Fig. 5. Exact solution (curve 1), approximate solution for integration step $\Delta\tau = 2.2$ (polygon 4) and curves, which present the exact solutions with initial conditions τ_k, u_k for $\tau \in [\tau_k, \tau_{k+1}]$, of the logistic differential equation.

Fig. 6. Exact solution (curve 1), approximate solution for integration step $\Delta\tau = 2.7$ (polygon 5) and curves, which present the exact solutions with initial conditions τ_k, u_k for $\tau \in [\tau_k, \tau_{k+1}]$, of the logistic differential equation.

corresponding u_{k+i} -values, $i = 1, 2, \dots$, can be interpreted as approximate solutions with numerical errors, which alternate irregularly.

Bifurcation and chaotic behaviour of solutions known for equation (7), which arise for integration steps $\Delta\tau > 2$, appear as properties of numerical errors of approximate solutions, provided the results of equation (7), determined with such large integration steps and after a large number of integration steps, can be still considered as approximate solutions of differential equation (1).

5. Conclusions

When the logistic difference equation (7) is considered as an equation for the calculation of approximate solutions of the logistic differential equation (1) by Euler's polygon method, then bifurcation of solutions, and transition to periodic or chaotic solutions, observed on equation (7), appear as properties of numerical errors in case of inappropriate (i.e. too large) integration steps. Thus, it is questionable, whether or not properties of numerical errors can be interpreted as real phenomena of the dynamical system governed by the considered differential equation.

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Address: Dr. rer. nat. habil. HORST J. KLEPP, Ruhr-Universität Bochum, Institut für Mechanik, Universitätsstraße 150, W-4630 Bochum 1, Germany

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WETTERLING, W.

Application of Chebyshev Approximation to Curve Fitting

Dedicated to the memory of LOTHAR COLLATZ

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1. Introduction

The standard methods of curve and surface fitting are based upon least squares approximation of given data. An alternative is the principle of uniform (Chebyshev) approximation. This approach can be advantageous and even essential in certain technical applications, such as the engineering problem described in section 2. The general problem of uniform curve fitting to given data with some generalizations and two special cases are discussed in section 3.

It should be noted that these problems are different from those of standard Chebyshev approximation where a given function is approximated by an element of a linear or nonlinear class of functions. However, as will be shown in section 4, certain principles known from approximation theory such as optimality conditions and alternation properties can be retrieved here.

2. An approximation problem in railway engineering

The work described in this section has been done in collaboration between the Department for Infrastructure of the Dutch Railway Company (Nederlandse Spoorwegen N.V.) and the Systems and Optimization Group of the Applied Mathematics Department at the University of Twente. Here we give a short account of the technical and mathematical aspects of the problem. More details can be found in the graduate thesis of E. JONKER [2].

The technical problem is as follows: It is required that the upper contour of the rails on a railway bridge is a C^2 -curve (second derivatives bounded and continuous). The permitted maximum speed on the bridge depends upon a lower bound for the radius of curvature of that curve. In order to achieve the required smoothness, after construction of a new railway bridge or in the case of renewal of tracks on existing bridges the irregularities in the bridge surface must be compensated, e.g. by adjusting the heights of the rail supports.

In the mathematical model the upper contour of the rails is represented by a cubic spline function which approximates a set of data (x_i, y_{Li}, y_{Ri}) , $i = 0, 1, \dots, m$, describing the bridge surface. The x_i are the horizontal coordinates (locations of rail supports), $A = x_0 < x_1 < \dots < x_m = B$, and y_{Li}, y_{Ri} are vertical coordinates for the left and right rail (reference heights, obtained by measurements of the bridge surface). A set of (fixed) knots p_j is chosen, $A = p_0 < p_1 < \dots < p_k = B$ (in practice k between $m/20$ and $m/8$, each p_j coincident with an x_i).

The approximating cubic spline function with knots p_j is denoted by $s = s(x, a_1, \dots, a_{k+3})$. The linear parameters a_j have to be determined such that the following conditions are satisfied: The deviation between $s(x_i, a_j)$ and the reference heights y_{Li} and y_{Ri}

should be less than or equal to a tolerance h ,

$$\left. \begin{aligned} |s(x_i, a_j) - y_{Li}| &\leq h \\ |s(x_i, a_j) - y_{Ri}| &\leq h \end{aligned} \right\}, \quad i = 1, 2, \dots, m. \tag{1}$$

The radius of curvature should be bounded from below by some r_0 ,

$$r = [1 + (\partial s / \partial x)^2]^{1/2} / |\partial^2 s / \partial x^2| \geq r_0, \quad A \leq x \leq B. \tag{2}$$

(we allow a zero denominator). Moreover, boundary conditions for

$$s, \quad \partial s / \partial x \quad \text{and} \quad \partial^2 s / \partial x^2, \quad x = A, B, \tag{3}$$

possibly also values in some intermediate points, are prescribed.

The nonlinear condition (2) in all points of the interval $[A, B]$ can be reduced to a linear condition on the finite set of knots. (2) will be satisfied if

$$1 / |\partial^2 s / \partial x^2| \geq r_0, \quad A \leq x \leq B, \tag{2a}$$

holds, and sufficient for (2a) is

$$|\partial^2 s / \partial x^2| \leq 1 / r_0, \quad x = p_0, p_1, \dots, p_k, \tag{2b}$$

since the second derivative of a cubic spline is a linear spline and therefore has its local extrema in the knots.

The original problem is now reduced to solving a finite system of linear inequalities (1), (2b) and equations (3). In the actual computations as described in [2] the problem is solved as an approximation problem by taking $h = c_1 \varepsilon$, $1 / r_0 = c_2 \varepsilon$ with constants c_1, c_2 and minimizing ε under the conditions (1) and (2b) after a suitable matrix transformation to ensure (3). A modified simplex method, essentially STIEFEL's algorithm [3], is used to solve this problem of discrete linear Chebyshev approximation. The numbers c_1 and c_2 are adapted appropriately in order to find a solution satisfying all requirements, possibly with an r_0 too small to allow maximum speed. In such cases E. JONKER's program gives information how to obtain a better surface by suitable modification of the data (e.g. grinding off some sections of the surface, or using leveling pads in the rail supports). The program can also calculate spline approximations for railway bridges with prescribed banking in horizontal curves.

The program calculated successfully several practical examples: A bridge over the Strobossvaart ($m = 46$, $r_0 = 8.8$ km), a steel bridge near Veendam ($m = 42$, $r_0 = 4.6$ km) and a concrete fly-over near Lage Zwaluwe ($m = 589$, $r_0 = 12.3$ km).

3. Uniform approximation by curves in $\mathbb{R}^n$

We consider a generalization of the problem above. Suppose a finite set of points $\{y_1, y_2, \dots, y_m\} \subset \mathbb{R}^n$ and a family of curves $x(t, a)$ in $\mathbb{R}^n$, $t \in \mathbb{R}$, $a \in A \subset \mathbb{R}^k$, are given. The problem is to find an $a^* \in A$ such that the curve $x(t, a^*)$ is a best approximation to the points y_i . That means we have to determine, with $\|\cdot\|$ the Euclidean norm,

$$\min_a \max_i \min_t \|y_i - x(t, a)\|. \tag{4}$$

Constraints like the condition for the radius of curvature above can be given, possibly included in the definition of the set A .

As an applied example we mention the problem where both the vertical and the horizontal contour of a railway section are modeled as smooth curves. There are several other possible technical applications, for instance in robotics.

We can regard problem (4) as a problem of curve fitting in the Chebyshev sense, where the maximum of the distances and not as usually the sum of squares is minimized. Even more general problems might be worth considering, for instance the semi-infinite problem with a general compact set Y instead of the finite set $\{y_1, \dots, y_m\}$, or the problem where t is in a set $T \subset \mathbb{R}^d$ ($d = 1$ curve fitting, $d = 2$ surface fitting).

We mention two special cases of this general problem. In the first case $x(t, a) = a = \text{const.}$ is independent of t . For a given finite set of points $\{y_1, \dots, y_m\}$ the unique point a^* is determined which minimizes $\max_i \|y_i - a\|$. This problem is known as the *minimum covering sphere problem*. The optimal point a^* is called the *Chebyshev center* of the set $\{y_1, \dots, y_n\}$. In [1] the problem is written as a convex

optimization problem and an algorithm based on WOLFE's duality theory is given which determines the Chebyshev center in a finite number of steps.

The second case which will be the subject of a forthcoming paper is approximation by a straight line, $x(t) = c + bt$, $c, b \in \mathbb{R}^n$, $t \in \mathbb{R}$. Here the approximation problem can be written as a non-convex optimization problem. Several local minima are possible. An iterative method has been developed which determines alternately an improved c using the finite algorithm for the Chebyshev center and an improved b as solution of a quadratic optimization problem.

4. Optimality conditions

Let $x(t, a)$ in (4) be sufficiently smooth. If $t = t_i$ minimizes $\|y_i - x(t, a)\|$ then

$$\left(\frac{dx}{dt}(t_i, a) \right)^T (y_i - x(t_i, a)) = 0. \tag{5}$$

We assume that the conditions of the implicit function theorem are satisfied and hence (5) has a solution $t_i = t_i(a)$ which is continuously differentiable in some neighborhood of a^* . We can write (4) as an optimization problem:

Minimize $F(r, a) = r$, subject to

$$\left. \begin{aligned} g_i(r, a) &= r - (y_i - x(t_i(a), a))^T (y_i - x(t_i(a), a)) \geq 0, \\ i &= 1, \dots, m. \end{aligned} \right\} \tag{6}$$

The 1st order optimality conditions for this problem are

$$\left. \begin{aligned} 1 - \sum_{i \in J} \lambda_i &= 0, \quad \lambda_i \geq 0, \quad i \in J, \\ \sum_{i \in J} \lambda_i \left(\frac{\partial x(t_i, a)}{\partial a} \right)^T (y_i - x(t_i, a)) &= 0, \end{aligned} \right\} \tag{7}$$

where the set J contains the indices of active constraints in (6) and λ_i are the Lagrange multipliers. Note that due to (5) a term containing dx/dt could be omitted. Geometrically (7) describes how the active points with distance $\sqrt{r}$ are situated along the curve $x(t, a)$.

As an example consider approximation by a straight line:

$$x(t) = c + bt, \quad a = \begin{pmatrix} c \\ b \end{pmatrix} \in \mathbb{R}^{2n}, \quad \frac{\partial x(t, a)}{\partial a} = (I, tI). \tag{8}$$

With $t_i = b^T(y_i - c)/b^T b$ condition (7) becomes

$$\left. \begin{aligned} \sum_{i \in J} \lambda_i &= 1, \quad \lambda_i \geq 0, \quad i \in J, \\ \sum_{i \in J} \lambda_i (y_i - c - bt_i) &= 0, \quad \sum_{i \in J} \lambda_i t_i (y_i - c - bt_i) = 0. \end{aligned} \right\} \tag{9}$$

It is easy to see that in the case $n = 2$ (approximation by a straight line in the plane) this condition implies the alternation property for Chebyshev approximation by first degree polynomials. Evidently we can consider (7) as a generalized alternation property.

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Address: Prof. Dr. WOLFGANG WETTERLING, Twente University of Technology, Department of Applied Mathematics, Postbus 217, NL-7500 AE Enschede, The Netherlands