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ANALYTICAL DESCRIPTION OF THE BASIN AND THE TRANSIENTS OF A POINT ATTRACTOR OF THE HÉNON MAPPING

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Extended abstract

Explicit expressions for closed curves C_n about the attractor that enclose a part of its basin are obtained. If $n \to \infty$ the interior of C_n covers the complete basin. A sequence of invertible polynomial transformations is given, that converges rapidly towards the normal transformation, that transforms the nonlinear motion in the basin to a linear one. Using these transformations we obtain some practical tools to describe one aspect of the transient behavior in the basin.

1. Results

Consider a quadratic invertible mapping of the real plane, $x \to x'$, with

$$x' = Hx, \quad Hx \equiv Px + Q(x), \tag{1}$$

with constant positive Jacobian b, smaller than unity. $Q(x)$ is a homogeneous quadratic expression in the components x_1 and x_2. The matrix P is assumed to have two different complex eigenvalues, such that the origin is a spiral attractor.

We obtain a sequence of polynomial expressions, representing closed curves C_n about the origin, which enclose a part of its basin of attraction. For each enclosed domain we shall obtain a polynomial Lyapunov function $L_n(x)$, which controls the rate of convergence,

$$\begin{aligned} &L_n(Hx) \leqq \theta_n(L_n(x))L_n(x), \\ &\theta_n(L_n) \equiv b^{1/2} + b^{n/2}qL_n, \end{aligned} \tag{2}$$

where q is some positive constant representing the 'strength' of the nonlinear term (cf. sec. 2). At sufficiently large n, each x in the basin is enclosed by a contour C_n. Fig. 1 shows some contours for the Hénon mapping [1].

To describe the transient behavior we introduce a functional $n_\varepsilon(x)$ that represents the number of steps necessary to map x into a well defined ε-neighborhood of the origin: below a Lyapunov function $L(x)$ is defined, for which

$$L(Hx) = b^{1/2}L(x) \tag{3}$$

for each x in the basin. Eq. (3) shows that a level line $L(x) = L_0$ is mapped onto the level line $L(x) = b^{1/2}L_0$. Thus the number of steps, necessary to map x into a small region given by $L(x) \leqq \varepsilon$, depends on L only and is

$$n_\varepsilon(x) \equiv \mathrm{INT}\left[(\ln L(x) - \ln \varepsilon)/\ln b^{-1/2}\right], \tag{4}$$

i.e. the integer part of the expression in brackets.

The irregular shape of the area enclosed by a level line (cf. fig. 2) indicates irregular transient behavior. One aspect of this behavior is formulated more precisely in terms of the gradient of L: in a (sufficiently) small neighborhood of some x_0 the average value of $n_\varepsilon(x)$ equals $n_\varepsilon(x_0)$ and one readily shows that its root mean square is proportional to the length of $(\nabla L/L)_{x=x_0}$ (cf. fig. 3). Both for the level lines and the gradient we have explicit approximate expressions.

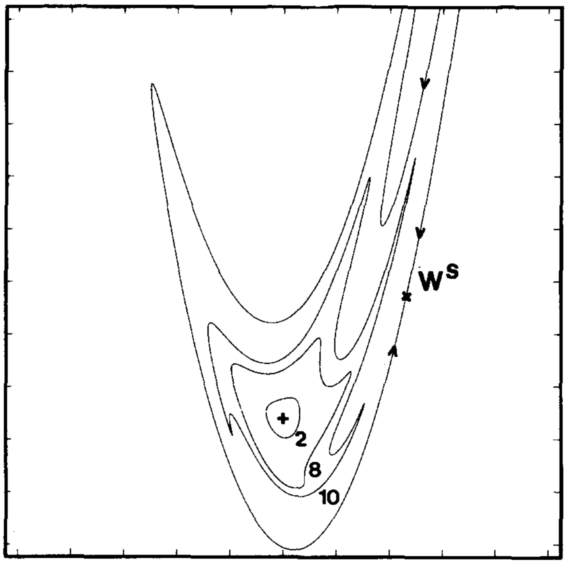

Fig. 1. Contours C_2, C_8 and C_{10} for the Hénon mapping (1) with $P = \begin{pmatrix} 2c & -b \\ 1 & 0 \end{pmatrix}$ and $Q(x) = \begin{pmatrix} 2x_1^2 \\ 0 \end{pmatrix}$ (cf. [4]), $b = 0.7$ and $c = 0$. + denotes the attractor and × is a saddle fixed point. The curve through × is the stable manifold W^s of the saddle. The basin of attraction is certainly within the region enclosed by this stable manifold. There can be other attracting orbits within this region however [5].

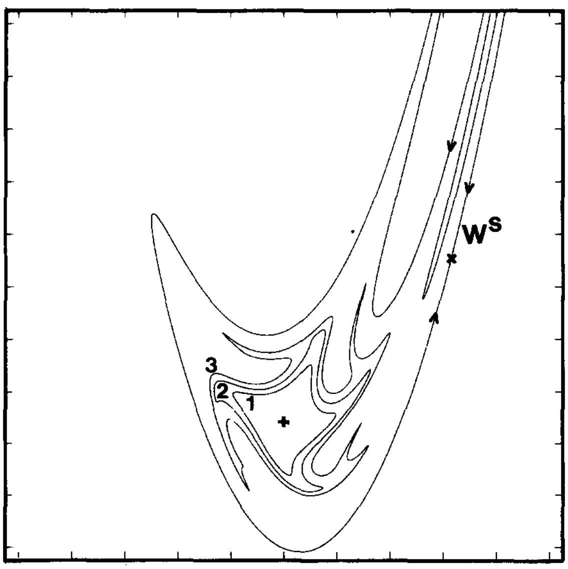

Fig. 2. Level lines of L for $b = 0.9$ and $c = 0$. Curve 3 is mapped by H in 5 steps onto curve 2, etc. Observe that the shape of the enclosed area is more complicated for larger values of L.

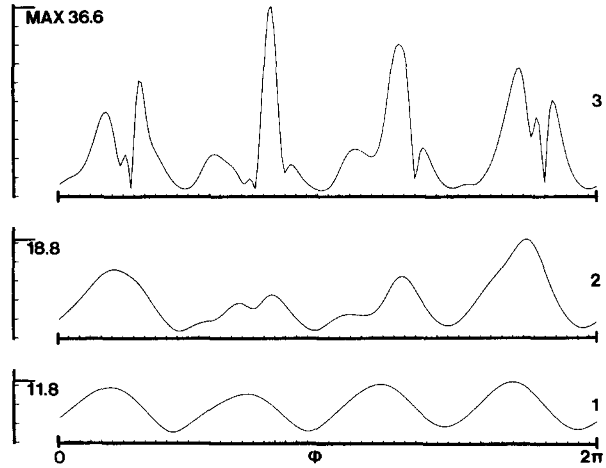

Fig. 3. $\|\nabla L\|/L$ along the level lines of fig. 2. Observe that the maximal value and the complexity of the graphs increase with L.

2. Method: transformation to normal form

To obtain the results above a sequence of invertible coordinate transformations $x_n(u)$, with inverse $u_n(x)$,

$$x_n(u) \equiv H^{-n}(P^n u), \quad u_n(x) \equiv P^{-n} H^n(x), \tag{5}$$

is introduced. It can be proved [2] that the sequence $\{x_n(u)\}$ converges to an analytic function $x(u)$, that is defined for $u \in \mathbf{R}^2$ and whose range is the basin of attraction of the origin. Furthermore, $x(u)$ transforms the mapping $x' = H(x)$ restricted to this basin, to the linear mapping $u' = Pu$. The existence of this function and its inverse $u(x)$ is guaranteed by an easy extension of a theorem of Poincaré [3].

With these transformations the functionals L_n and L are defined,

$$\begin{aligned} L_n(x) &\equiv (u_n(x), Au_n(x))^{1/2}, \\ L(x) &\equiv (u(x), Au(x))^{1/2}, \end{aligned} \tag{6}$$

where A is a real symmetric positive matrix such that $(Pu, APu) = b(u, Au)$. Such an A exists since

$\dot{P}$ has complex eigenvalues. One readily proves (2) and (3), with q being the smallest positive number such that $(Q(x), AQ(x)) \leqq q^2(x, Ax)^2$.

Furthermore, observe that a level line $L_n(x) = L_0$ is the image in the x plane of an ellipse $(u, Au) = L_0$ in the u plane by the mapping $x_n(u)$. Such an ellipse is parametrized by $u = L_0^{1/2}(a_1 \cos\phi + a_2 \sin\phi)$, a_1 and a_2 being eigenvectors of A of appropriate length. Consequently we have explicit expressions for the level lines of L_n. Since $L_n(x)$ has only one stationary point, which is a minimum at the origin, it is clear from (2) that curves C_n for which $\theta_n = 1$ enclose a domain in the basin of attraction. It can be proved that a level line $L_n(x) = L_0$ converges to a level line $L(x) = L_0$ [2], and that $\nabla L_n(x)$ along a level line converges to $\nabla L(x)$.

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