tfrac{1}{2}s_i^{(k)}G(r_i^{(k)}, R_{i+1}^{(k)})\} - \{\tfrac{1}{2}(\bar{x}_i^{(k)} + \bar{x}_{i+1}^{(k)}) + \tfrac{1}{2}\bar{s}_i^{(k)}G(\bar{r}_i^{(k)}, \bar{r}_{i+1}^{(k)})\}| \\
&\quad \leqslant \tfrac{1}{2}|x_i^{(k)} - \bar{x}_i^{(k)}| + \tfrac{1}{2}|x_{i+1}^{(k)} - \bar{x}_{i+1}^{(k)}| + |\tfrac{1}{2}s_i^{(k)}G(r_i^{(k)}, R_{i+1}^{(k)}) - \tfrac{1}{2}\bar{s}_i^{(k)}G(\bar{r}_i^{(k)}, \bar{r}_{i+1}^{(k)})|.
\end{aligned} \tag{37}
$$

Using the triangle inequality, the last contribution is written as

$$
\begin{aligned}
&|s_i^{(k)}G(r_i^{(k)}, R_{i+1}^{(k)}) - \bar{s}_i^{(k)}G(\bar{r}_i^{(k)}, \bar{R}_{i+1}^{(k)})| \\
&\quad = |s_i^{(k)}G(r_i^{(k)}, R_{i+1}^{(k)}) - \bar{s}_i^{(k)}G(r_i^{(k)}, R_{i+1}^{(k)}) + \bar{s}_i^{(k)}G(r_i^{(k)}, R_{i+1}^{(k)}) - \bar{s}_i^{(k)}G(r_i^{(k)}, R_{i+1}^{(k)}) \\
&\qquad + \bar{s}_i^{(k)}G(r_i^{(k)}, R_{i+1}^{(k)}) - \bar{s}_i^{(k)}G(\bar{r}_i^{(k)}, \bar{R}_{i+1}^{(k)})| \\
&\quad \leqslant |s_i^{(k)} - \bar{s}_i^{(k)}||G(r_i^{(k)}, R_{i+1}^{(k)})| + |\bar{s}_i^{(k)}||G(r_i^{(k)}, R_{i+1}^{(k)}) \\
&\qquad - G(r_i^{(k)}, R_{i+1}^{(k)})| + |\bar{s}_i^{(k)}||G(r_i^{(k)}, R_{i+1}^{(k)}) - G(\bar{r}_i^{(k)}, \bar{R}_{i+1}^{(k)})|.
\end{aligned}
$$

In the appendix, we show convergence of the following contributions:

$$\tfrac{1}{2}\max_i |s_i^{(k)} - \bar{s}_i^{(k)}| \max_i |G(r_i^{(k)}, R_{i+1}^{(k)})| \leqslant A_1\mu_1^k h^4, \tag{38}$$

$$\tfrac{1}{2}\max_i |\bar{s}_i^{(k)}| \max_i |G(r_i^{(k)}, R_{i+1}^{(k)}) - G(r_i^{(k)}, R_{i+1}^{(k)})| \leqslant A_2\mu_2^k h^4, \tag{39}$$

$$\tfrac{1}{2}\max_i |\bar{s}_i^{(k)}| \max_i |G(r_i^{(k)}, R_{i+1}^{(k)}) - G(\bar{r}_i^{(k)}, \bar{R}_{i+1}^{(k)})| \leqslant A_3\mu_3^k h^4, \tag{40}$$

where

$$\mu_j < 1 \quad \text{and} \quad A_j < \infty, \quad j = 1, 2, 3,$$

are numbers depending on derivatives of the original function g. It is easily checked that these inequalities are indeed sufficient for conditions (38)–(40). The proofs of some of the estimates are given by induction in k. Using Taylor series it is derived that the initial data indeed satisfy the estimates.

Using induction hypothesis (36) together with the estimates (38)–(40), Eq. (37) yields

$$
\begin{aligned}
\max_i |x_{2i+1}^{(k+1)} - \bar{x}_{2i+1}^{(k+1)}| &\leqslant \tfrac{1}{2}C_k h^4 + \tfrac{1}{2}C_k h^4 + A_1\mu_1^k h^4 + A_2\mu_2^k h^4 + A_3\mu_3^k h^4 \\
&\leqslant C_k h^4 + \bar{A}\bar{\mu}^k h^4,
\end{aligned}
$$

where $\bar{A} = A_1 + A_2 + A_3$ and $\bar{\mu} = \max\{\mu_1, \mu_2, \mu_3\}$. The coefficient C_{k+1} now satisfies

$$C_{k+1} \leqslant \max\{C_k, C_k + \bar{A}\bar{\mu}^k\} = C_k + \bar{A}\bar{\mu}^k.$$

Since $C_0 = 0$, this gives

$$C_k \leqslant \bar{A}\frac{1 - \bar{\mu}^k}{1 - \bar{\mu}} \;\Rightarrow\; \lim_{k\to\infty} C_k \leqslant \bar{A}\frac{1}{1 - \bar{\mu}} =: A < \infty.$$

Having proved the induction step, hypothesis (36) holds. Therefore

$$\|x_h^{(\infty)} - g\|_{I,\infty} \leqslant \|x_h^{(\infty)} - \bar{x}_h^{(\infty)}\|_{I,\infty} + \|\bar{x}_h^{(\infty)} - g\|_{I,\infty} \leqslant (A+B)h^4,$$

which completes the proof, i.e., subdivision scheme (7) with (21) has approximation order four. □

Remark 25. Observe that this analysis is only valid for strictly monotone data, i.e., data drawn from a function g with $g'(\tau) > 0$, $\forall \tau \in I$. Numerical experiments show that if $g'(\tau) = 0$ for some $\tau \in I$, the approximation order in the max-norm decreases to 3.

Remark 26. More recently, see [12, 17], an alternative proof for the approximation order of subdivision schemes is provided. This approach uses the notion of stability of stationary subdivision schemes.

10. Generalisations

In this section, we briefly describe an extension and look forward to generalisations of the monotonicity preserving subdivision scheme discussed in this article.

10.1. Piecewise monotonicity

The subdivision scheme discussed in this article can be extended to a *piecewise monotonicity* preserving subdivision scheme suited for interpolation of piecewise monotone data.

We first observe that any monotonicity preserving subdivision scheme of the form (7) is also directly applicable to monotone decreasing data. It is therefore only necessary to examine regions in the initial data where the differences $s_i^{(0)}$ change sign and to split the domain in monotone increasing parts and monotone decreasing parts.

If one of the differences in the initial data is zero, i.e., the case that the differences satisfy $s_0^{(0)} = 0$, $s_{-1}^{(0)} < 0$ and $s_1^{(0)} > 0$, a simple and natural way to split the monotonicity regions is provided by the data: the solution in the interval $[t_0^{(0)}, t_1^{(0)}]$ becomes constant. The monotonicity preserving subdivision scheme can be applied both on the left-hand side of $t_0^{(0)}$ and on the right-hand side of $t_1^{(0)}$. The limit function is then monotone decreasing left to $t_1^{(0)}$ and monotone increasing right to $t_0^{(0)}$. In fact, the scheme is piecewise monotonicity preserving for these data.

The general case is characterised by data with differences satisfying $s_{-2}^{(0)} < 0$, $s_{-1}^{(0)} < 0$, $s_0^{(0)} > 0$ and $s_1^{(0)} > 0$, since all other cases degenerate to this situation after one iteration. We define the split point as the point where the differences change sign, i.e., in this case $t_0^{(0)}$.

A simple method to adapt the subdivision scheme to piecewise monotonicity preservation is as follows:

- Apply the monotonicity preserving subdivision scheme to determine $x_j^{(k)}$ for $j < 0$, where $s_0^{(0)}$ is replaced by 0.
- Apply the monotonicity preserving subdivision scheme to determine $x_j^{(k)}$ for $j > 0$, where $s_{-1}^{(0)}$ is replaced by 0.

The corresponding function G_{PM} of this piecewise monotonicity preserving subdivision scheme can be written as

$$G_{\mathrm{PM}}(r,R) = G(\max\{0,r\},\max\{0,R\}),$$

with for example G defined as in (21). It is clear that the limit function is monotone decreasing for $t \leqslant t_0^{(0)}$ and monotone increasing for $t \geqslant t_0^{(0)}$. The derivative of the limit function in $t_0^{(0)}$ is equal to zero. Convergence to a continuously differentiable limit function is achieved if conjecture 22 is true.

10.2. Relation with monotonicity preserving splines

An alternative way to derive subdivision scheme (7) with G as in (25) originates from Hermite interpolation using quadratic splines, see [18, 11].

Consider a strictly monotone data set $\{(2^{-k}i, x_i^{(k)})\}_i$ and define Bézier points as follows

$$b_{2i}^{(k)} = x_i^{(k)}, \quad b_{2i+1}^{(k)} = x_i^{(k)} + \frac{2^{-k}}{4}\xi_i^{(k)}, \quad b_{2i+3}^{(k)} = x_{i+1}^{(k)} - \frac{2^{-k}}{4}\xi_{i+1}^{(k)}, \quad b_{2i+4}^{(k)} = x_{i+1}^{(k)},$$

where the $\xi_i^{(k)}$ are suitable derivative estimates. Define now the subdivision points $x_{2i+1}^{(k+1)}$ as follows:

$$x_{2i+1}^{(k+1)} = b_{2i+2}^{(k)} = \tfrac{1}{2}(b_{2i+1} + b_{2i+3}) = \tfrac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)}) + \frac{2^{-k}}{8}(\xi_i^{(k)} - \xi_{i+1}^{(k)}). \tag{41}$$

If the derivatives are estimated using Butland slopes, see [1, 9], i.e.,

$$\xi_i^{(k)} = 2\frac{y_{i-1}^{(k)} y_i^{(k)}}{y_{i-1}^{(k)} + y_i^{(k)}} \quad \text{where } y_j^{(k)} = 2^k s_j^{(k)},$$

then subdivision scheme (41) preserves monotonicity and is written as

$$\begin{aligned} x_{2i+1}^{(k+1)} &= \frac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)}) + \frac{1}{4}s_i^{(k)}\left(\frac{s_{i-1}^{(k)}}{s_{i-1}^{(k)} + s_i^{(k)}} - \frac{s_{i+1}^{(k)}}{s_i^{(k)} + s_{i+1}^{(k)}}\right) \\ &= \frac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)})d + \frac{1}{2}s_i^{(k)} G_B(r_i^{(k)}, R_{i+1}^{(k)}), \end{aligned}$$

with

$$\begin{aligned} G_B(r_i^{(k)}, R_{i+1}^{(k)}) &= \frac{1}{2}\left(\frac{s_{i-1}^{(k)}}{s_{i-1}^{(k)} + s_i^{(k)}} - \frac{s_{i+1}^{(k)}}{s_i^{(k)} + s_{i+1}^{(k)}}\right) = \frac{1}{2}\left(\frac{s_i^{(k)}}{s_i^{(k)} + s_{i+1}^{(k)}} - \frac{s_i^{(k)}}{s_{i-1}^{(k)} + s_i^{(k)}}\right) \\ &= \frac{1}{2}\left(\frac{1}{1 + s_{i+1}^{(k)}/s_i^{(k)}} - \frac{1}{1 + s_{i-1}^{(k)}/s_i^{(k)}}\right) = \frac{1}{2}\left(\frac{1}{1 + R_{i+1}^{(k)}} - \frac{1}{1 + r_i^{(k)}}\right), \end{aligned}$$

which coincides with the special case G_C in (25), see Remark 14.

In fact this approach does not only define a subdivision scheme, but the construction using Bézier points also provides an explicit interpolation method using quadratic (B-)splines that is monotonicity preserving.

Remark 27. Note that the Butland slope is the harmonic average of two adjacent differences, where the convexity preserving subdivision scheme in [16] is a scheme that contains the harmonic average of two adjacent second-order differences.

A relation between rational interpolation and convexity preserving subdivision is observed in [8]. There exists also a connection between *monotonicity preserving rational interpolation* and subdivision scheme (7) with (21). In [10], a class of rational splines is defined on an interval $[t_i, t_{i+1}]$ as follows:

$$x_i(t) = \frac{y_i x_{i+1}\theta^2 + (x_i\xi_{i+1} + x_{i+1}\xi_i)\,\theta(1-\theta) + y_i x_i(1-\theta)^2}{y_i\theta^2 + (\xi_{i+1} + \xi_i)\theta(1-\theta) + y_i(1-\theta)^2},$$

where

$$\theta = \frac{t - t_i}{t_{i+1} - t_i} \quad \text{and} \quad y_i = \frac{x_{i+1} - x_i}{t_{i+1} - t_i},$$

and the ξ_j are suitable chosen derivative estimates at t_j.

The construction is restricted to equidistant data, and a subdivision scheme is obtained by halfway evaluation of $x_i(t)$, i.e., at $t_{2i+1}^{(k+1)} = \frac{1}{2}(t_i^{(k)} + t_{i+1}^{(k)})$. This results in the subdivision scheme

$$x_{2i+1}^{(k+1)} = x(t_{2i+1}^{(k+1)}) = \frac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)}) + \frac{1}{2}h_i^{(k)}\frac{\xi_{i+1}^{(k)} - \xi_i^{(k)}}{\xi_i^{(k)} + 2\Delta x_i^{(k)} + \xi_{i+1}^{(k)}}.$$

Two choices for the derivative estimates $\xi_j^{(k)}$ are presented: the arithmetic mean and the harmonic mean:

$$\xi_j^{(k)} = \tfrac{1}{2}(\Delta x_{j-1}^{(k)} + \Delta x_j^{(k)}), \quad \text{or} \quad \xi_j^{(k)} = \frac{2\Delta x_{j-1}^{(k)}\,\Delta x_j^{(k)}}{\Delta x_{j-1}^{(k)} + \Delta x_j^{(k)}},$$

and both choices yield a subdivision scheme as in (7), and the functions G_L and G_B are respectively given by

$$G_L(r,R) = \frac{r-R}{6+r+R} \quad \text{and} \quad G_B(r,R) = \frac{r-R}{1+2(r+R)+3rR}.$$

Both functions are contained in class (21), as $(\ell_1,\ell_2,\ell_3) = (6,0,0) \in \Omega$, and $(\ell_1,\ell_2,\ell_3) = (1,1,3) \in \Omega$.

10.3. Application to grid refinement

The application area of the subdivision scheme discussed in this article is not restricted to monotonicity preserving uniform subdivision. In this section we briefly describe the importance of monotonicity preserving subdivision to *grid refinement.*

We examined uniform subdivision schemes applied to data points $(t_i^{(k)}, x_i^{(k)})$ that preserve strict monotonicity. This is exactly the same as subdivision scheme for a grid $x_i^{(k)}$ that keeps the grid ordered. In addition, the schemes discussed in this article have the property that ratios of first-order differences converge to 1, see Section 7. Applying such a scheme to $x_i^{(k)}$ yields that the grid becomes homogeneous, i.e., ratios of the size of two adjacent cells tends to 1. This is important in applications such as grid generation and finite element methods.

In case of functional nonuniform data $(x_i^{(k)}, f_i^{(k)})$, we propose to apply a suitable monotonicity preserving scheme to the data $x_i^{(k)}$, e.g., scheme (7) with (21), which makes the grid homogeneous. Another subdivision scheme is applied to the data $f_i^{(k)}$. This leads to stationary nonuniform subdivision schemes for functional nonuniform data, which are discussed in a separate report [14]. We describe nonuniform extensions of the convexity preserving scheme [16] and the monotonicity preserving scheme discussed in this article. The approach can also be applied to the linear four-point scheme [5].

Appendix A

In this appendix the proof concerning the order of approximation of subdivision scheme (7) with G in (21) is treated. A complete proof is given in [13], which is algebraically much involved.

Lemma A.1 (Approximation order). *Let the same conditions hold as in Theorem 24. Data $x_i^{(k)}$, first-order differences $s_i^{(k)}$ (see (2)) and ratios of differences $r_i^{(k)}$ (see (6)), are obtained by repeated application of subdivision scheme (7) with (21) on the data $x_i^{(0)}$. In addition, data $\bar{x}_i{}^{(k)}$, first-order differences $\bar{s}_i{}^{(k)}$ and ratios of differences $\bar{r}_i{}^{(k)}$ are defined by repeated application of the linear four-point scheme given in (34) on the initial data $\bar{x}_i{}^{(0)} = x_i^{(0)}$.*

Then, provided h is small enough, the following estimates hold:

$$\max_i \max\left\{r_i^k, \frac{1}{r_i^k}\right\} - 1 \leqslant C_0\rho_0^k h,$$

$$\max_i s_i^{(k)} \leqslant C_1\rho_1^k h,$$

$$\max_i |\bar{s}_i{}^{(k)}| \leqslant C_2\rho_2^k h,$$

$$\max_i |r_i^{(k)} - r_{i+1}^{(k)}| \leqslant C_3\rho_3^k h^2,$$

$$\max_i |G(r_i^{(k)}, R_{i+1}^{(k)})| \leqslant C_4\rho_4^k h,$$

$$\max_i |G(r_i^{(k)}, R_{i+1}^{(k)}) - \bar{G}(r_i^{(k)}, R_{i+1}^{(k)})| \leqslant C_5\rho_5^k h^3,$$

$$\max_i |r_i^{(k)} - \bar{r}_i{}^{(k)}| \leqslant C_6\rho_6^k h^3,$$

$$\max_i |\bar{G}(r_i^{(k)}, R_{i+1}^{(k)}) - \bar{G}(\bar{r}_i{}^{(k)}, \bar{R}_{i+1}{}^{(k)})| \leqslant C_7\rho_7^k h^3,$$

$$\max_i |s_i^{(k)} - \bar{s}_i{}^{(k)}| \leqslant C_8\rho_8^k h^4,$$

where

$$\rho_j < 1 \quad \text{and} \quad C_j < \infty, \qquad j = 0, 1, \ldots, 8.$$

Proof. It is easily checked that these inequalities are indeed sufficient for conditions (38)–(40), i.e., subdivision scheme (7) with (21) has approximation order four. □

The proofs of some of the estimates are given by induction in k. It is easily derived that the initial data indeed satisfy the following estimates:

Lemma A.2 (Taylor initial data). *The initial data satisfy*

$$\max_i \max\left\{r_i^{(0)}, \frac{1}{r_i^{(0)}}\right\} - 1 \leqslant \|g''\| \cdot \|(g')^{-1}\| h,$$

$$\max_i \bar{s}_i^{(0)} = \max_i s_i^{(0)} \leqslant \|g'\| h,$$

$$\max_i |r_{i+1}^{(0)} - r_i^{(0)}| \leqslant \|(g')^{-1}\|^2 (\|g'\| \cdot \|g'''\| + \|g''\|^2) h^2,$$

$$\bar{r}_i^{(0)} = r_i^{(0)}, \quad \forall i,$$

$$\bar{s}_i^{(0)} = s_i^{(0)}, \quad \forall i,$$

where the norm $\|f\|$ *is defined as*

$$\|f\| := \max_{\tau \in \tilde{I}} |f(\tau)|.$$

Proof. The results are obtained by applying Taylor series around $h = 0$ on the initial data, where

$$s_i^{(0)} = g((i+1)h) - g(ih),$$

$$r_i^{(0)} = \frac{g(ih) - g((i-1)h)}{g((i+1)h) - g(ih)},$$

$$R_{i+1}^{(0)} = \frac{g((i+2)h) - g((i+1)h)}{g((i+1)h) - g(ih)}.$$

As an example, we show that

$$|r_i^{(0)} - 1| = \left|\frac{g(ih) - g((i-1)h)}{g((i+1)h) - g(ih)} - 1\right| = \left|-\frac{g((i+1)h) - 2g(ih) + g((i-1)h)}{g((i+1)h) - g(ih)}\right|$$
$$\leqslant \frac{\max_{\tau \in \tilde{I}} |g''(\tau)|}{\min_{\tau \in \tilde{I}} g'(\tau)} h = \|g''\| \|(g')^{-1}\| h,$$

is the estimate for $r_i^{(0)}$. □

In order to be able to prove the other estimates we claim in Lemma A.1, we need the following technical lemma:

Lemma A.3. *Let* $z^{(0)}$ *satisfy*

$$0 \leqslant z^{(0)} \leqslant Ah^n \quad \text{with} \quad A < \infty \quad \text{and} \quad n > 0,$$

and let the following estimate hold for $z^{(k)}$:

$$z^{(k+1)} \leqslant \lambda(h) z^{(k)} + \sum_{m=2}^{M} B_m(h)(z^{(k)})^m + \sum_{i=1}^{N} C_i(h)\mu_i^k h^n,$$

with $M, N < \infty$, $0 \leqslant \mu_i < 1$ *and rational functions* λ, B_m *and* C_i *satisfying*

$$0 \leqslant \lambda(0) < 1 \quad \text{and} \quad 0 \leqslant B_m(0), C_i(0) < \infty.$$

Then, provided h is small enough, we have

$$z^{(k)} \leqslant \bar{A}\bar{\lambda}kh^n, \quad \forall k, \text{ with } \bar{\lambda} < 1,\ \bar{A} < \infty.$$

The claims in Lemma A.1 have been proved in [13] by repeated application of Lemma A.3.

References

[1] J. Butland, A method of interpolating reasonably-shaped curves through any data, Proc. Comput. Graphics 80 (1980) 409–422.

[2] Z. Cai, Convergence, error estimation and some properties of four-point interpolation subdivision scheme, Comput. Aided Geom. Design 12 (1995) 459–468.

[3] A.S. Cavaretta, C.A. Micchelli, The design of curves and surfaces by subdivision algorithms, in: T. Lyche, L.L. Schumaker (Eds.), Mathematical Methods in CAGD, Academic Press, Boston, 1989, pp. 115–153.

[4] B.W. Char et al., Maple V Language Reference Manual, Springer, New York, 1991.

[5] N. Dyn, J.A. Gregory, D. Levin, A 4-point interpolatory subdivision scheme for curve design, Comput. Aided Geom. Design 4 (1987) 257–268.

[6] N. Dyn, J.A. Gregory, D. Levin, Analysis of uniform binary subdivision schemes for curve design, Constr. Approx. 7(2) (1991) 127–147.

[7] N. Dyn, D. Levin, Interpolating subdivision schemes for the generation of curves and surfaces, in: W. Haussmann, K. Jetter (Ed.), Multivariate Approximation and Interpolation, International Series of Numerical Mathematics, vol. 4, Birkhäuser, Basel, 1990, pp. 91–106.

[8] M.S. Floater, C.A. Micchelli, Nonlinear stationary subdivision, in: Approximation Theory, Monogr. Textbooks Pure Appl. Math., vol. 212, Dekker, New York, 1998, pp. 209–224.

[9] F.N. Fritsch, J. Butland, A method for constructing local monotone piecewise cubic interpolants, SIAM J. Sci. Statist. Comput. 5 (1984) 300–304.

[10] J.A. Gregory, R. Delbourgo, Piecewise rational quadratic interpolation to monotonic data, IMA J. Numer. Anal. 2 (1982) 123–130.

[11] R. Iqbal, A one-pass algorithm for shape-preserving quadratic spline interpolation, J. Sci. Comput. 7 (1992) 359–376.

[12] F. Kuijt, Convexity preserving interpolation – stationary nonlinear subdivision and splines, Ph.D. Thesis, University of Twente, 1998.

[13] F. Kuijt, R. van Damme, Monotonicity preserving interpolatory subdivision schemes, Memorandum no. 1402, University of Twente, Faculty of Mathematical Sciences, 1997.

[14] F. Kuijt, R. van Damme, Shape preserving interpolatory subdivision schemes for nonuniform data, Memorandum no. 1423, University of Twente, Faculty of Mathematical Sciences, 1997, J. Approx. Theory, submitted.

[15] F. Kuijt, R. van Damme, Smooth interpolation by a convexity preserving nonlinear subdivision algorithm, in: A. Le Méhauté, C. Rabut, L.L. Schumaker (Eds.), Surface Fitting and Multiresolution Methods, Vanderbilt University Press, Nashville, TN, 1997, pp. 219–224.

[16] F. Kuijt, R. van Damme, Convexity preserving interpolatory subdivision schemes, Constr. Approx. 14(4) (1998) 609–630.
[17] F. Kuijt, R.M.J. van Damme, Stability of subdivision schemes, J. Comput. Appl. Math., submitted.
[18] L.L. Schumaker, On shape preserving quadratic spline interpolation, SIAM J. Numer. Anal. 20 (1983) 854–864.
[19] I. Yad-Shalom, Monotonicity preserving subdivision schemes, J. Approx. Theory 74 (1993) 41–58.