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Label-connected graphs and the gossip problem

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Abstract

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A graph with m edges is called label-connected if the edges can be labeled with real numbers in such a way that, for every pair (u, v) of vertices, there is a (u, v)-path with ascending labels. The minimum number of edges of a label-connected graph on n vertices equals the minimum number of calls in the gossip problem for n persons, which is known to be $2n-4$ for $n \geqslant 4$. A polynomial characterization of label-connected graphs with n vertices and $2n-4$ edges is obtained. For a graph G, let $\phi(G)$ denote the minimum number of edges that have to be added to $E(G)$ in order to create a graph with two edge-disjoint spanning trees. It is shown that for a graph G to be label-connected, $\phi(G) \leqslant 2$ is necessary and $\phi(G) \leqslant 1$ is sufficient. For $i = 1, 2$, the condition $\phi(G) \leqslant i$ can be checked in polynomial time. Yet recognizing label-connected graphs is an NP-complete problem. This is established by first showing that the following problem is NP-complete: Given a graph G and two vertices u and v of G, does there exist a (u, v)-path P in G such that $G - E(P)$ is connected?

1. Introduction

All graphs considered are finite and undirected. They may have multiple edges, but no loops. We use [1] for basic graph theoretic terminology and notation. In describing problems and their complexity the terminology of [4] is applied.

By a *labeling* of a graph G we will mean a function $f: E(G) \to \mathbb{R}$. A trail $v_0e_1v_1e_2v_2 \cdots e_kv_k$ in a graph G with a given labeling f is *ascending* if $f(e_{i+1}) > f(e_i)$ for $i = 1, \ldots, k-1$. A labeling of G is *admissible* if, for every pair (u, v) of vertices of G, there exists an ascending (u, v)-path. The graph G is *label-connected* if there exists an admissible labeling of G. Note that if there exists

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an admissible labeling of G, then there also exists an admissible labeling which is injective.

Clearly, if a spanning subgraph of G is label-connected, then so is G. The minimal label-connected graphs on at most three vertices are K_1, K_2, K_3 and the graph obtained from P_3 by duplicating one edge.

In Section 2 the problem of determining the minimum number of edges of a label-connected graph with a given number of vertices is solved by translating it into the so-called gossip problem. For later use, some known results around the gossip problem are also stated. In Section 3 a sufficient condition and a necessary condition for a graph to be label-connected are obtained in terms of the minimum number of edges that have to be added in order to create a graph with two edge-disjoint spanning trees. Although the two conditions are quite close and can be checked in polynomial time, recognizing label-connected graphs is an NP-complete problem, as proved in Section 7. In Section 4 the problem is shown to be well-solvable for a graph G if $|E(G)|$ equals the minimum number of edges of a label-connected graph with $|V(G)|$ vertices. In Section 5 label-connected graphs are characterized in terms of properties of their blocks. In Section 6 two ways of constructing new label-connected graphs out of known ones are exhibited.

2. The gossip problem

Let G be a label-connected graph with an admissible labeling. Interpret the vertices of G as persons each having a piece of information, and the edges of G as telephone calls, ordered in time according to the labeling of G. In each call, two persons exchange all the information they have. After all calls have been made, everybody knows all the information of everybody else, since there is an ascending path in G from each person to each other person. If follows that asking for the minimum number of edges of a label-connected graph on n vertices is equivalent to asking for the minimum number of calls that allows n persons to obtain each other's information. The latter problem is known as the *gossip problem* and has been solved by several people. See, for example, [2] and its references. We state its solution, as well as the solution (in the affirmative) of the so-called *4-cycle conjecture* [6], in the terminology of label-connected graphs.

Theorem 1 ([2, 7]). *Let G be a label-connected graph on n vertices. Then $|E(G)| \geqslant 2n-4$. Furthermore, $|E(G)| = 2n-4$ only if G contains a 4-cycle.*

Let T be a labeled tree and let $u \in V(T)$. Then T is called an *ascending* (*descending*) *u-tree* if, for each $v \in V(T) - \{u\}$, the unique (u, v)-path ((v, u)-path) in T is ascending. It is easily seen that a label-connected graph G contains a spanning ascending u-tree and a spanning descending u-tree for each vertex u of G. A *tree-pair* is a forest with two components.

Suppose a graph G contains two edge-disjoint spanning trees T_1 and T_2. We show that G is label-connected. Choose an arbitrary vertex u of G and assign labels $1, \ldots, n-1$ to the edges of T_1 in such a way that T_1 becomes a descending u-tree. Assign labels $n, \ldots, 2n-2$ to the edges of T_2 in such a way that T_2 becomes an ascending u-tree. Then in $T_1 \cup T_2$ there is an ascending (v_1, v_2)-trail containing u for all $v_1, v_2 \in V(G)$. Hence G is label-connected. A slight refinement of this argument yields the following results, stated in [5] within the context of the gossip problem.

Theorem 2 ([5]). *If a graph G contains a spanning tree T and a spanning tree-pair S with $E(T) \cap E(S) = \emptyset$, then G is label-connected.*

Theorem 3 ([5]). *If a graph G contains two edge-disjoint spanning tree-pairs S_1 and S_2 such that there exists a 4-cycle $uvwxu$ with uv, wx in distinct components of S_1 and vw, xu in distinct components of S_2, then G is label-connected.*

Note that if a graph G with n vertices satisfies the hypothesis of Theorem 2, then $|E(G)| \geq 2n-3$, whereas $|E(G)|$ may equal $2n-4$ if G satisfies the hypothesis of Theorem 3. If G satisfies the hypothesis of Theorem 3 and $G = S_1 \cup S_2$, then G is called a *C_4-graph*. In particular, C_4 is a C_4-graph. Theorem 9 below states that the label-connected graphs with n vertices and $2n-4$ edges are exactly the C_4-graphs, thus improving the second part of Theorem 1.

3. A sufficient condition and a necessary condition

By $\phi(G)$ we denote the minimum number of edges that have to be added to $E(G)$ in order to create a graph with two edge-disjoint spanning trees. In particular, $\phi(G) = 0$ if and only if G has two edge-disjoint spanning trees. An immediate consequence of Theorem 2 is the following.

Corollary 4. *If G is a graph with $\phi(G) \leq 1$, then G is label-connected.*

On the other hand the following is true.

Theorem 5. *If G is a label-connected graph, then $\phi(G) \leq 2$.*

For our proof of Theorem 5 we need some preliminary definitions and results.

Following [1], we say that an edge of a graph is *contracted* if it is deleted and its ends are identified. When an edge is contracted, multiple edges may arise; such edges are not identified. On the other hand, loops are deleted whenever they arise. If a graph H can be obtained from a graph G by a (possibly empty) sequence of edge-contractions, then we call H a *contraction* of G and write

$H \leqq G$. We define

$$\phi'(G) = \max_{H \leqq G}\{2(|V(H)| - 1) - |E(H)|\}.$$

Equivalently we have

$$\phi'(G) = \max_{F \subseteq E(G)}\{2(\omega(G - F) - 1) - |F|\},$$

where $\omega(H)$ denotes the number of components of a graph H. Taking $F = \emptyset$ shows that $\phi'(G) \geqslant 0$ for every graph G.

It is easily seen that if a graph G has two edge-disjoint spanning trees, then so has every contraction of G, implying that $|E(H)| \geqslant 2(|V(H)| - 1)$ for all $H \leqslant G$. Hence, if $\phi(G) = 0$, then $\phi'(G) = 0$. The converse was proved independently by Tutte [10] and Nash-Williams [9]. More generally, the following holds.

Theorem 6. *$\phi(G) = \phi'(G)$ for every graph G.*

The nontrivial part of the proof of Theorem 6 follows immediately from a result of Catlin.

Lemma 7 ([3]). *If G is a connected graph, then G has edge-disjoint spanning forests T and U such that $\omega(T) = 1$ and $\omega(U) = \phi'(G) + 1$.*

Proof of Theorem 6. We only prove the theorem for connected graphs. The result is then easily extended to disconnected graphs. Let G be a connected graph and H_0 a contraction of G such that $\phi'(G) = 2(|V(H_0)| - 1) - |E(H_0)|$. Then $\phi(G) \geqslant \phi(H_0) \geqslant 2(|V(H_0)| - 1) - |E(H_0)| = \phi'(G)$. On the other hand Lemma 7 immediately gives $\phi(G) \leqslant \phi'(G)$. □

Lemma 8. *Every contraction of a label-connected graph is label-connected.*

Proof. Let G be a label-connected graph, H a contraction of G and f an admissible labeling of G. Then the restriction of f to the edges of H is an admissible labeling of H. □

Proof of Theorem 5. Let G be a label-connected graph. Combination of Lemma 8 and Theorem 1 yields $|E(H)| \geqslant 2|V(H)| - 4$ for all $H \leqslant G$, or equivalently, $2(|V(H)| - 1) - |E(H)| \leqslant 2$. Hence $\phi'(G) \leqslant 2$. Application of Theorem 6 completes the proof. □

The maximum number of edge-disjoint spanning trees of a graph G can be computed in polynomial time by matroid partitioning algorithms [8]. Hence, in particular, it can be checked in polynomial time whether $\phi(G) = 0$. Since $\phi(G) \leqslant i$ if and only if there is an edge set F with $|F| = i$ such that $F \cap E(G) = \emptyset$ and $\phi(G + F) = 0$, the sufficient condition of Corollary 4 and the necessary condition of Theorem 5 can also be checked in polynomial time. Although the two conditions are quite close, we prove in Section 7 that recognizing label-connected graphs is an NP-complete problem. Note that by Corollary 4 and Theorem 5 the problem is hard only for graphs G with $\phi(G) = 2$.

4. A well-solvable case

A special case in which it can be decided in polynomial time whether a graph G on n vertices with $\phi(G) = 2$ is label-connected, occurs when $|E(G)| = 2n - 4$. The proof of the following result uses Lemma 8 and relies heavily on assertions in the proof of [7, Theorem 2] and on [2, Theorem 3].

Theorem 9. *Let G be a graph with n vertices and $2n - 4$ edges. Then G is label-connected if and only if G is a C_4-graph.*

Proof. C_4-graphs are label-connected by Theorem 3.

Conversely, assume G is a label-connected graph with n vertices and $2n - 4$ edges. Then $n \geqslant 4$. For a partition $\{E_1, E_2\}$ of $E(G)$ with $|E_1| = |E_2| = n - 2$, let $G_1 = G[E_1]$ and $G_2 = G[E_2]$. At least two components of G_i are trees $(i = 1, 2)$. Let S_1, S_2 be two tree components of G_1 and T_1, T_2 two tree components of G_2. By arguments in the proof of [7, Theorem 2], $\{E_1, E_2\}$ can be chosen to have the following two properties:

(1) $G_2 = T_1 \cup T_2$.

(2) There exists a 4-cycle $x_1y_1y_2x_2x_1$ with $x_1x_2 \in E(S_1)$, $y_1y_2 \in E(S_2)$, $x_1y_1 \in E(T_1)$ and $x_2y_2 \in E(T_2)$.

By [2, Theorem 3], among the partitions $\{E_1, E_2\}$ of $E(G)$ satisfying (1) and (2) there is one that also has one of the following properties:

(3) $G_1 = S_1 \cup S_2$.

(3′) Apart from S_1 and S_2 the graph G_1 contains exactly one component U, which is unicyclic.

G is a C_4-graph if (and only if) there is a partition of $E(G)$ satisfying (1), (2) and (3). Assume no partition of $E(G)$ satisfies (1), (2) and (3) and $\{E_1, E_2\}$ is a partition satisfying (1), (2) and (3′) for which $|V(U)|$ is minimum.

Suppose $G_2[V(U)]$ is connected. Then $|E(G[V(U)])| = 2|V(U)| - 1$. Obtain G' from G by contracting $G[V(U)]$ to a single vertex (i.e., by contracting all edges of a spanning tree of $G[V(U)]$). Then $|V(G')| = |V(G)| - |V(U)| + 1$ and $|E(G')| = |E(G)| - |E(G[V(U)])| = |E(G)| - 2|V(U)| + 1$. Since $|E(G)| = 2|V(G)| - 4$, it follows that $|E(G')| = 2|V(G')| - 5$. By Lemma 8, G' is label-connected. This contradiction with Theorem 1 shows that, in fact, $G_2[V(U)]$ is disconnected.

Let u_1u_2 be an edge of U such that u_1 and u_2 are in different components of $G_2[V(U)]$ and let F be the set of all edges of G with exactly one end in $V(U)$. Note that $F \subseteq E(G_2)$. The proof is now completed by deriving contradictions in two cases.

Case 1: $G_2 + u_1u_2$ *is connected.*

Then $G_2 + u_1u_2$ is a spanning tree of G. Let P be the unique (x_1, x_2)-path in $G_2 + u_1u_2$, where x_1, x_2 are the vertices referred to in (2). Since x_1 and x_2 are in different components of G_2, P contains u_1u_2 and hence at least two edges of F. The edge u_1u_2 does not belong to the unique cycle of U, otherwise the partition $\{(E_1 \cup \{e\}) - \{u_1u_2\}, (E_2 \cup \{u_1u_2\}) - \{e\}\}$, where e is an arbitrary edge in $F \cap E(P)$, satisfies (1), (2) and (3), contradicting our assumptions. Hence u_1u_2 is a cut edge of U. Let U_1 be the component of $U - u_1u_2$ containing the unique cycle of U, and U_2 the other component (which is a tree). The set $F \cap E(P)$ contains an edge e_1 incident with a vertex of U_2. Since $|V(U_1)| < |V(U)|$, the partition $\{E_1', E_2'\}$ of $E(G)$ with $E_1' = (E_1 \cup \{e_1\}) - \{u_1u_2\}$ now contradicts the minimality of $|V(U)|$ in the choice of $\{E_1, E_2\}$.

Case 2: $G_2 + u_1u_2$ *is disconnected.*

Then $G_2 + u_1u_2$ contains a unique cycle C. By the way u_1u_2 was chosen, C contains at least one vertex not in U and hence at least two edges of F. With P replaced by C, the rest of the argument can be copied from Case 1. □

5. Characterization by properties of blocks

The following result implies a characterization of label-connected graphs in terms of properties of their blocks.

Theorem 10. *Let G_1 and G_2 be connected graphs with $|V(G_1) \cap V(G_2)| = 1$ and $E(G_1) \cap E(G_2) = \emptyset$. Then $G_1 \cup G_2$ is label-connected if and only if there exists an integer $i \in \{1, 2\}$ such that G_i is label-connected and $\phi(G_{3-i}) = 0$.*

Proof. Let G_1 and G_2 be edge-disjoint connected graphs with $V(G_1) \cap V(G_2) = \{u\}$.

To establish sufficiency, assume without loss of generality that G_1 is label-connected and $\phi(G_2) = 0$. Set $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$ $(i = 1, 2)$. Let T_1 and T_2 be two edge-disjoint spanning trees of G_2. Choose an admissible labeling of G_1 with labels $n_2, \dots, n_2 + m_1 - 1$. Assign labels $1, \dots, n_2 - 1$ to $E(T_1)$ and labels $n_2 + m_1, \dots, 2n_2 + m_1 - 1$ to $E(T_2)$ in such a way that T_1 becomes a descending u-tree and T_2 an ascending u-tree. Having thus obtained an admissible labeling of a spanning subgraph of $G_1 \cup G_2$, we conclude that $G_1 \cup G_2$ is label-connected.

Conversely, assume $G_1 \cup G_2$ is label-connected. Since G_1 and G_2 are contractions of $G_1 \cup G_2$, G_1 and G_2 are label-connected by Lemma 8. If $\phi(G_1) = \phi(G_2) = 0$, then we are done. Hence assume, without loss of generality, that $\phi(G_2) > 0$. Let f be an admissible labeling of $G_1 \cup G_2$. Since there exists an ascending (v, u)-path for all $v \in V(G_1)$, G_1 contains a spanning descending u-tree. Defining $\mu(H) = \max\{f(e) \mid e \in E(H)\}$ for a subgraph H of $G_1 \cup G_2$, let S_1 be a spanning descending u-tree of G_1 such that $\mu(S_1)$ is as small as possible. The choice of S_1 implies that S_1 contains a vertex u_1 such that every ascending (u_1, u)-path contains an edge with label $\mu(S_1)$. Since there is an ascending (u_1, v)-path for all $v \in V(G_2)$, G_2 must contain a spanning ascending u-tree T_2 such that $f(e) > \mu(S_1)$ for all $e \in E(T_2)$. Analogously, G_2 contains a spanning descending u-tree S_2 and G_1 a spanning ascending u-tree T_1 such that $f(e) > \mu(S_2)$ for all $e \in E(T_1)$. Now since $\phi(G_2) > 0$, S_2 and T_2 have a common edge e_0. Hence, whenever $e_1 \in E(S_1)$ and $e_2 \in E(T_1)$, we have $f(e_1) < f(e_0) \leqslant \mu(S_2) < f(e_2)$. It follows that S_1 and T_1 are edge-disjoint, so $\phi(G_1) = 0$. □

Corollary 11. *For a connected graph G the following two statements are equivalent.*

(i) *G is label-connected.*

(ii) *At most one block of G does not have two edge-disjoint spanning trees. If such a block exists, it is label-connected.*

Corollary 11 generalizes a result of Harary and Schwenk [6]. We call the graph obtained from a graph G by replacing all multiple edges by single edges the *underlying simple graph* of G.

Corollary 12 ([6]). *Let T be a nontrivial tree with n vertices. If a label-connected graph G has T as its underlying simple graph, then* $|E(G)| \geqslant 2n - 3$. *There exist label-connected graphs with exactly* $2n - 3$ *edges that have T as their underlying simple graph.*

6. Constructions

The following results yield two ways of constructing new label-connected graphs out of known ones.

Theorem 13. *Let* v_1 *and* v_2 *be vertices of a label-connected graph G. Obtain the graph G' from G by adding a new vertex u and the edges* uv_1 *and* uv_2. *Then G' is label-connected.*

Proof. Let f be an admissible labeling of G using the labels $1, \ldots, m$, where $m = |E(G)|$. Extend f to a labeling f' of G' by assigning label 0 to uv_1 and label $m + 1$ to uv_2. Then f' is an admissible labeling of G'. □

Theorem 14. *If G and H are label-connected graphs, then the graph* $G \times H$ *is label-connected.*

Proof. Set $m = |E(G)|$ and $k = |E(H)|$. Assign identical admissible labelings to all copies of G in $G \times H$, using labels $1, \ldots, m$. Also assign identical admissible labelings to all copies of H in $G \times H$, now using labels $m+1, \ldots, m+k$. The resulting labeling of $G \times H$ is admissible. □

7. NP-completeness

In our proof that recognizing label-connected graphs is an NP-complete problem we use the NP-completeness of the Satisfiability problem (SAT). We start with a description of this problem.

Let $X = \{x_1, \ldots, x_m\}$ be a set of boolean variables. Following [4], we call a function $t: X \to \{T, F\}$ a *truth assignment* for X. If $t(x) = T$, we say that x is true under t; if $t(x) = F$, we say that x is false. If x is a variable in X, then x and $\bar{x}$ are *literals* over X. The literal x is true under t if and only if the variable x is true under t; the literal $\bar{x}$ is true if and only if the variable x is false. A *clause* over X is a set of literals over X. It represents the disjunction of those literals and is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment. If A is a collection of clauses over X, then a truth assignment for X that simultaneously satisfies all clauses in A is called a *satisfying truth assignment* for A. Hence a satisfying truth assignment can be viewed as a solution of a set of simultaneous boolean equations.

We are now ready to state SAT.

SAT
Instance. Set X of variables, collection A of clauses over X.
Question. Is there a satisfying truth assignment for A?

We state three more problems.

PT
Instance. Graph G, vertices u and v of G.
Question. Does G contain a (u, v)-path P such that $G - E(P)$ is connected?

Equivalent to PT is the question whether G contains a (u, v)-path P and a spanning tree T such that $E(P) \cap E(T) = \emptyset$.

PTT
Instance. Graph G, vertices u and v of G.
Question. Does G contain a (u, v)-path P and two spanning trees T_1, T_2 such that P, T_1 and T_2 are pairwise edge-disjoint?

LC
Instance. Graph G.
Question. Is G label-connected?

Clearly, all three problems are in NP. We now show that they are NP-complete by transforming SAT to PT, PT to PTT, and PTT to LC.

Theorem 15. PT *is* NP-*complete.*

Proof. We transform SAT to PT. Let $X = \{x_1, \ldots, x_m\}$ be a set of boolean variables and $A = \{a_1, \ldots, a_n\}$ a collection of clauses over X. The NP-completeness of PT will be established by exhibiting a graph G with two vertices u and v such that G contains a (u, v)-path P for which $G - E(P)$ is connected if and only if there is a satisfying truth assignment for A. The structure of G should be clear from the example in Fig. 1 and the partial description below.

For each clause a_i $(i = 1, \ldots, n)$ there is a subset A_i of $V(G)$ (nonsolid vertices on the left in Fig. 1); there is a vertex a_{ij} in A_i when the variable x_j occurs in a_i. On the other hand, for each variable x_i $(i = 1, \ldots, m)$ there is a subset X_i of $V(G)$ (nonsolid vertices in the middle of Fig. 1); the vertices of X_i occur in pairs (x_{ij1}, x_{ij2}), one pair for each clause a_j containing the variable x_i. The vertices x_{ij1} and x_{ij2} of X_i are both adjacent to the vertex a_{ji} of A_j; they are called T-*vertices* if A_j contains the literal $\bar{x}_i$, and F-*vertices* if A_j contains the literal x_i. (In Fig. 1 T-vertices appear on the left side of u, F-vertices on the right.)

Now there is a bijection between truth assignments for X and (u, v)-paths consisting of nonsolid vertices only; if $t: X \to \{T, F\}$ is a truth assignment, then the corresponding (u, v)-path contains the $t(x_i)$-vertices of X_i $(i = 1, \ldots, m)$.

Let t be a truth assignment for X and P the corresponding (u, v)-path. Due to the presence of the solid vertices, $G - E(P)$ is disconnected if and only if there exists an integer $i \in \{1, \ldots, n\}$ such that P contains all vertices of A_i. Now P contains all vertices of A_i if and only if every literal in a_i is false under t. It follows that $G - E(P)$ is disconnected if and only if t is not a satisfying truth assignment for A.

Noting that $G - E(Q)$ is disconnected for every (u, v)-path Q that contains a solid vertex, we conclude that there exists a satisfying truth assignment for A if and only if there exists a (u, v)-path P in G such that $G - E(P)$ is connected. Given that the construction of the graph G can be carried out in polynomial time, the result follows. □

Theorem 16. PTT *is* NP-*complete.*

Proof. We transform PT to PTT. Let G be a graph and u and v two vertices of G. Obtain the graph G' from G by first subdividing each edge of G and then duplicating each edge of the resulting graph. See Fig. 2(a). (If one wishes to establish the NP-completeness of PTT within the class of simple graphs, then the transformation of Fig. 2(b) will do.)

There is a natural correspondence between (u, v)-paths in G and (u, v)-paths

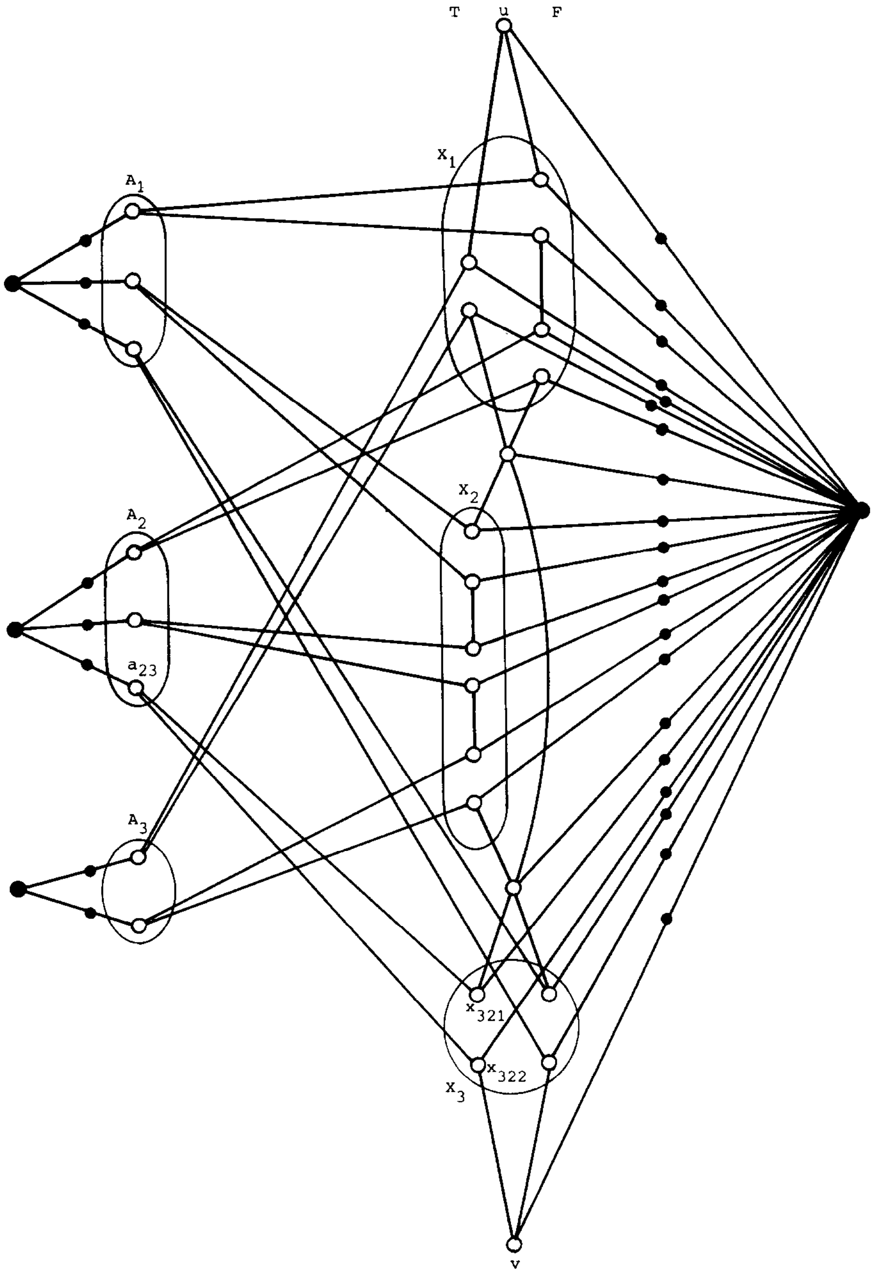

Fig. 1. The graph G in case $X=\{x_1, x_2, x_3\}$, $A=\{a_1, a_2, a_3\}$, $a_1=\{x_1, \bar{x}_2, x_3\}$, $a_2=\{x_1, \bar{x}_2, \bar{x}_3\}$, $a_3=\{\bar{x}_1, \bar{x}_2\}$.

Fig. 2.

in G' in the sense that every (u, v)-path in G' is a subdivision of a (u, v)-path in G. Let P and P' be corresponding (u, v)-paths in G and G', respectively. Noting that G' can be constructed from G in polynomial time we establish the NP-completeness of PTT by showing that $G - E(P)$ is connected if and only if $G' - E(P')$ has two edge-disjoint spanning trees.

If $G - E(P)$ is connected, then clearly $G' - E(P')$ has two edge-disjoint spanning trees.

Conversely, assume $G - E(P)$ is disconnected. Let S be the set of vertices in G' that subdivide the edges of P. Every vertex of S has degree 2 in $G' - E(P')$. If F is any subset of $E(G')$ obtained by selecting, for each vertex of S, exactly one incident edge, then $G' - (E(P') \cup F)$ is disconnected as $G - E(P)$ is. An arbitrary spanning tree T of $G' - E(P')$ contains such an edge set F, implying that $G' - (E(P') \cup E(T))$ is disconnected. It follows that $G' - E(P')$ does not contain two edge-disjoint spanning trees. □

Theorem 17. LC *is* NP-*complete.*

Proof. We transform PTT to LC. Let G be a graph and u and v two vertices of G. Obtain the graph G' from G by adding three new vertices x_1, x_2, x_3 and the edges $ux_1, x_1x_2, x_2x_3, x_3v$. We establish the NP-completeness of LC by showing that G contains a (u, v)-path P with $\phi(G - E(P)) = 0$ if and only if G' is label-connected.

Assume first that G contains a (u, v)-path P with $\phi(G - E(P)) = 0$. Let T_1 and T_2 be two edge-disjoint spanning trees of $G - E(P)$. We describe an admissible labeling of a spanning subgraph of G'. Assign label 1 to ux_1 and x_2x_3, label 2 to x_1x_2 and x_3v. Label T_1 as a descending u-tree using labels smaller than 1. Label T_2 as an ascending v-tree using labels greater than 2. Label P as an ascending path using labels greater than 1 and smaller than 2. It follows that G' is label-connected.

Conversely, assume G' is label-connected and let f be an admissible labeling of G'. Set $r_1 = f(ux_1)$, $r_2 = f(x_1x_2)$, $r_3 = f(x_2x_3)$ and $r_4 = f(x_3v)$. From the existence of an ascending (x, y)-path for all $x, y \in \{u, x_1, x_2, x_3, v\}$ one easily deduces that either $\max\{r_1, r_3\} < \min\{r_2, r_4\}$ or $\max\{r_2, r_4\} < \min\{r_1, r_3\}$. Assume without loss of generality that $\max\{r_1, r_3\} < \min\{r_2, r_4\}$. Then for all $w \in V(G)$ an ascending (w, x_1)-path has ux_1 as its last edge. Hence G contains a descending u-tree T_1 with $f(e) < r_1$ for all $e \in E(T_1)$. Similarly, the existence of an ascending (x_3, w)-path for all $w \in V(G)$ implies that G contains an ascending v-tree T_2 with $f(e) > r_4$ for all

$e \in E(T_2)$. An ascending (x_1, x_3)-path necessarily contains x_1u and vx_3. Hence G contains an ascending (u, v)-path P with $r_1 < f(e) < r_4$ for all $e \in E(P)$. Since the sets of labels used for T_1, T_2 and P are pairwise disjoint, T_1, T_2 and P are pairwise edge-disjoint. □

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