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Evolving Solitons in Bubbly Flows

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Abstract. At the end of the sixties, it was shown that pressure waves in a bubbly liquid obey the KdV equation, the nonlinear term coming from convective acceleration and the dispersive term from volume oscillations of the bubbles.

For a variable u, proportional to $-p$, where p denotes pressure, the appropriate KdV equation can be casted in the form $u_t - 6uu_x + u_{xxx} = 0$. The theory of this equation predicts that, under certain conditions, solitons evolve from an initial profile $u(x,0)$. In particular, it can be shown that the number N of those solitons can be found from solving the eigenvalue problem $\psi_{xx} - u(x,0)\psi = 0$, with $\psi(0) = 1$ and $\psi'(0) = 0$. N is found from counting the zeros of the solution of this equation between $x = 0$ and $x = Q$, say, Q being determined by the shape of $u(x,0)$. We took as an initial pressure profile a shockwave, followed by an expansion wave. This can be realised in the laboratory and the problem, formulated above, can be solved exactly.

In this contribution the solution is outlined and it is shown from the experimental results that from the said initial disturbance, indeed solitons evolve in the predicated quantity.

Mathematics Subject Classification (1991): 58F07.

Key words: waves, solitons.

1. Introduction

In the 1960s and 1970s it was recognised that it was not only gravity waves on the free surface of a liquid of finite depth that obeyed the Korteweg–de Vries (KdV) equation: other waves do, too.

Armed with experience, we can today safely predict that waves displaying both nonlinearity and dispersion are governed by a KdV equation, provided dissipation is sufficiently small. Pressure waves in bubbly fluids belong to this class, as was shown by Van Wijngaarden [11] who derived the corresponding KdV equation.

Experiments to verify this were carried out in Russia, as reported in the book by Nakoryakov *et al.* [5] and in our laboratory [6, 7]. While there is no question that these experiments validated the theory qualitatively, it cannot be said that they did this in a completely quantitative way. Various reasons can be given for this. One of them is that many of these experiments concern transitions of the bore or shock-wave type, from one constant level of pressure to another. The Korteweg–de Vries equation has no steady solution for such a transition. In practice, seemingly steady, or quasi-steady pressure profiles are obtained due to an inevitable, though small, dissipation. Such a small dissipation is much less important when the disturbance, from a uniform quiescent state, is of finite spatial extent. Such experiments have, for example, been reported by Nakoryakov *et al.* [5]. However, a mathematical solution of the associated KdV equation was not given these. Also, as is emphasized by Watanabe and Prosperetti [8], thermal effects are extremely important in waves evolving over a long time, obscuring the dynamics.

With a view to these, rather recently gained, insights it seems worthwhile, and especially now that we are celebrating the 100th anniversary of the seminal paper by Korteweg and de Vries [3], to draw attention to a series of experiments which my student P. M. Roelofsen and I did about ten years ago, and which allow a comparison with the exact solution of the pertinent KdV equation.

2. Description of Experiment

The experiments discussed here were carried out in the facility described in detail by Noordzij and Van Wijngaarden [7]. The basic elements are shown in Figure 1.

A long tube of 11 cm diameter contains a mixture of water and small gas bubbles, mainly CO_2 in these experiments. Above the free surface of the mixture, for times $t' < 0$ is a low pressure region with height h_1 and a high pressure region of height h_2.

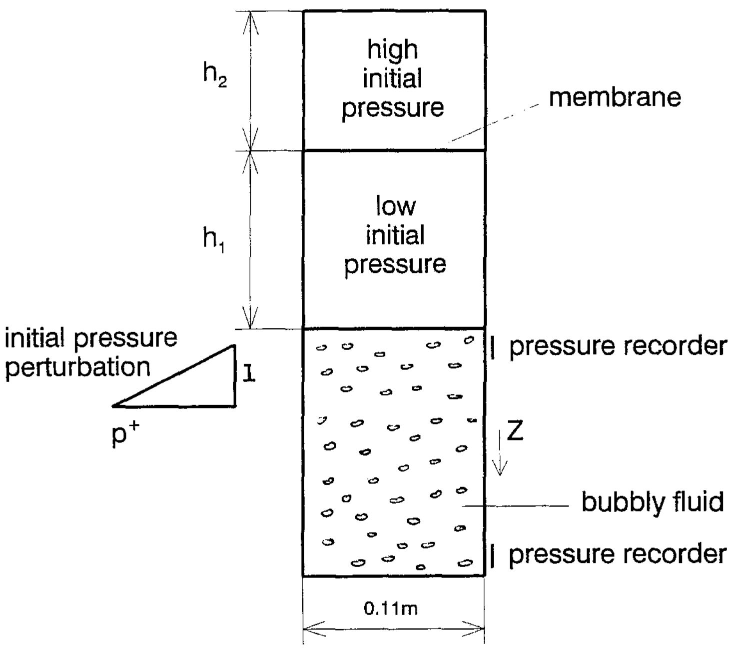

Fig. 1. Schematic representation of shock tube.

Let the pressure for $t' < 0$ be p_0 in the low pressure region and in the mixture. At $t' = 0$ the membrane, which separates the low pressure region from the high pressure region, is punctured. By an adroit choice of h_1, h_2 and the pressure in the high pressure region a pressure perturbation is generated which, at the entrance to the mixture, $z = 0$ in Figure 1, has the shape of a right-angled triangle.

First comes an instantaneous rise of the pressure to a value p^+, followed by an expansion wave in which the pressure linearly returns to p_0. Hence the perturbation is of finite extent, as opposed to a shock wave which produces a semi-infinite perturbation. Since the propagation into the bubbly water is both nonlinear (at sufficiently large $p^+ - p_0$) and dispersive, the initial shape is not preserved but evolves according to the KdV equation.

Before passing on to the way in which this happens, we discuss the most important physical quantities in the experiments:

Bubble radius a, in the undisturbed conditions a_0. In our experiments these are all of the same radius, which varies from one experiment to the other, but is always of the order of 1 mm.

Concentration of gas by volume α. The initial value α_0 can be measured from determining the static pressure distribution along the tube.

Velocity of sound c_0. Denoting the density of the fluid by ρ, the undisturbed pressure as before by p_0, the ratio of specific heats of the gas by γ we have for this the relation [12]:

$$c_0^2 = \frac{\gamma p_0}{\rho \alpha_0 (1 - \alpha_0)} . \tag{2.1}$$

Note that for values of α_0 of a few percent, c_0 is of the order of 10^2 m/s, under atmospheric conditions.

The natural frequency for small amplitude volume oscillations of a bubble ω_B. This has been found by Minnaert [4] to be given by

$$\omega_B^2 = \frac{3\gamma p_0}{\rho a_0^2} . \tag{2.2}$$

At frequencies ω of external disturbances which are far below ω_B, bubbles just follow the external pressure variations, and linear waves pass undeformed through the mixture with wavelength $2\pi c_0/\omega$. The closer the frequencies approach ω_B, the more this changes. The speed of propagation depends on ω for linear waves, long waves travelling faster than short waves. By nonlinear effects, the speed of propagation depends on the wave amplitude. The two phenomena together give rise to interesting behaviour.

3. Some Experimental Results

At various places along the tube (Figure 1), pressure transducers are placed, enabling the recording of the pressure as a function of time. We have done experiments in which the pressure was recorded, generated by the initial signal, in $z = 0$ in the form of a shock wave followed by an expansion wave.

Figures 2 and 3 show some results. In Figure 2 the pressure transducer is located at 47 cm from the free surface; in Figure 3 it is at 97 cm. The particulars of the experiments are given in Table I. In Figure 2 both the profile in $z = 0$ and the following profile in $z = 0.47$ m are shown; in Figure 3 only the profile in $z = 0.97$ m. For this experiment $l = 30$ cm and $p^+ - p_0 = 34$ kPa. Clearly, in both experiments the original disturbance, in the shape of a right-angled triangle, breaks up into a distinct number of wave structures. This is precisely what solution of the KdV equation predicts. The outstanding feature of these experiments is that for this particular initial shape the number of evolving distinct structures, which we shall recognise as solutions, can be exactly calculated and compared with the experimental results.

4. Theory

We denote the dimensionless pressure perturbation by v,

$$v = \frac{p - p_0}{p_0}. \qquad (4.1)$$

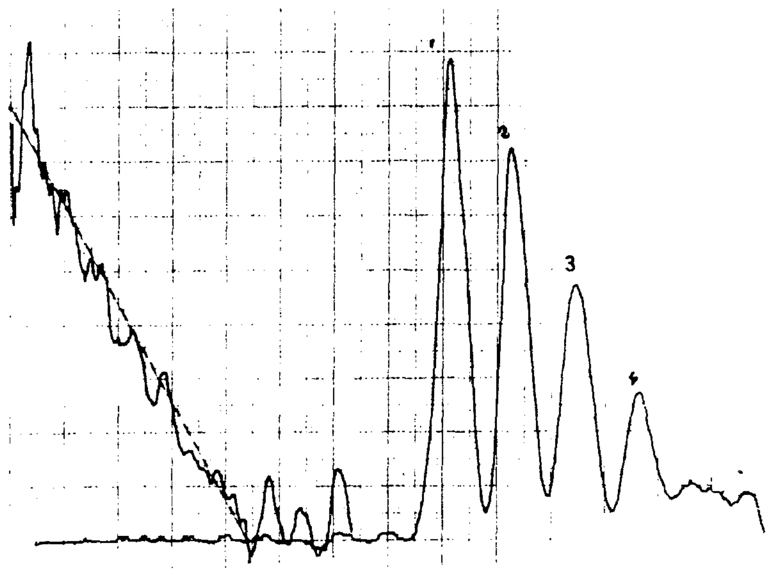

Fig. 2. The pressure profile on entering the mixture (left) and as it appears at the pressure transducer in $z = 0.47$ m below the interface. Data are given in Table I under 'exp 1'.

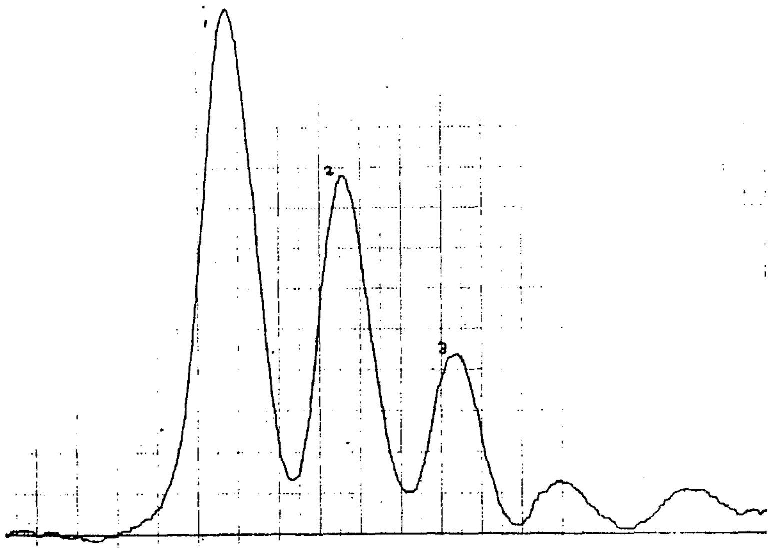

Fig. 3. Pressure profile measured by a transducer in $z = 0.97$ m below the interface following from the original profile with data as given in Table I under 'exp 4'.

Using the quantities defined in Section 2, and in addition the dispersion parameter σ,

$$\sigma = \frac{a_0^2}{\alpha_0(1-\alpha_0)} = \frac{3c_0^2}{\omega_B^2}, \tag{4.2}$$

Van Wijngaarden [11] showed that $v(z, t')$ evolves according to the equation

$$\frac{\partial v}{\partial t'} + c_0 \frac{\partial v}{\partial z} + \frac{\gamma+1}{2\gamma} c_0 v \frac{\partial v}{\partial z} + \frac{\sigma c_0}{6} \frac{\partial^3 v}{\partial z^3} = 0. \tag{4.3}$$

It is convenient to choose a new coordinate $x' = z - c_0 t'$, moving with the undisturbed speed of sound, and in addition the dimensionless variables x, t and η, defined by

$$x' = \left\{\frac{\sigma l \gamma}{3(\gamma+1)}\right\}^{1/3} x, \tag{4.4}$$

$$t' = \frac{2\gamma l}{(\gamma+1)c_0} t, \tag{4.5}$$

$$v = -\left\{\frac{72\sigma\gamma}{l^2(\gamma+1)}\right\}^{1/3} u. \tag{4.6}$$

We recall (see Figure 1) that l is the length of the initial pressure disturbance.

In terms of the variables x, t and u, (4.3) takes the standard form of the KdV equation employed in the literature (e.g., [9, p. 586])

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{4.7}$$

This equation is valid according to the theory, provided:

(i) dispersion is weak;
(ii) dissipation is negligible; and
(iii) nonlinearity is weak.

The first condition, (i), means that a typical frequency c_0/l must be small with respect to ω_B. The ratio $c_0/\omega_B l$ is about 0.1 in our experiments. The second condition, (ii), is also fulfilled. When we take dissipation (viscous, but mainly thermal) into account, this would in Equation (4.7) lead to an additional term $0.06\,\partial^2 u/\partial x^2$ on the left-hand side. Such a term is negligible in our case, where the scale is determined by (4.4) but becomes important for phenomena of large extent. In other words, the approximation underlying (4.7) is not a uniformly valid one. Regarding (iii) we have chosen pressure disturbances of moderate value, 0.2–0.5, guaranteeing a not-too-strong nonlinearity.

Theory predicts that out of the initial profile $u(x, 0)$ evolve

(a) N solitons, N to be found from inverse scattering theory, and travelling in the positive x direction;
(b) a continuous wave train travelling in the negative x direction.

We recall that the x-axis moves with speed c_0 in the z direction. This means that the continuous wave train moves in the laboratory frame rapidly in the negative z direction. These waves can easily be distinguished from the solitons, since they have small amplitude and move quickly. We shall therefore restrict our attention here to the solitons. These emerge from the spectrum of discrete eigenvalues of the Schrödinger equation

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + \{\lambda - u(x,t)\}\varphi = 0, \tag{4.8}$$

in which u serves as potential through Miura's transformation

$$u = \frac{1}{\varphi}\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + \lambda. \tag{4.9}$$

The eigenvalues λ_n of (4.8) are negative, $\lambda_n = -k_n^2$, and, most importantly, independent of time. They are ordered such that $k_1 > k_2 > k_3 > \cdots > k_N > 0$, where N is finite. Because they are independent of time, they can be found from solution of

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + \{\lambda - u(x,0)\}\varphi = 0. \tag{4.10}$$

Asymptotically – that is for large x and t – each eigenvalue corresponds to a soliton. The problem is therefore to find the total number, N, of eigenvalues. In the present problem, this can be found exactly.

First we insert the appropriate form of $u(x,0)$ into (4.10). In physical coordinates x' or z (which are the same at $t' = 0$), the pressure profile is

$$\begin{aligned} v = \frac{p - p_0}{p_0} &= \frac{p^+ - p_0}{p_0}\frac{x'}{l}, \quad 0 < x' < l \\ &= 0, \quad x' \geqslant l \quad \text{and} \quad x' \leqslant 0. \end{aligned} \tag{4.11}$$

Using (4.4) and (4.6) this gives for $u(x,0)$

$$\begin{aligned} u(x,0) &= -\frac{p^+ - p_0}{p_0}\frac{x}{6}, \quad x < \left\{\frac{3l^2(\gamma+1)}{\gamma\sigma}\right\}^{2/3} \\ &= 0, \quad x \leqslant 0 \quad \text{and} \quad x \geqslant \left\{\frac{3l^2(\gamma+1)}{\gamma\sigma}\right\}^{2/3}. \end{aligned} \tag{4.12}$$

We consider solutions of (4.10), $u(x,0)$ being given by (4.12), and with boundary conditions

$$x = 0\colon \ \frac{\partial\varphi}{\partial x} = 0, \quad \varphi = 1. \tag{4.13}$$

Sturm's First Comparison Theorem (see, e.g., [2, Ch. 10]) states the following: suppose a solution, φ_a, corresponding with λ_a, has two zeros, in x_1 and x_2, say, on the x-axis. Then a solution φ_b, for $\lambda_b > \lambda_a$, has an additional zero in between x_1 and x_2. Now we apply this to our problem. For $\lambda \to -\infty$ the solution of (4.10), subjected to (4.13), is cosh λx which has no zeros for real x. Hence the solution φ_1 corresponding with the largest value of k, k_1, has one zero on the x-axis, the solution φ_2 associated with k_2 has two zeros, etc. Since all eigenvalues are negative, the solution corresponding with $\lambda = 0$ has as many zeros as the one with the nearest eigenvalue which is $\lambda_N = -k_N^2$. Therefore, we have to determine the number of zeros of the solution of (4.10) with $\lambda = 0$ and $u(x,0)$ given by (4.12) and with (4.13) as boundary conditions. (This method has been used before by Hammack and Segur [1] in a problem concerning waterwaves.)

We take $\lambda = 0$ and insert (4.12) into (4.10). It is convenient to use instead of x a new coordinate y, given by

$$x = \left(\frac{6p_0}{p^+ - p_0}\right)^{1/3} y. \tag{4.14}$$

Upon using this in inserting (4.12) into (4.10), the latter equation is transformed into the Airy equation [10],

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}y^2} + y\varphi = 0, \quad 0 < y < Q, \tag{4.15}$$

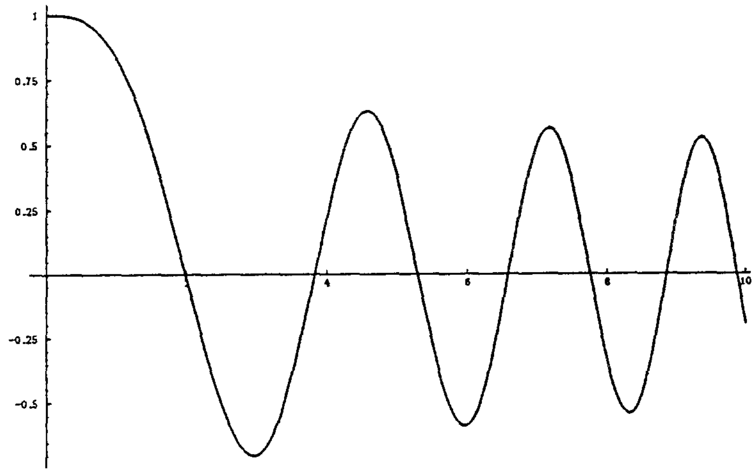

Fig. 4. The function $1.408Ai(-y) + 0.813Bi(-y)$, in the right-hand side of (4.20).

where

$$Q = \left\{ \frac{p^+ - p_0}{p_0} \frac{(\gamma + 1)l^2}{2\sigma\gamma} \right\}^{1/3} \tag{4.16}$$

and

$$\frac{d^2\varphi}{dy^2} = 0, \qquad y \leqslant 0 \quad \text{and} \quad y \geqslant Q. \tag{4.17}$$

The boundary conditions (4.13) become

$$y = 0\text{:}\ \frac{\partial\varphi}{\partial y} = 0, \quad \varphi = 1. \tag{4.18}$$

The system (4.15)–(4.18) has the solution

$$y < 0, \qquad \varphi = 1; \tag{4.19}$$

$$0 \leqslant y \leqslant Q, \qquad \varphi = 1.408Ai(-y) + 0.813Bi(-y), \tag{4.20}$$

$$y > Q, \qquad \bar{\varphi} = \varphi = \left(\frac{d\varphi}{dy}\right)_{y=Q}(y - Q) + \varphi(Q), \tag{4.21}$$

φ being given by (4.20).

Figure 4 shows a sketch of the function on the right-hand side of (4.20). For a given value of Q we have to count the zeros in $0 < y < Q$. The part in $y < 0$

has no zeros, whereas the part of the solution in $y > Q$ contributes an additional zero when

$$\left\{\frac{\mathrm{d}\varphi}{\mathrm{d}y}/\varphi\right\}_{y=Q} < 0.$$

5. Comparison with Experiment

In Table I, we collect data of a number of our experiments. In the final columns the number N_t of solitons is given, as following from the theory, expounded above, and the number N_e from the corresponding experiment.

From the table it follows that in all experiments the number of solitons in both theory and experiment agree completely. The experiments shown in Figures 2 and 3 show this for experiments 1 and 4 in the table, respectively. The same agreement holds also for other experiments, not displayed here, verifying the validity of the KdV equation for these waves.

6. Shape of Solitons

We have labelled the evolving structures somewhat loosely as solitons. In fact, the distinct pressure humps shown in Figures 2 and 3 are not yet solitons, because they are still interacting and will only further down the tube become true solitons. If the peak pressure in a soliton is p_s, and accordingly $v_s = (p_s - p_0)/p_0$, a soliton has the shape

$$v = v_s \mathrm{sech}^2\left[\left(\frac{v_s}{2}\frac{\gamma+1}{\sigma\gamma}\right)^{1/2}\left(x' - \frac{1}{3}\frac{\gamma+1}{2\gamma}v_s t'\right)\right]$$

In some experiments, with the pressure transducer at $z = 0.97$ m or further down, this shape could indeed be verified. At closer distance solitons are still undergoing evolution to their final shape. Their number, however, does not change, anymore.

TABLE I.

Exp.	Location press. transd. m	Void fraction α %	l m	C_0 m s^{-1}	$\frac{p^+-p_0}{p_0}$	Q	N_t	N_e
1	0.47	0.74	0.24	107	0.52	6.6	4	4
2	0.47	0.74	0.19	107	0.30	4.7	3	3
3	0.47	0.74	0.16	107	0.16	3.4	2	2
4	0.97	0.43	0.30	157	0.34	5.8	3	3
5	0.97	1.32	0.13	90	0.13	2.7	1	1
6	0.97	0.74	0.18	118	0.25	4.3	2	2

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