THE SQUEEZE EFFECT IN NON-INTEGRABLE HAMILTONIAN SYSTEMS

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Received 15 May 1987

In non-integrable Hamiltonian systems (represented by mappings of the plane) the stable island around an elliptic fixed point is generally squeezed into the fixed point by three saddle points, when the rotation number ρ of the motion at the fixed point approaches $\frac{1}{3}$. At $\rho = \frac{1}{3}$ the island is reduced to one single point.

A detailed investigation of this squeeze effect, and some of its global implications, is presented by means of a typical two-dimensional area-preserving map. In particular, it turns out that the squeeze effect occurs in any mapping for which the Taylor expansion around the fixed point contains a quadratic term, whereas it does not occur if the first non-linear term is cubic. We illustrate this with two physical examples: a compass needle in an oscillating magnetic field, showing the squeeze effect, and a ball which bounces on a vibrating plane, for which the squeeze effect does not occur.

1. Introduction

Most Hamiltonian systems of coupled differential equations are non-integrable^{1,2}). That is, there are less constants of the motion than there are degrees of freedom. Take for instance an autonomous system (for which the Hamiltonian itself is a constant of the motion) with two degrees of freedom, or a non-autonomous system with one degree of freedom. A convenient way to study these systems is to make a two-dimensional section through the phase space and to look at the successive intersections of the orbit with the surface of section. This generates a two-dimensional map

$$\begin{cases} x_{n+1} = f(x_n, y_n, C), & (1.1a) \\ y_{n+1} = g(x_n, y_n, C), & (1.1b) \end{cases}$$

which must be area-preserving as a consequence of the Hamiltonian (non-dissipative) nature of the system, i.e.

$$\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} = 1. \quad (1.2)$$

The parameter C is an essential ingredient of the map (1.1). For an autonomous system C usually represents the energy. In a non-autonomous system C will be related to the amplitude and/or frequency of the external force. If we vary the value of C the system will show a rich variety in (regular and chaotic) behaviour. In this paper we will focus upon one special feature, which we shall call the squeeze effect, and which appears in many physical systems.

In order to describe the squeeze effect explicitly, we take (1.1) to be of the form

$$\begin{cases} x_{n+1} = 2Cx_n - y_n + \sum_{k \geq 2} a_k x_n^k, \\ y_{n+1} = x_n. \end{cases} \quad (1.3a)$$

$$(1.3b)$$

For this map the origin is a fixed point and its stability is governed by the linearized map with eigenvalues

$$\lambda_{\pm} = C \pm \sqrt{C^2 - 1}. \quad (1.4)$$

Note that $\lambda_+ \lambda_- = 1$, in agreement with the area preservation. For $-1 \leq C \leq 1$ the eigenvalues lie on the unit circle, complex conjugate to each other. In that case the origin is elliptic. It is convenient to write the eigenvalues as $\lambda_{\pm} = \exp(\pm 2\pi i \rho_0)$, where

$$\rho_0 = \frac{1}{2\pi} \arccos C \quad (1.5)$$

is the (local) rotation number of the elliptic motion around the origin.

For rational values of ρ_0 , i.e. for $\rho_0 = p/q$ with p and q relatively prime integers, generically two period q cycles (one elliptic and one hyperbolic) with rotation number p/q bifurcate at the origin. This happens at $C = C_{p/q}$, with

$$C_{p/q} = \cos(2\pi p/q). \quad (1.6)$$

Thus, as C is decreased from 1 to -1 the origin undergoes an infinite number of p/q resonances at successive values $C_{p/q}$. At $C = -1$ (where $p/q = \frac{1}{2}$) the

fixed point bifurcates into a 2 cycle. For $C < -1$ the origin is an unstable saddle point with real eigenvalues $\lambda_+ = 1/\lambda_-$. The trajectories of the eigenvalues through the complex plane are shown in fig. 1.

Typically, an elliptic fixed point is (arbitrarily close to the fixed point) surrounded by an assembly of KAM curves, periodic and other bounded (chaotic) orbits. Together they form a stable island. However, at the $\frac{1}{3}$ resonance this island does not exist anymore (see ref. 3, page 222). In fact, when C goes to $C_{1/3}$ a hyperbolic 3 cycle moves towards the origin and reaches it at $C = C_{1/3}$, thereby squeezing the enclosed KAM curves and other orbits into the origin. We call this the squeeze effect. Under certain conditions this squeezing also occurs at the $1/4$ resonance (see ref. 4, page 307, and also refs. 5 and 6). We will see that the $1/3$ squeeze effect is generic for all area-preserving maps for which the lowest order non-linearity (locally around the fixed point) is quadratic. If the dominant non-linear term is cubic, then the squeeze effect does not occur.

The process in which orbits are absorbed by the origin for decreasing C takes place in a range $C_0 > C > C_{1/3}$. For the system (1.3) we derive an expression for C_0 , which tends to $C_{1/3}$ if the coefficient of the quadratic term is very small compared to the coefficient of the cubic term.

The squeeze effect (for $p/q = 1/3$) can be observed in a number of physical systems⁷⁻⁹). In section 2 we will discuss a particularly simple example: a compass needle in an oscillating magnetic field¹⁰), which is a non-autonomous Hamiltonian system with one degree of freedom. In section 3 we present a physical example of such a system: a ball bouncing on a vibrating plane^{11,12}). In section 4 we discuss the squeeze effect on the basis of a bifurcation analysis.

Fig. 1. Trajectories of λ_+ and λ_- through the complex plane. The dominant resonances p/q up to $q = 6$ are indicated along the unit circle.

Finally, in section 5, we apply the results to a typical mapping. In particular, we describe the global aspects of the squeeze effect.

2. The compass needle

Consider a compass needle in a homogeneous magnetic field which is oscillating in time, $B = B_0 \sin \omega t$. This system¹⁰⁾ is shown in fig. 2. Let μ be the dipole moment of the compass needle, and I its moment of inertia. Then the Hamiltonian is

$$H(\theta, p, t) = \frac{p^2}{2I} + \mu B_0 \sin \omega t \cos \theta, \quad (2.1)$$

where θ is the angle between the compass needle and the magnetic field direction, and $p = I d\theta/dt$ is the angular momentum.

The motion through the three-dimensional phase space (with coordinates θ , p and t) is governed by the Hamilton equations:

$$\left\{ \begin{array}{l} \frac{d\theta}{dt} = \frac{\partial H}{\partial p} = \frac{p}{I}, \\ \frac{dp}{dt} = -\frac{\partial H}{\partial \theta} = \mu B_0 \sin \omega t \sin \theta. \end{array} \right. \quad (2.2a)$$

$$\left\{ \begin{array}{l} \frac{d\theta}{dt} = \frac{\partial H}{\partial p} = \frac{p}{I}, \\ \frac{dp}{dt} = -\frac{\partial H}{\partial \theta} = \mu B_0 \sin \omega t \sin \theta. \end{array} \right. \quad (2.2b)$$

The equations are periodic in θ and in t , with period 2π and $T = 2\pi/\omega$, respectively. Thus it is convenient to normalize the system in such a way that $\theta \rightarrow x = \theta/2\pi$, $p \rightarrow y = p/I\omega$ and $t \rightarrow \tau = (\omega/2\pi)t$, making the system periodic in x and τ with period 1. Then the Hamilton equations (2.2) become

$$\left\{ \begin{array}{l} \frac{dx}{d\tau} = y, \\ \frac{dy}{d\tau} = A \sin(2\pi\tau) \sin(2\pi x), \end{array} \right. \quad (2.3a)$$

$$\left\{ \begin{array}{l} \frac{dx}{d\tau} = y, \\ \frac{dy}{d\tau} = A \sin(2\pi\tau) \sin(2\pi x), \end{array} \right. \quad (2.3b)$$

Fig. 2. A compass needle moving under the influence of a homogeneous, alternating magnetic field.

where A is given by

$$A = \frac{2\pi\mu B_0}{I\omega^2}. \quad (2.4)$$

For $A=0$ the system is integrable, and for growing A the non-integrable behaviour becomes more and more apparent. Therefore we call A the chaos parameter.

Solving eqs. (2.3) numerically, a natural surface of section can then be made by concentrating only on the discrete times $\tau = 0, 1, 2, \dots$. This gives an area preserving map like (1.1). In an experimental setup this map can be generated by a stroboscopic light which is synchronized to the magnetic field, measuring θ (or $x = \theta/2\pi$) at each flash. The experiment has actually been done by Meissner and Schmidt¹⁰⁾.*

In figs. 3–5 we show the surface of section plots at three different values of A . In each case we used a fourth order Runge–Kutta scheme to do the numerical integration of (2.3), with 50 steps per section. Note that the section patterns are point-symmetric with respect to the point $(x, y) = (\frac{1}{2}, 0)$. This reflects the fact that the Hamilton equations are invariant under the joint transformation $x \rightarrow 1-x$, $y \rightarrow -y$. The patterns are also symmetric with respect to the vertical lines $x = \frac{1}{4}$ and $x = \frac{3}{4}$, corresponding to the $x \rightarrow \frac{1}{2} - x$, $\tau \rightarrow -\tau$ (and $x \rightarrow \frac{3}{2} - x$, $\tau \rightarrow -\tau$) invariance of the system.

In fig. 3, at $A=3$, the islands of two elliptic 1 cycles (surrounded by secondary cycles of period 5) are clearly visible. We also see two blank regions in the chaotic sea around $y=0$, indicating the presence of an elliptic 2 cycle. It should be noted that these cycles appear in the form of Poincaré–Birkhoff chains¹⁾; a cycle of elliptic islands (except those born by a period doubling bifurcation) is interlaced with a hyperbolic cycle of the same period.

In fig. 4a, at $A=5.5$, we see that the island of the 1 cycle is being squeezed by a hyperbolic 3 cycle. This is shown in more detail in fig. 4b. The stable and unstable separatrices of the three saddle points mark the boundary of the island. The squeezing is complete at a slightly lower value of A , when the three saddle points coincide with the central fixed point.

In fig. 5, at $A=7$, the island of the (still stable) 1 cycle has grown again. The squeeze effect has just been a peculiar intermezzo during the stable life of the 1 cycle. This stable life does not end before $A \approx 8.6$, when the 1 cycle bifurcates into a 2 cycle, which in turn bifurcates into a 4 cycle at $A \approx 10.9$, and so on. This is the beginning of the well known period doubling sequence to chaos^{13,14)}.

* Of course in a real experiment one always has some dissipation, and strictly speaking the Hamiltonian system described by (2.1) cannot be realized. Meissner and Schmidt call their system “near-Hamiltonian”.

Fig. 3. Surface of section plot for the compass system (3a)–(3b), with $A = 3$, showing 11 different orbits. All the points in the chaotic sea belong to one single orbit.

Fig. 4a. Surface of section plot for the compass system (3a)–(3b), with $A = 5.5$. All the points in the chaotic sea belong to one single orbit.

Fig. 4b. Detail of fig. 4a. The island of the 1 cycle is squeezed by the hyperbolic 3 cycle, which we have indicated by dots.

Fig. 5. Surface of section plot for the compass system (3a)–(3b), with $A = 7$. All the points in the chaotic sea belong to one orbit.

The squeeze effect is a *local* effect around an elliptic fixed point (although it has global implications, see section 5). Therefore, to make a generic model for it we do not need all the details of the mapping, but only the dominant local behaviour. Usually, the dominant non-linear terms of the mapping will be quadratic. In that case, the map can always be transformed (up to quadratic terms) in the following standard form^{15,16}):

$$\begin{cases} x_{n+1} = 2Cx_n + 2x_n^2 - y_n, \\ y_{n+1} = x_n, \end{cases} \quad (2.5a)$$

$$\begin{cases} x_{n+1} = 2Cx_n + 2x_n^2 - y_n, \\ y_{n+1} = x_n, \end{cases} \quad (2.5b)$$

cf. (1.3). As discussed in section 1 we have an elliptic fixed point in the origin for $-1 < C < 1$. At $C = -1$ this fixed point becomes unstable, bifurcating into a 2 cycle, which bifurcates into a 4 cycle at $C = 1 - \sqrt{5} = -1.23607$, and so on.

In order to describe the squeeze effect we now concentrate on the 3 cycles. There are two of them, given by

$$x_0 = -\frac{1}{2}C \pm \frac{1}{2}\sqrt{C^2 - 2C - 1}, \quad y_0 = x_0; \quad (2.6a)$$

$$x_1 = -(x_0 + C + \frac{1}{2}) \quad , \quad y_1 = x_0; \quad (2.6b)$$

$$x_2 = x_0 \quad , \quad y_2 = x_1; \quad (2.6c)$$

which we denote by the *plus* cycle and the *minus* cycle, respectively. It may be noted^{14,17}) that these cycles have one point on each of the three symmetry curves $x = y$, $y = Cx + x^2$ and $x = Cy + y^2$. One easily checks that the surface of section pattern of (2.5) must be symmetric for reflection with respect to these curves.

The two 3 cycles (2.6) only exist in the real plane if $C^2 - 2C - 1 \geq 0$, i.e. for*

$$C \leq 1 - \sqrt{2} = -0.41421. \quad (2.7)$$

They are born together at $C = 1 - \sqrt{2}$, as twins, and move apart (along the symmetry curves) as C is lowered. The *plus* cycle moves outward, away from the origin. For $-\frac{1}{2} < C < 1 - \sqrt{2}$ it is stable (elliptic), but at $C = -\frac{1}{2}$ it becomes unstable and bifurcates into a 6 cycle, which in turn bifurcates into a 12 cycle at $C = -0.51060$, and so on.

* Or for $C \geq 1 + \sqrt{2}$. Incidentally, all phenomena for (2.5) appear twice, at C values that lie symmetric with respect to $C = 1$ ¹⁶). So we may restrict ourselves to $C \leq 1$.

The *minus* cycle (consisting of three saddle points) moves inward as C is lowered, thereby squeezing the island of the fixed point in the origin. In figs. 6–8 we show the situation. In fig. 6, at $C = 1 - \sqrt{2} = -0.41421$, we see the 1 cycle at the origin while the two 3 cycles are just being born (as twins) at the points of the sharply-pointed triangular orbit, well within the island of the 1 cycle.

We have indicated the three symmetry curves $x = y$, $y = Cx + x^2$ and $x = Cy + y^2$ by dashed lines. Note that the points in the chaotic region outside the island escape to infinity for the map (2.5), unlike those for the (periodic) compass system in figs. 3–5. Thus, the chaotic “sea” is quite white.

In fig. 7, at $C = -0.43$, the two 3 cycles have moved apart. The unstable minus cycle, indicated by three arrows, closes in upon the origin.

In figs. 8a–c we concentrate on the squeeze effect at the origin. In fig. 8a, at $C = -0.499$, we see that the three saddle points have nearly reached the origin. For decreasing C they move as indicated by the arrows.

In fig. 8b, at $C = -0.5$, the squeezing is complete. The three saddle points coincide at the origin, forming a so-called monkey saddle (“Affensattel-punkt”). Meanwhile, the elliptic 3 cycle (outside the figure) is on the verge of bifurcating into a 6 cycle.

In fig. 8c, at $C = -0.501$, we see that the three saddle points have gone through the origin and are now heading away from it along the arrows. In their wake the (still stable) 1 cycle gets an island again. Note that the orientation of the triangle enclosed by the saddle points has been reversed.

Although figs. 6–8 have been worked out for the specific map (2.5), the squeeze effect (i.e. the local properties around the fixed point at C values close

Fig. 6. The quadratic map (2.5) for $C = 1 - \sqrt{2} = -0.41421$. The 3 cycles are just being born by a tangent bifurcation, at the cusp-shaped corners of the sharp-pointed triangle within the island of the 1 cycle. The dashed lines are the symmetry curves $x = y$, $y = Cx + x^2$ and $x = Cy + y^2$.

Fig. 7. The quadratic map (2.5) for $C = -0.43$. The elliptic 3 cycle moves away from the origin as C is lowered, while the hyperbolic 3 cycle (indicated by arrows) closes in upon the origin. The dashed lines are the symmetry curves $x = y$, $y = Cx + x^2$ and $x = Cy + y^2$.

Fig. 8a. The quadratic map (2.5) for $C = -0.499$, just before the complete squeeze. For decreasing C the three saddle points move towards the origin as indicated by the arrows.

Fig. 8b. The quadratic map (2.5) for $C = -0.5$, where the squeezing of the origin is complete. The three saddle points coincide with the origin, which is still (marginally) stable, reducing the island of the 1 cycle to one single point. The origin is now a monkey saddle, with three incoming and three outgoing arms.

Fig. 8c. The quadratic map (2.5) for $C = -0.501$, just after the complete squeeze. For decreasing C the three saddle points move away from the origin as indicated by the arrows.

to $-\frac{1}{2}$) has universal validity for any map with a non-vanishing quadratic part. In fact, the cubic and higher order terms of such a map may be neglected in a small neighbourhood of the origin. On the other hand, the global phenomena (outside this small neighbourhood) *do* depend on the details of the mapping.

For the quadratic map (2.5) we do not have a squeeze effect at the $1/4$ resonance, which occurs at $C = 0$, cf. (1.6). Nevertheless this resonance has an interest of its own and we will come back to it in section 5.

The squeeze effect is not restricted to the elliptic 1 cycle at the origin, but it generally occurs also at other elliptic period q orbits whenever they are in $1/3$ resonance. In general, when the eigenvalues of the q times iterated map (or

Fig. 9a. The quadratic map (2.5) for $C = -0.94$, showing the squeeze effect for the 5 cycle with rotation numbers $p/q = 2/5$. Each point of the 5 cycle has three arms associated with the unstable manifolds of the hyperbolic 15 cycle; each point has also three incoming arms, but these are not visible in this picture (see however figs. 9b and 9c).

Fig. 9b. A magnification of the island in the lower left quadrant of fig. 9a, at $C = -0.94$, just before the complete squeeze.

Fig. 9c. A magnification of the island in the lower left quadrant of fig. 9a, but now at $C = -0.941$, just after the complete squeeze.

rather its linearized version) are equal to $\lambda_{\pm} = \exp(\pm 2\pi i/3)$, or equivalently when the trace $\lambda_+ + \lambda_-$ of this map is equal to -1 , then the q cycle is squeezed by a hyperbolic $3q$ cycle. That is, each elliptic point (island) of the q cycle is squeezed separately by three saddle points. This multiple squeezing is illustrated in fig. 9, for the map (2.5), at the $1/3$ resonance for the 5 cycle with $p/q = 2/5$. This 5 cycle is born from the origin (as a Poincaré–Birkhoff pair) at $C = \cos(4\pi/5) = -0.809017$, cf. (1.6), and bifurcates into a 10 cycle at $C = -0.955245$. The $1/3$ resonance, where the squeezing is complete, occurs at $C = -0.940394$.

3. The bouncing ball

Consider a little ball bouncing on a plane which is vibrating vertically, $z = z_0 \cos \omega t$. This system^{11,12}) is shown in fig. 10. The mass of the ball m is negligibly small compared to the mass of the plane M . Also, the amplitude z_0 of the vibration of the plane is much smaller than the height of the jumps of the ball. If there is no energy loss during the bouncing process then the velocity just after the bounce (v_{n+1}) is related to the velocity just before the bounce (v_n) via

$$v_{n+1} = v_n + 2z_0\omega \sin \omega t_{n+1} \quad (3.1)$$

and the time interval between two bounces is, for small z_0 ,

$$t_{n+1} = t_n + \frac{2v_n}{g}. \quad (3.2)$$

Rescaling $x_n = \omega t_n$, $y_n = x_{n-1}$ these two equations (3.1) and (3.2) give an area-preserving map:

Fig. 10. A ball (with mass m) bouncing on a vibrating plane (with mass $M \gg m$), under the influence of gravitation.

$$\begin{cases} x_{n+1} = 2x_n + \lambda \sin(x_n) - y_n, \\ y_{n+1} = x_n, \end{cases} \quad (3.3a)$$

$$(3.3b)$$

where λ is the chaos parameter

$$\lambda = \frac{4\omega^2 z_0}{g}. \quad (3.4)$$

The map (3.3) has a fixed point in the origin which is elliptic for $0 > \lambda > -4$, and which is in $1/3$ resonance for $\lambda = -3$. However, the local behaviour around this fixed point is now governed by a cubic term instead of a quadratic term. In fig. 11 we show the neighbourhood of the origin at $\lambda = -3.1$, just after the $1/3$ resonance. The lack of a triangular structure (such as in fig. 8) demonstrates that there has been no squeezing. Instead, a Poincaré–Birkhoff chain consisting of *two* elliptic and *two* hyperbolic 3 cycles has emerged from the origin. In section 5 we will see that the absence of squeezing at the $1/3$ resonance is a direct consequence of the fact that the Taylor expansion of the mapping (around the elliptic fixed point under consideration) is dominated by a cubic non-linearity. On the other hand, if the map around the fixed point contains a quadratic term, the squeeze effect will *always* occur.

Fig. 11. Bounce map at $\lambda = -3.1$, showing the two elliptic 3 cycles, which have bifurcated from the origin. The points of the two hyperbolic 3 cycles are located between the islands of the elliptic cycles, where the two surrounding KAM curves nearly touch each other.

4. Bifurcation analysis

To describe the behaviour of an elliptic fixed point at a p/q resonance (including the squeeze effect) on a somewhat more general level, we perform a bifurcation analysis. A slightly different bifurcation analysis, based on earlier work by Meyer¹⁸), has been presented by MacKay¹⁹). Other bifurcation treatments can be found in the mathematical literature, for instance in refs. 20 and 21.

Consider again the general area preserving map (1.3), or equivalently

$$x_{n+1} + x_{n-1} = 2Cx_n + F(x_n), \quad (4.1)$$

with

$$F(x) = \sum_{k \geq 2} a_k x^k \quad (4.2)$$

We are interested in period q cycles (with rotation number p/q) that bifurcate from the origin. Writing such a q cycle as a vector x with q components

$$x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{q-1} \end{pmatrix} \quad (4.3)$$

the map (4.1) takes the form

$$(\mathbf{S} + \mathbf{S}^{-1} - 2C\mathbf{1}) \cdot x = F(x). \quad (4.4)$$

Here $\mathbf{S}$ is the shift operator, relabelling all the points of the q cycle $x_n \rightarrow x_{n+1}$:

$$\mathbf{S} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ \vdots & & 0 & 1 & \\ & & & \ddots & \ddots \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{pmatrix}, \quad (4.5)$$

with eigenvalues

$$\lambda_p = e^{2\pi i p/q} \quad (4.6)$$

and corresponding (non-normalized) eigenvectors

$$v_p = (1, \lambda_p, \lambda_p^2, \dots, \lambda_p^{q-1}). \quad (4.7)$$

The eigenvalues and eigenvectors of $\mathbf{S}^{-1}$ (the transposed of $\mathbf{S}$) are just the complex conjugates $\lambda_p^* = \lambda_{-p}$ and $\mathbf{v}_p^* = \mathbf{v}_{-p}$.

We decompose the bifurcating vector $\mathbf{x}$ (with rotation number p/q) in one part that lies in the plane spanned by the eigenvectors $\mathbf{v}_p$ and $\mathbf{v}_{-p}$, and one part χ that is orthogonal to $\mathbf{v}_p$ and $\mathbf{v}_{-p}$:

$$\mathbf{x} = \varepsilon \mathbf{v}_p + \varepsilon^* \mathbf{v}_{-p} + \chi, \quad (4.8)$$

where

$$\varepsilon = |\varepsilon| e^{i\phi}.$$

It will turn out that $|\chi|$ is of order $|\varepsilon^2|$, so $|\mathbf{x}|$ is of order $|\varepsilon|$. That is, $|\varepsilon|$ measures the distance of the q cycle from the origin. The angle ϕ determines the direction of the elements of the period q cycle in the (x_n, y_n) plane with respect to the origin, with $y_n = x_{n-1}$:

$$\frac{x_n}{y_n} \approx \frac{\cos(\phi + 2\pi n p/q)}{\cos(\phi + 2\pi(n-1)p/q)}, \quad n = 0, \dots, q-1.$$

The distance $|\varepsilon|$ is a small quantity as long as we are close to the resonant C value, i.e. for small values of

$$\delta = C - C_{p/q} \quad (4.9)$$

with $C_{p/q}$ given by (1.6). In fact, the relation between δ and ε gives the key to the behaviour at the resonance. So our goal is to find this relation, for any p/q .

Let $\mathbf{P}$ be the projection operator on the space spanned by $\mathbf{v}_p$ and $\mathbf{v}_{-p}$, then

$$\mathbf{P} \cdot \mathbf{x} = \varepsilon \mathbf{v}_p + \varepsilon^* \mathbf{v}_{-p}, \quad (4.10a)$$

$$(\mathbf{1} - \mathbf{P}) \cdot \mathbf{x} = \chi. \quad (4.10b)$$

The operators $\mathbf{P}$ and $\mathbf{S}$ commute, so if we let $\mathbf{P}$ act on (4.4) we get:

$$-2\delta(\varepsilon \mathbf{v}_p + \varepsilon^* \mathbf{v}_{-p}) = \mathbf{P} \cdot \mathbf{F}(\varepsilon \mathbf{v}_p + \varepsilon^* \mathbf{v}_{-p} + \chi), \quad (4.11a)$$

and if we let $(\mathbf{1} - \mathbf{P})$ act on (4.4):

$$-2\delta\chi + (\mathbf{S} + \mathbf{S}^{-1} - 2C_{p/q}) \cdot \chi = (\mathbf{1} - \mathbf{P}) \cdot \mathbf{F}(\varepsilon \mathbf{v}_p + \varepsilon^* \mathbf{v}_{-p} + \chi). \quad (4.11b)$$

In principle, we can solve χ uniquely from (4.11b) (for given p/q) in terms of ε , ε^* and δ . Note that $\chi = \mathcal{O}(|\varepsilon|^2)$ since F only contains quadratic and higher order terms. In the case that the lowest order non-linearities in F are of order z we have $\chi = \mathcal{O}(|\varepsilon|^z)$.

Having determined χ one can proceed to solve δ , from (4.11a), in terms of ε and ε^* . To do this we first substitute the expression for χ into (4.11a). Next we project (4.11a) onto the eigenvector $\mathbf{v}_p$, by means of a second projection operator $\mathbf{P}_p$ (note that $\mathbf{P}_p \cdot \mathbf{P} = \mathbf{P}_p$):

$$-2\delta\varepsilon \mathbf{v}_p = \mathbf{P}_p \cdot \mathbf{F}[\varepsilon \mathbf{v}_p + \varepsilon^* \mathbf{v}_{-p} + \chi(\varepsilon, \varepsilon^*, \delta)], \quad (4.12)$$

which is an implicit relation between δ , ε and ε^* .

Now, $\mathbf{P}_p \cdot \mathbf{F}$ can be uniquely expressed as a power series in ε and ε^* , with terms like $\varepsilon^k \varepsilon^{*l} R(k, l)$, $k + l \geq 2$, where $R(k, l)$ is a q -dimensional vector which can be written in terms of k vectors $\mathbf{v}_p$ and l vectors $\mathbf{v}_{-p}$ (cf. appendix). But in the series of $\mathbf{P}_p \cdot \mathbf{F}$ not all combinations of k and l are allowed, as can be seen as follows (see also ref. 22). On the one hand, the operator $\mathbf{S}$ shifts an arbitrary term of the series as follows:

$$\varepsilon^k \varepsilon^{*l} R(k, l) \xrightarrow{\mathbf{S}} e^{2\pi i(k-l)p/q} \varepsilon^k \varepsilon^{*l} R(k, l). \quad (4.13)$$

On the other hand, since $\mathbf{P}_p \cdot \mathbf{F}$ is proportional to $\mathbf{v}_p$, it is an eigenvector of $\mathbf{S}$ with eigenvalue $\lambda_p = \exp(2\pi i p/q)$. This leads to the selection rule

$$k - l = 1 \pmod{q}, \quad q \geq 3. \quad (4.14)$$

Thus we may write (4.12) as follows, putting all the dependence on $\mathbf{v}_p$ and $\mathbf{v}_{-p}$ into the coefficients $C_{k,l}$,

$$-2\delta\varepsilon = \sum_{\substack{k-l=1 \\ \pmod{q}}} C_{k,l} \varepsilon^k \varepsilon^{*l}, \quad (4.15)$$

or equivalently, separating the terms with $k > l$ and $k < l$,

$$-2\delta\varepsilon = \sum_{n=0}^{\infty} \varepsilon^{nq+1} A_n(|\varepsilon|^2) + \sum_{n=1}^{\infty} \varepsilon^{*nq-1} B_n(|\varepsilon|^2). \quad (4.16)$$

Here $A_n(|\varepsilon|^2)$ and $B_n(|\varepsilon|^2)$ are shorthand for

$$A_n(|\varepsilon|^2) = \sum_l |\varepsilon|^{2l} C_{l+1+nq,l} = A_{n0} + A_{n1} |\varepsilon|^2 + A_{n2} |\varepsilon|^4 + \dots, \quad (4.17a)$$

$$B_n(|\varepsilon|^2) = \sum_k |\varepsilon|^{2k} C_{k, k-1+nq} = B_{n0} + B_{n1}|\varepsilon|^2 + B_{n2}|\varepsilon|^4 + \cdots, \quad (4.17b)$$

in which all the coefficients are real. This can be seen as follows. Observe that (4.16) equals the inner product $(\mathbf{v}_p \cdot \mathbf{F})$, up to a normalization constant, cf. (4.12). Then, on the one hand, the derivation above shows that $(\mathbf{v}_{-p} \cdot \mathbf{F})$ is identical to $(\mathbf{v}_p \cdot \mathbf{F})$ but with ε replaced by ε^* and vice versa. On the other hand, $(\mathbf{v}_{-p} \cdot \mathbf{F})$ equals $(\mathbf{v}_p \cdot \mathbf{F})^*$ since $\mathbf{F}$ is real. Equating both expressions for $(\mathbf{v}_{-p} \cdot \mathbf{F})$ then shows that A_n and B_n are real functions.

Since $\mathbf{F}$ contains only quadratic and higher order terms the right-hand side of (4.16) must be $\mathcal{O}(|\varepsilon|^2)$, or in general $\mathcal{O}(|\varepsilon|^z)$ if z is the lowest order appearing in $\mathbf{F}$. This means that $A_0(|\varepsilon|^2)$ cannot have a constant term, i.e. $A_{00} = 0$. Working out (4.16) we get:

$$\begin{aligned} q = 3: \\ -2\delta\varepsilon = B_{10}\varepsilon^{*2} + A_{01}|\varepsilon|^2\varepsilon + A_{10}\varepsilon^4 + B_{11}|\varepsilon|^2\varepsilon^{*2} + A_{02}|\varepsilon|^4\varepsilon \\ + B_{20}\varepsilon^{*5} + A_{11}|\varepsilon|^2\varepsilon^4 + B_{12}|\varepsilon|^4\varepsilon^{*2} + \mathcal{O}(|\varepsilon|^7), \end{aligned} \quad (4.18a)$$

$$\begin{aligned} q = 4: \\ -2\delta\varepsilon = A_{01}|\varepsilon|^2\varepsilon + B_{10}\varepsilon^{*3} + A_{02}|\varepsilon|^4\varepsilon + A_{10}\varepsilon^5 + B_{11}|\varepsilon|^2\varepsilon^{*3} + \mathcal{O}(|\varepsilon|^7), \end{aligned} \quad (4.18b)$$

$$\begin{aligned} q \geq 5: \\ -2\delta\varepsilon = A_{01}|\varepsilon|^2\varepsilon + B_{10}\varepsilon^{*q-1} + A_{02}|\varepsilon|^4\varepsilon + A_{10}\varepsilon^{q+1} + B_{11}|\varepsilon|^2\varepsilon^{*q-1} \\ + \mathcal{O}(|\varepsilon|^7). \end{aligned} \quad (4.18c)$$

Assuming that none of the relevant coefficients vanishes (which is generally true if $\mathbf{F}$ contains a quadratic term) we thus find the standard bifurcation behaviour, which we discuss now.

For $q = 3$ we have

$$\delta = -\frac{1}{2}B_{10} \frac{\varepsilon^{*2}}{\varepsilon} + \mathcal{O}(|\varepsilon|^2). \quad (4.19)$$

Setting $\varepsilon = |\varepsilon|e^{i\phi}$ we obtain from (4.19)

$$\delta = -\frac{1}{2}B_{10}|\varepsilon|e^{-3i\phi} + \mathcal{O}(|\varepsilon|^2). \quad (4.20)$$

Since B_{10} and $|\varepsilon|$ are real, a necessary condition for δ to be real is $e^{-3i\phi} = \pm 1$. Substitution in (4.16) shows that it is also sufficient. Thus we obtain:

$$\phi = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \quad \text{for } \delta = -\frac{1}{2}B_{10}|\varepsilon| + \mathcal{O}(|\varepsilon|^2), \quad (4.21a)$$

$$\phi = \frac{\pi}{3}, \pi, \frac{5\pi}{3}, \quad \text{for } \delta = \frac{1}{2}B_{10}|\varepsilon| + \mathcal{O}(|\varepsilon|^2). \quad (4.21b)$$

Under the shift operator **S** the three ϕ values in (4.21a) are transformed into each other, and similarly for (4.21b). So, unless $B_{10} = 0$, we have one 3 cycle for $\delta < 0$ and one for $\delta > 0$, which is typical for the squeeze effect. The corresponding relation between δ and $|\varepsilon|$ is sketched in fig. 12.

Eq. (4.21) also explains the reversal of the orientation of the 3 cycle when C passes $C_{1/3}$, as shown in figs. 8a–c and fig. 9.

For $q = 4$ there are two competing dominant terms,

$$\delta = -\frac{1}{2}\left(A_{01}|\varepsilon|^2 + B_{10}\frac{\varepsilon^{*3}}{\varepsilon}\right) + \mathcal{O}(|\varepsilon|^4). \quad (4.22)$$

This complicates the bifurcation behaviour (see also ref. 4, section 35). With $\varepsilon = |\varepsilon|e^{i\phi}$ we get

$$\delta = -\frac{1}{2}|\varepsilon|^2(A_{01} + B_{10}e^{-4i\phi}). \quad (4.23)$$

Fig. 12. A graphical classification of the bifurcation behaviour for $q = 3$, $q = 4$ and $q \geq 5$, assuming that the dominant terms in (4.18) have non-zero coefficients. The three possibilities for $q = 4$ are illustrated in figs. 14, 15 and 16.

Thus $e^{-4i\phi} = \pm 1$, provided $B_{10} \neq 0$, yielding two 4 cycles with

$$\phi = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \quad \text{for } \delta = -\frac{1}{2}|\varepsilon|^2(A_{01} + B_{10}), \quad (4.24a)$$

$$\phi = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \quad \text{for } \delta = -\frac{1}{2}|\varepsilon|^2(A_{01} - B_{10}). \quad (4.24b)$$

If $A_{01}^2 < B_{10}^2$ then the δ 's in (4.24a) and (4.24b) have opposite signs, and we will have a squeeze effect with one 4 cycle for $\delta > 0$ and one for $\delta < 0$ (just as in the case $q = 3$). The corresponding relation between δ and $|\varepsilon|$ is shown in fig. 12, and the actual squeezing for a specific map is shown in fig. 16.

On the other hand, if $A_{01}^2 > B_{10}^2$, then the δ 's have the same sign and we get a Poincaré–Birkhoff pair of two 4 cycles at one side of $C_{1/4} = 0$, and none at the other side. The corresponding relation between δ and $|\varepsilon|$ is again shown in fig. 12, and a specific example is shown in fig. 14.

In the marginal case $A_{01}^2 = B_{10}^2$, which includes the quadratic map (2.5), we have to take into account higher order terms in the expression (4.18b). We will come back to this in section 5.3. For future reference we have drawn the corresponding relation between δ and $|\varepsilon|$ in fig. 12.

For $q \geq 5$ we have

$$\delta = -\frac{1}{2}A_{01}|\varepsilon|^2 - \frac{1}{2}B_{10} \frac{\varepsilon^{*q-1}}{\varepsilon} + \mathcal{O}(|\varepsilon|^4). \quad (4.25)$$

The term with A_{01} tells us that for decreasing C a Poincaré–Birkhoff chain of two q cycles is being born (if $A_{01} > 0$) or is dying (if $A_{01} < 0$) at $C = C_{p/q}$. The corresponding relation between δ and $|\varepsilon|$ is shown in fig. 12.

The term with B_{10} determines the phase ϕ of $\varepsilon = |\varepsilon|e^{i\phi}$, i.e. the directions in which the points of the q cycle bifurcate from the origin. In fact (4.25) implies, since $e^{-q i \phi} = \pm 1$, that ϕ is a multiple of π/q . This is again a necessary and sufficient condition for $\delta(\varepsilon)$ to be real. In the even directions $\phi = 2l\pi/q$ we observe a hyperbolic q cycle (with the smaller value of $|\varepsilon|$, at fixed δ) when $B_{10}/A_{01} > 0$, and an elliptic q cycle when $B_{10}/A_{01} < 0$. In the odd directions $\phi = (2l+1)\pi/q$ it is the other way around. In the limit $\delta \rightarrow 0$ the radial distance $|\varepsilon|$ for both cycles is proportional to $|\delta|^{1/2}$.

5. Summary and conclusions

5.1. A model map

In this concluding section we apply the above results on the special (but typical) case

$$F(x) = ax^2 + bx^3, \quad (5.1)$$

which is convenient for studying the competition between quadratic and cubic terms in a map. In this case the relevant coefficients are (see (4.18) and the appendix):

$$A_{01} = \begin{cases} \frac{4}{3}a^2 + 3b, & \text{for } q = 3, \\ a^2 + 3b, & \text{for } q = 4, \\ \frac{(4C_{p/q} + 1)}{(1 - C_{p/q})(2C_{p/q} + 1)} a^2 + 3b, & \text{for } q \geq 5, \end{cases} \quad (5.2)$$

$$B_{10} = \begin{cases} a^2, & \text{for } q = 3, \\ -a^2 + b, & \text{for } q = 4. \end{cases} \quad (5.3)$$

The same expressions are obtained if higher order terms are added to $F(x)$ in (5.1), i.e. for any map with a non-vanishing quadratic and/or cubic part.

The coefficient B_{10} for $q \geq 5$ can be evaluated via a recursive scheme, but we do not need it here. Also, it may be noted that the expression for $q \geq 5$ in (5.2) also covers the case $q = 4$.

5.2. The case $q = 3$, and the influence of dissipation

If $a \neq 0$ we get for $q = 3$, cf. (4.19),

$$\delta = -\frac{1}{2}a^2 \frac{\varepsilon^{*2}}{\varepsilon} + \mathcal{O}(|\varepsilon^2|), \quad (5.4)$$

implying that we have a $q = 3$ squeeze effect in any mapping with a non-vanishing quadratic part.

We now discuss to what extent this remains true in the presence of a small dissipation, as one often finds in real experiments. As an example one may consider the map^{16,17)}

$$\begin{cases} x_{n+1} = 2Cx_n + 2x_n^2 - y_n, \\ y_{n+1} = Bx_n, \end{cases} \quad (5.5)$$

where B , the Jacobian of the map, is a measure of the dissipation (area-contraction):

$$\iint dx_{n+1} dy_{n+1} = |B| \iint dx_n dy_n. \quad (5.6)$$

For $B = 1$ we have an area-preserving map (without dissipation), but for $|B| < 1$ we have a dissipative map. In that case the squeeze effect is not complete but one still observes the change in the orientation of the unstable 3 cycle. This is illustrated in fig. 13, where we have shown the (incomplete) squeeze effect for $B = 0.99$. For decreasing C the three saddle points move as indicated by the arrows. Just before the origin is reached they rapidly (within a small C interval) change their orientation ϕ from the odd multiples of $\pi/3$ to the even multiples, and subsequently move away from the origin. The minimal distance of the unstable 3 cycle to the origin can be regarded as a measure of the dissipation. Some more illustrations of this behaviour can be found in ref. 4.

On the other hand, if $a = 0$ (and $b \neq 0$) there is no squeeze effect (see also ref. 23). Instead, two Poincaré–Birkhoff chains are born simultaneously. This can be seen from (4.18a) in which the terms with even powers of ε (B_{10} , A_{10} , B_{11} , A_{11} , B_{12}) vanish, as a consequence of the fact that $F(x)$ only contains odd powers. Furthermore we have $A_{01} = 3b$ and $A_{02} = B_{20} = b^2$ (see appendix), so (4.18a) reduces to

$$\delta = -\frac{1}{2}b \left[3|\varepsilon|^2 + b \left(\frac{\varepsilon^{*5}}{\varepsilon} + |\varepsilon|^4 \right) \right]. \quad (5.7)$$

This equation with $\varepsilon = |\varepsilon| e^{i\phi}$ implies that $e^{-6i\phi} = \pm 1$ provided $b \neq 0$. In the directions $\phi = 2l\pi/6$ (associated with the smaller $|\varepsilon|$ value) we have two

Fig. 13a. The map (5.5) with $B = 0.99$, at $C = -0.49$, just before the maximal squeezing. For decreasing C the three saddle points (solid dots) move as indicated by the arrows.

Fig. 13b. The map (5.5) with $B = 0.99$, at $C = -0.498$, where the squeezing is maximal. For decreasing C the three saddle points (solid dots) move as indicated by the arrows.

hyperbolic 3 cycles, and in the directions $\phi = (2l + 1)\pi/6$ two elliptic 3 cycles. These features are evident from fig. 11 for the bouncing ball.

5.3. The case $q = 4$

For $q = 4$, as discussed below (4.24), we have a squeeze effect if $|A_{01}| < |B_{10}|$, and bifurcation of a Poincaré–Birkhoff pair if $|A_{01}| > |B_{10}|$. Or, with (5.2) and (5.3),

Fig. 13c. The map (5.5) with $B = 0.99$, at $C = -0.51$, just after the maximal squeezing. For decreasing C the three saddle points (solid dots) as indicated by the arrows.

Fig. 14. The map (5.1) with $a=2$ and $b=2$ at $C=-0.01$. Around the origin we see a Poincaré-Birkhoff pair of two 4 cycles, one elliptic and the other hyperbolic. Further away from the origin, towards the edge of the island we also see higher periodic orbits, e.g. a 9 cycle with rotation number $p/q = 2/9$.

$$8b(a^2 + b) \begin{cases} > 0, & \text{Poincaré-Birkhoff chain,} \\ = 0, & \text{marginal case,} \\ < 0, & \text{squeeze effect.} \end{cases} \quad (5.8)$$

These three possibilities are illustrated in figs. 14, 15 and 16a-c for $a=2$ and

Fig. 15. The map (5.1) with $a=2$ and $b=0$, which is just the standard quadratic map (2.5), at $C=-0.01$. We see a peculiar Poincaré-Birkhoff pair of two 4 cycles; the elliptic cycle lies significantly further away from the origin than the hyperbolic cycle.

Fig. 16a. The map (5.1) with $a = 2$ and $b = -2$ at $C = 0.005$, just before the complete squeeze. For decreasing C the four saddle points close in upon the origin along the x - and y -axis, as indicated by the arrows.

Fig. 16b. The map (5.1) with $a = 2$ and $b = -2$ at $C = 0$, where the squeezing of the origin is complete.

Fig. 16c. The map (5.1) with $a = 2$ and $b = -2$ at $C = -0.005$, just after the complete squeeze. For decreasing C the four saddle points move away from the origin as indicated by the arrows.

$b = 2, 0, -2$ respectively. Note that the marginal case shown here is just the standard quadratic map (2.5).

To understand the marginal case we have to take into account more terms of the expression (4.18b). Up to order $|\varepsilon^5|$ we get:

$$\begin{aligned} -2\delta\varepsilon = & |\varepsilon|^2\varepsilon(a^2 + 3b) + \varepsilon^{*3}(-a^2 + b) + |\varepsilon|^4\varepsilon(\tfrac{5}{2}a^4 + \tfrac{9}{2}a^2b) \\ & + \varepsilon^{*3}|\varepsilon|^2(\tfrac{5}{4}a^4 - \tfrac{15}{4}a^2b) + \varepsilon^5(\tfrac{1}{4}a^4 - \tfrac{3}{4}a^2b). \end{aligned} \quad (5.9)$$

For instance, for the standard quadratic map (2.5) with $a = 2$ and $b = 0$ we find, with $\varepsilon = |\varepsilon|e^{i\phi}$,

$$-2\delta = 4|\varepsilon|^2(1 - e^{-4i\phi}) + 16|\varepsilon|^4(\tfrac{5}{2} + \tfrac{5}{4}e^{-4i\phi} + \tfrac{1}{4}e^{4i\phi}). \quad (5.10)$$

Since δ and $|\varepsilon|$ are both real we require that $e^{4i\phi} = \pm 1$. If $e^{4i\phi} = +1$ we have a 4 cycle which bifurcates from the origin along $\phi = 0, \pi/2, \pi, 3\pi/2$ (along the x - and y -axis) with $-\delta = 32|\varepsilon|^4$, i.e. for negative δ or equivalently $C < 0$, since $C_{1/4} = 0$. If $e^{4i\phi} = -1$ the 4 cycle bifurcates along $\phi = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ with $-\delta = 4|\varepsilon|^2$, i.e. also for $C < 0$.

These two 4 cycles form a peculiar Poincaré–Birkhoff pair, as can also be seen from fig. 15. The cycle which bifurcates along the axis has radius $|\varepsilon| = (-\delta/32)^{1/4} = (-C/32)^{1/4}$, which is significantly larger than the radius $|\varepsilon| = (-C/4)^{1/2}$ of the other cycle.

5.4. Global aspects of the squeeze effect

For $q \geq 5$ we always have a Poincaré–Birkhoff bifurcation at the origin, as discussed below (4.25). If $A_{01} > 0$ we have a normal *outward* bifurcation, i.e. a Poincaré–Birkhoff pair of one elliptic and one hyperbolic q cycle is born at the origin at $C = C_{p/q}$ and moves outward as C is decreased. On the other hand, if $A_{01} < 0$ we have an *inward* bifurcation. In that case the Poincaré–Birkhoff pair is born outside the origin at some C value above $C_{p/q}$, moving inward for decreasing C until it dies at the origin at $C = C_{p/q}$.

These inward bifurcations are closely related to the squeeze effect. In fact, when $a \neq 0$ we have from (5.2)

$$A_{01} \rightarrow \begin{cases} -\infty, & \text{for } C = -\frac{1}{2} - 0, \\ +\infty, & \text{for } C = -\frac{1}{2} + 0, \end{cases} \quad (5.11)$$

and there is always an interval where A_{01} is negative, for

$$-\frac{1}{2} < C < C_0. \quad (5.12)$$

Here the value of C_0 , for which $A_{01} = 0$, is given by

$$C_0 = \frac{1}{4} + \gamma - \gamma \sqrt{1 + \frac{1}{\gamma} + \frac{9}{16\gamma^2}}, \quad (5.13)$$

where $\gamma = a^2/3b$. For $b = 0$ we have $C_0 = -\frac{1}{4}$, and for $a = 0$ we get $C_0 = -\frac{1}{2}$ which confirms the absence of squeezing (and inward bifurcations) in that case.

As an example let us again consider the quadratic map (2.5), with $a = 2$, $b = 0$ and consequently $C_0 = -\frac{1}{4}$. In particular, we calculate the rotation number ρ of the orbits within the stable island around the origin. The rotation number is defined as

$$\rho = \lim_{n \rightarrow \infty} \frac{\phi_n - \phi_0}{2\pi n}, \quad 0 \leq \phi_n - \phi_{n-1} < 2\pi, \quad (5.14)$$

where ϕ_n is the angle between the vector (x_n, y_n) and the positive x -axis, which increases with n . Thus, ρ is the mean number of rotations per iteration, for a given orbit (starting point) and a given value of C . Taking the starting point (x_0, y_0) on the symmetry line $x = y$ we calculated the rotation number $\rho(x_0)$ as a function of x_0 for several values of C . The results are shown in fig. 17. Note that the curves are shaped like staircases, where the steps (at rational values $\rho = p/q$) indicate the presence of a Poincaré–Birkhoff chain with rotation number p/q . Only the largest steps, with small values of q , are visible. The

Fig. 17. Rotation number $\rho(x_0)$ for the map (2.5), at several values of C , as a function of x_0 . Here x_0 measures the distance d from the origin along the line $x = y$, in fact $d = \sqrt{2}x_0$. The curves are staircases, with plateaus at every rational value $\rho = p/q$, although only the dominant plateaus can be seen in this figure. For some of these we have given the corresponding value of p/q . The curves end when $x = y$ reaches the edge of the stable island. It is instructive to compare the curves for $C = 1 - \sqrt{2}$ and $C = -0.43$ with figs. 6 and 7.

curves start at the origin with $\rho = \rho_0$ given by (1.6), and they end at the right when x_0 reaches the edge of the stable island. Beyond this edge (5.14) is no longer a well-behaved quantity.

If we let C decrease from $+1$ downward, we first (until $C = C_0 = -\frac{1}{4}$) observe only outward bifurcations, since $A_{01} > 0$. The origin gives birth to KAM curves and Poincaré–Birkhoff chains with ever increasing rotation numbers, pushing the (older) KAM curves and periodic chains with lower rotation numbers towards the edge of the stable island. For these C values the curve $\rho(x_0)$ is a descending staircase; see the curves $C = 0$ and $C = -0.23$ in fig. 17.

For $-\frac{1}{2} < C < -\frac{1}{4}$, or $\frac{1}{3} > \rho_0 > \arccos(-\frac{1}{4})/2\pi = 0.290215$, we have $A_{01} < 0$ and the curves $\rho(x_0)$ develop a maximum outside the origin. Accordingly, all the KAM curves and periodic chains with rotation numbers in this interval $0.290215 < \rho \leq \frac{1}{3}$ are born outside the origin, at the x_0 value corresponding to the top of the curve. They appear to be born as twins, one chain/curve moving

towards the origin as C is decreased and one moving towards the outer edge of the island.* For instance, in the curve $C = -0.40$ in fig. 17 we have one chain with $\rho = 0.32$ at $x_0 \sim 0.14$ and one at $x_0 \sim 0.30$. The ingoing chain/curve dies at the origin at $C = \cos(2\pi\rho)$. Thus, for this interval of rotation numbers the origin is not a place of birth but a place of dying.

The top of the $\rho(x_0)$ curve gets flattened when it hits the value $\frac{1}{3}$, at $C = 1 - \sqrt{2} = -0.41421$, representing the fact that no new KAM curves or periodic chains are being born between $C = 1 - \sqrt{2}$ and $C = -\frac{1}{2}$. This enables the hyperbolic 3 cycle to tunnel through the origin instead of dying there. (Otherwise, this would have been prevented by a surrounding curve with $\rho > \frac{1}{3}$.) For C slightly larger than $-\frac{1}{2}$ the curve $\rho(x_0)$ has a small rising part, corresponding to the stable island around the origin, and a large plateau $\rho(x_0) = \frac{1}{3}$. At the left, or inner, edge of this plateau we have the hyperbolic 3 cycle. Then there is also a dashed part of the plateau, where the orbit escapes to infinity, while the solid part at the right corresponds to the stable island of the elliptic 3 cycle (see also fig. 7). For $C = -\frac{1}{2}$, at the complete squeeze, the plateau extends from $x_0 = 0$ to $x_0 \approx 0.54$.

For $C < -\frac{1}{2}$ we have $A_{01} > 0$ again, with outward bifurcations and a descending $\rho(x_0)$ curve. In fig. 17 we can see that there is a newborn island around the origin, with rotation numbers strictly larger than $\frac{1}{3}$. If we let C decrease further we eventually get $\rho_0 = \frac{1}{2}$ at $C = -1$. Then the origin becomes a saddle point, giving birth to an elliptic 2 cycle.

The global aspects of the squeeze effect described above are of course not restricted to our specific example. Similar behaviour is to be expected for any mapping with a non-vanishing quadratic part.

Acknowledgement

This investigation is part of the research program of the Stichting voor Fundamenteel Onderzoek der Materie (FOM) which is financially supported by the Nederlandse Organisatie voor Zuiver-Wetenschappelijk Onderzoek (ZWO).

Appendix

In this appendix we present some calculations for the map (5.1). In particular, we derive the expressions (5.2), (5.3) and (5.6). We will do this by working out (4.12), bringing it into the form (4.18).

* The birth of twin Poincaré–Birkhoff chains will be treated in a forthcoming article²⁴).

First we introduce the star product $P * Q$ between two vectors on the basis (4.3) of the q cycle:

$$\begin{pmatrix} P_0 \\ P_1 \\ \vdots \\ P_{q-1} \end{pmatrix} * \begin{pmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_{q-1} \end{pmatrix} = \begin{pmatrix} P_0 Q_0 \\ P_1 Q_1 \\ \vdots \\ P_{q-1} Q_{q-1} \end{pmatrix}. \quad (\text{A.1})$$

For the star product, with $\mathbf{v}_p$ given by (4.7), we have the property

$$\mathbf{v}_p * \mathbf{v}_{p'} = \mathbf{v}_{p+p'}, \quad (\text{A.2})$$

which is a direct consequence of $\lambda_p \lambda_{p'} = \exp(2\pi i(p + p')/q)$, cf. (4.6).

Using the star product the right-hand side of (4.12) becomes, with $\mathbf{F}$ as in (5.1):

$$\begin{aligned} & \mathbf{P}_p \cdot \mathbf{F}(\varepsilon \mathbf{v}_p + \varepsilon^* \mathbf{v}_{-p} + \chi) \\ &= \mathbf{P}_p \{ a(\varepsilon \mathbf{v}_p + \varepsilon^* \mathbf{v}_{-p} + \chi) * (\varepsilon \mathbf{v}_p + \varepsilon^* \mathbf{v}_{-p} + \chi) \\ & \quad + b(\varepsilon \mathbf{v}_p + \varepsilon^* \mathbf{v}_{-p} + \chi) * (\varepsilon \mathbf{v}_p + \varepsilon^* \mathbf{v}_{-p} + \chi) * (\varepsilon \mathbf{v}_p + \varepsilon^* \mathbf{v}_{-p} + \chi) \}, \end{aligned} \quad (\text{A.3})$$

where χ , as defined by (4.8), may be expressed in terms of the eigenvectors $\mathbf{v}_r$ ($r \neq p, -p$) of $\mathbf{S}$:

$$\chi = \sum_{r \neq p, -p} \eta_r \mathbf{v}_r. \quad (\text{A.4})$$

The vector χ , or equivalently the coefficients η_r , can be solved uniquely from (4.11b). Here the coefficients η_r are (at least) of order $|e^2|$, since $\chi = \mathcal{O}(|e^2|)$ for the mapping (5.1).

Now in (A.3), after the projection $\mathbf{P}_p$ only those products $\mathbf{v}_i * \mathbf{v}_j$ (and $\mathbf{v}_i * \mathbf{v}_j * \mathbf{v}_k$) remain which have $i + j = p \pmod q$ (and $i + j + k = p \pmod q$). This follows from (A.2).

We first discuss the case $q = 3$. In this case the only allowed value of r is zero, corresponding to the eigenvalue 1. Thus (4.12) becomes, with (A.3) and $p = 1$,

$$\begin{aligned} -2\delta\varepsilon \mathbf{v}_1 &= a(2\varepsilon\eta_0 \mathbf{v}_1 * \mathbf{v}_0 + \varepsilon^{*2} \mathbf{v}_{-1} * \mathbf{v}_{-1}) + b(3|\varepsilon|^2 \varepsilon \mathbf{v}_1 * \mathbf{v}_1 * \mathbf{v}_{-1} \\ & \quad + 3\varepsilon\eta_0^2 \mathbf{v}_1 * \mathbf{v}_0 * \mathbf{v}_0 + 3\varepsilon^{*2} \eta_0 \mathbf{v}_{-1} * \mathbf{v}_{-1} * \mathbf{v}_0) \\ &= \{a(2\varepsilon\eta_0 + \varepsilon^{*2}) + 3b(|\varepsilon|^2 \varepsilon + \varepsilon\eta_0^2 + \varepsilon^{*2} \eta_0)\} \mathbf{v}_1. \end{aligned} \quad (\text{A.5})$$

The coefficient η_0 can be determined from (4.11b) with $\lambda_0 + \lambda_0^* = 2$, and neglecting the small term $-2\delta\chi$ this equation becomes (keeping only the star products $\mathbf{v}_i * \mathbf{v}_j$ with $i + j = 0 \bmod 3$, and $\mathbf{v}_i * \mathbf{v}_j * \mathbf{v}_k$ with $i + j + k = 0 \bmod 3$):

$$\begin{aligned} 2(1 - C_{1/3})\eta_0\mathbf{v}_0 &= a(2|\varepsilon|^2\mathbf{v}_1 * \mathbf{v}_{-1} + \eta_0^2\mathbf{v}_0 * \mathbf{v}_0) \\ &\quad + b(\varepsilon^3\mathbf{v}_1 * \mathbf{v}_1 * \mathbf{v}_1 + \varepsilon^{*3}\mathbf{v}_{-1} * \mathbf{v}_{-1} * \mathbf{v}_{-1} \\ &\quad + \eta_0^3\mathbf{v}_0 * \mathbf{v}_0 * \mathbf{v}_0 + 6|\varepsilon|^2\eta_0\mathbf{v}_1 * \mathbf{v}_{-1} * \mathbf{v}_0) \\ &= 2a|\varepsilon|^2\mathbf{v}_0 + \mathcal{O}(|\varepsilon^3|). \end{aligned} \quad (\text{A.6})$$

So, with $C_{1/3} = -\frac{1}{2}$,

$$\eta_0 = \frac{2}{3}a|\varepsilon|^2 + \mathcal{O}(|\varepsilon^3|). \quad (\text{A.7})$$

Inserting (A.7) in (A.5) we get:

$$-2\delta\varepsilon = a\varepsilon^{*2} + (\frac{4}{3}a^2 + 3b)|\varepsilon|^2\varepsilon + \mathcal{O}(|\varepsilon^4|). \quad (\text{A.8})$$

Comparing this expression with (4.18b) we find $A_{01} = \frac{4}{3}a^2 + 3b$ and $B_{10} = a$, which are just the $q = 3$ results given in (5.2) and (5.3). In the case that $a = 0$ eq. (A.6) yields $\eta_0 = \frac{1}{3}b(\varepsilon^3 + \varepsilon^{*3}) + \mathcal{O}(\varepsilon^5)$. Inserting this into (A.5) we obtain (5.7).

For $q = 4$ we have $\chi = \eta_0\mathbf{v}_0 + \eta_2\mathbf{v}_2$ and in this case (4.12) becomes, with (A.3) and $p = 1$,

$$\begin{aligned} -2\delta\varepsilon\mathbf{v}_1 &= a(2\varepsilon\eta_0\mathbf{v}_1 * \mathbf{v}_0 + 2\varepsilon^*\eta_2\mathbf{v}_{-1} * \mathbf{v}_2) \\ &\quad + b(\varepsilon^{*3}\mathbf{v}_{-1} * \mathbf{v}_{-1} * \mathbf{v}_{-1} + 3|\varepsilon|^2\varepsilon\mathbf{v}_1 * \mathbf{v}_1 * \mathbf{v}_{-1} \\ &\quad + 3\varepsilon\eta_0^2\mathbf{v}_1 * \mathbf{v}_0 * \mathbf{v}_0 + 3\varepsilon\eta_2^2\mathbf{v}_1 * \mathbf{v}_2 * \mathbf{v}_2 + 6\varepsilon^*\eta_0\eta_2\mathbf{v}_{-1} * \mathbf{v}_0 * \mathbf{v}_2) \\ &= \{2a(\varepsilon\eta_0 + \varepsilon^*\eta_2) + b(\varepsilon^{*3} + 3\varepsilon|\varepsilon|^2 + 3\varepsilon\eta_0^2 + 3\varepsilon\eta_2^2 + 6\varepsilon^*\eta_0\eta_2)\}\mathbf{v}_1. \end{aligned} \quad (\text{A.9})$$

The coefficients η_0 and η_2 are determined from (4.11b). For $r = 0$, with $\lambda_0 + \lambda_0^* = 2$ we get:

$$\begin{aligned} 2(1 - C_{1/4})\eta_0\mathbf{v}_0 &= \{a(\eta_0^2 + \eta_2^2 + 2|\varepsilon|^2) \\ &\quad + b(\eta_0^3 + 3\varepsilon^2\eta_2 + 3\varepsilon^{*2}\eta_2 + 3\eta_0\eta_2^2 + 6|\varepsilon|^2\eta_0)\}\mathbf{v}_0. \end{aligned} \quad (\text{A.10})$$

And for $r = 2$, with $\lambda_2 + \lambda_2^* = -2$ we get:

$$-2(1 + C_{1/4})\eta_2 \mathbf{v}_2 = \{a(\varepsilon^2 + \varepsilon^{*2} + 2\eta_0 \eta_2) + b(\eta_2^3 + 3\varepsilon^2 \eta_0 + 3\varepsilon^{*2} \eta_0 + 3\eta_0^2 \eta_2 + 6|\varepsilon|^2 \eta_2)\} \mathbf{v}_2. \quad (\text{A.11})$$

From (A.10) and (A.11), together with the fact that η_0 and η_2 are of order $|\varepsilon^2|$, we get (with $C_{1/4} = 0$):

$$\eta_0 = a|\varepsilon|^2 + \mathcal{O}(|\varepsilon^4|), \quad (\text{A.12a})$$

$$\eta_2 = -\frac{1}{2}a(\varepsilon^2 + \varepsilon^{*2}) + \mathcal{O}(|\varepsilon^4|). \quad (\text{A.12b})$$

Inserting (A.12) in (A.9) we find

$$-2\delta\varepsilon = |\varepsilon|^2\varepsilon(a^2 + 3b) + \varepsilon^{*3}(-a^2 + b) + \mathcal{O}(|\varepsilon^5|), \quad (\text{A.13})$$

and comparing this expression with (4.18b) yields $A_{01} = a^2 + 3b$ and $B_{10} = -a^2 + b$, which are just the $q = 4$ results given in (5.2) and (5.3). In the marginal case that $|A_{01}| = |B_{10}|$, i.e. $8b(a^2 + b) = 0$, we must also take into account higher order terms. That is, we must expand η_0 and η_2 up to order $|\varepsilon^4|$, enabling us to work out (A.9) up to the $\mathcal{O}(|\varepsilon^5|)$ terms.

Inserting (A.12a) and (A.12b) into the right-hand side of (A.10) gives:

$$\eta_0 = a|\varepsilon|^2 + \frac{1}{2}(\frac{3}{2}a^3 + 3ab)|\varepsilon|^4 + \frac{1}{4}(\frac{1}{2}a^3 - 3ab)(\varepsilon^4 + \varepsilon^{*4}) + \mathcal{O}(|\varepsilon^6|). \quad (\text{A.14})$$

And similarly from (A.11):

$$\eta_2 = -\frac{1}{2}a(\varepsilon^2 + \varepsilon^{*2}) + \frac{1}{2}a^3|\varepsilon|^2(\varepsilon^2 + \varepsilon^{*2}) + \mathcal{O}(|\varepsilon^6|). \quad (\text{A.15})$$

Inserting these expressions into (A.9) we get the result (5.9) given in the main text. The scheme which we have used to treat the cases $q = 3$ and $q = 4$ can be worked out systematically for all $q \geq 5$.

For $q \geq 5$ the relation (4.12) becomes, keeping only the dominant $\mathcal{O}(|\varepsilon^3|)$ terms,

$$-2\delta\varepsilon \mathbf{v}_p = 2a(\varepsilon \eta_0 \mathbf{v}_p * \mathbf{v}_0 + \varepsilon^* \eta_{2p} \mathbf{v}_{-p} * \mathbf{v}_{2p}) + 3b|\varepsilon|^2 \varepsilon \mathbf{v}_p * \mathbf{v}_p * \mathbf{v}_{-p} + \dots \quad (\text{A.16})$$

The coefficients η_0 and η_{2p} are determined from (4.11b). Up to order $|\varepsilon^2|$ we get:

$$\eta_0 = \frac{a}{(1 - C_{p/q})} |\varepsilon|^2 + \dots, \quad (\text{A.17a})$$

$$\eta_{2p} = \frac{a}{2(C_{2p/q} - C_{p/q})} \varepsilon^2 + \dots. \quad (\text{A.17b})$$

Inserting (A.17) into (A.16) gives

$$-2\delta\varepsilon = \left[a^2 \left(\frac{2}{(1 - C_{p/q})} + \frac{1}{(C_{2p/q} - C_{p/q})} \right) + 3b \right] |\varepsilon|^2 \varepsilon + \dots. \quad (\text{A.18})$$

Comparing this expression with (4.18c) we find that the expression between square brackets is just A_{01} for $q \geq 5$, also given in (5.2). In order to rewrite it in terms of $C_{p/q}$ only, we have used $C_{2p/q} = 2C_{p/q}^2 - 1$.

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