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Numerical simulations of nonlinear thermally stratified spin-up in a circular cylinder

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We present a numerical study of incremental spin-up of a thermally stratified fluid enclosed within a right circular cylinder with rigid bottom and side walls and stress-free upper surface. This investigation reveals a feasibility for transition from an axisymmetric initial circulation to nonaxisymmetric flow patterns, and it is motivated by the desire to compare the spin-up for Dirichlet and Neumann thermal boundary conditions. The numerical simulations demonstrate that the destabilizing mechanism is not purely baroclinic but that vertical and horizontal shears may contribute to the instability. By characterizing the azimuthal instabilities without introducing any simplification, we were able to assess to what extent an insulating boundary condition changes the time-dependent emergence of the instability. Our results agree with previous experimental data and provide a framework for understanding the role played by the baroclinic vorticity in the development of instabilities in thermally stratified incremental spin-up flows. © 2010 American Institute of Physics. [doi:10.1063/1.3505025]

I. INTRODUCTION

Stratified spin-up flow—an impulsive change of the rotation rate of a rigid container filled with a liquid—is a classical fluid mechanics problem that has an established relevance to large-scale geophysical flows.^{1,2} Some examples are surface-wind-driven flows,³ currents in coastal regions,^{4,5} geophysical vortical flows such as ocean rings, arctic eddies and, in the atmosphere, the polar vortex.^{6–9} Spin-up of two-layer liquid systems is also relevant to the investigation of rotating gravity currents in the atmosphere and oceans^{2,10–13} (see Ref. 14 for a comprehensive review).

Because spin-up/down flows are transient problems, they cannot be analyzed for stability in the usual sense. The final state will depend on the final density profile: the curvature of the isotherms near the bottom/top of the insulated walls results in radial diffusive mass transport and the associated Sweet-Eddington azimuthal flow.^{15,16} This is a nonlinear question where the interest is in the state of the transient flow.

One of the earliest studies of spin-up leading to the formation of cyclonic and anticyclonic eddies is due to Greenspan,¹⁷ who considered stratified fluid in a cylindrical container. The qualitative observations made in Ref. 17 were extended and quantified in the experimental investigation of Ref. 18. Particular attention was paid to determining the criteria for axisymmetric breaking instabilities by adapting the stability analysis of Ref. 19. The initial phase of motion in spin-up reported by Refs. 18 and 20 was characterized by the formation of axisymmetric corner regions that accumulated

fluid drawn through the Ekman layer. The formation of corner regions due to upwelling coupled with the nonpenetration boundary conditions for density forced the isopycnals within the central core to be displaced vertically. This vertical stretching reduced the density gradient within the core, and its buoyancy frequency following the establishment of the corner regions decreased. The stability of the central vortex was observed to depend on its stratification and radius-to-height aspect ratio. For highly stratified flows, the axisymmetry was retained and the central vortex reached a state of near solid-body rotation. For relatively low stratification, the axisymmetry of the flow was broken due to baroclinic unstable waves that propagated along the interface of the core and corner regions.

Similar nonaxisymmetric instabilities in which cyclones near the outer wall propagated toward the center of the tank were also observed in two-layer stratification flows with a thin lower layer.² A region where the upper layer was in direct contact with the bottom of the tank (bare spot) was formed during the spin-up. The circular shape of the bare spot was distorted by the presence of waves, which grew to large amplitude on the front surrounding the bare spot. The flow patterns observed by these waves were similar to those produced by a front at the free surface of baroclinic instabilities of Ekman layers observed by Ref. 19.

Columnar baroclinic vortices generated by the instability of stratified fluids due to a change in rotation rate of its container (cylinder or annulus) from $\Omega_i = \Omega - \Delta\Omega$ to Ω were reported in Refs. 21–23. It was found that inside the corner regions, which develop at the intersection of the vertical sidewall and horizontal bottom boundary, the density gradi-

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ents were weak (the fluid was well-stirred), while above the corner regions, the density gradient was higher compared to the initial background value. The eddies grew in size and marched along the circumference until they occupied a large portion of the tank. These laboratory experiments (conducted at much later spin-up times) have stressed the importance of both baroclinic (vertical) and barotropic (horizontal) shears in the symmetry breaking process. However, despite the efforts to elucidate which mechanisms dominate the breakup of symmetry, at what time, and how these mechanisms interact among each other, they remain elusive.²⁴

On the other hand, numerical simulations of spin-up/down have been limited to investigating the axisymmetric flow evolution.^{10,20,25–35} Recent three-dimensional numerical simulations analyzed homogeneous swirling flows created by a differential rotation in cylinder-lid enclosures.^{36–38} However, notwithstanding the numerous numerical studies on spin-up flows, three-dimensional simulations are conspicuously lacking.

This work is concerned with the numerical investigation of nonlinear incremental spin-up of a thermally stratified fluid with kinematic viscosity ν and thermal diffusivity κ in a cylinder of radius R and height $2H$ rotating about a vertical axis. The rationale for this investigation is the comparison of spin-up for Dirichlet and Neumann thermal boundary conditions on the end-walls. Initially, its temperature varies linearly with height and is characterized by a constant buoyancy frequency N , which is proportional to the temperature gradient. The system undergoes an abrupt change in the rotation rate from its initial value $\Omega_i = \Omega - \Delta\Omega$, when the fluid is in a solid-body rotation state, to the final value Ω . Our investigation focuses on the regime that corresponds to the transient Ekman bottom boundary layer. Because the exact conditions for the transition to turbulence for stratified spin-up are unknown, we use the criterion of Lilly³⁹ to determine the region of the stability, i.e., $\text{Re}_\delta < 55$, where Re_δ is the Reynolds number based on the bottom Ekman layer depth. Laboratory experiments in Ref. 40 showed that the Ekman layer remains stable until $\text{Re}_\delta \approx 57$ and becomes fully turbulent at $\text{Re}_\delta \geq 150$. Recent measurements of the bottom friction law during homogeneous spin-up over a flat surface also confirmed these estimates.⁴¹ Since the parametric studies of Refs. 18 and 21–23 suggest that for tall containers ($\Gamma < 1$) the spin-up in the nonlinear regime is less prone to become unstable, we restrict our simulations to large aspect ratios ($\Gamma > 1$) and compare the instabilities arising under different boundary conditions on the horizontal walls. We consider five different runs listed in Table I where the following notation indicates the set of parameters used in the simulations: NEU and DIR correspond to the nonlinear stratified flow with Neumann and Dirichlet boundary conditions on the horizontal walls of the cylinder respectively; HOM stands for the spin-up of a homogeneous fluid (no stratification); and LBU and SRO designate the large-Burger number and small-Rossby number runs with Neumann boundary conditions, respectively. Using Fourier analysis of the velocity field in the azimuthal direction, we were able to identify the most unstable wave number n as the mode with largest growth rate. We computed the time evolution of each term of

the perturbation energy equation, which allowed us to determine the contribution to the instability of the barotropic, baroclinic, centrifugal, and dissipative terms. Our two objectives are to study the characteristics of the azimuthal instabilities without introducing any simplification and to determine the extent that an insulating, rather than fixed temperature, boundary condition would change the time-dependent emergence of the instability.

The three-dimensional numerical solutions allowed us to identify the mechanism responsible for the nature of the vortex structure at the late stages of flow development when using Neumann or Dirichlet boundary conditions. The simulations show a noticeable but expected difference in stratification between isolated and constant temperature boundary conditions (NEU and DIR), and this is accompanied by a peculiar change of the isobars, which in turn triggers different instabilities. Evidence of this conjecture is given at the end of the Sec. III.

II. NAVIER-STOKES EQUATIONS AND THE NUMERICAL SCHEME

A. Governing equations

The system of interest consists of a circular cylinder of radius R rotating counterclockwise about its axis of symmetry with the rotation rate $\Omega \mathbf{e}_z$, where $\mathbf{e}_z$ is the unit vector pointing in the positive direction of $\hat{z}$ axis (see Fig. 1). We consider the case when the gravity and rotation vectors are colinear, i.e., $\mathbf{g} = -g\mathbf{e}_z$. The fluid occupies the domain $0 \leq \hat{r} \leq R$ and $0 \leq \hat{z} \leq 2H$ so that the total height of the cylinder is $2H$. At time $\hat{t} = 0$, the system is instantly accelerated by the amount $\Delta\Omega$ to a new rotation rate Ω from its initial state $\Omega_i = \Omega(1 - \epsilon)$, where $\epsilon = \Delta\Omega/\Omega$. We describe the fluid motion relative to the cylindrical coordinate system $\hat{\mathbf{r}} = (\hat{r}, \hat{\theta}, \hat{z})$ rotating with the final rotation rate Ω . The components of the velocity vector $\hat{\mathbf{u}} = (\hat{u}_r, \hat{u}_\theta, \hat{u}_z)$ represent the radial, azimuthal, and vertical velocities, respectively, with respect to this frame. Note that in this reference frame, the final state will depend on the final density profile: the curvature of the isotherms near the bottom/top of the insulated walls results in radial diffusive mass transport and the associated Sweet-Eddington azimuthal flow.^{15,16}

The governing equations that describe the motion of an incompressible flow of density ρ in the rotational (noninertial) reference frame in the Boussinesq limit have the following form:^{42,43}

$$\hat{\nabla} \cdot \hat{\mathbf{u}} = 0, \quad (1a)$$

$$\frac{D\hat{\mathbf{u}}}{D\hat{t}} + 2\Omega \times \hat{\mathbf{u}} = -\frac{1}{\rho_o} \hat{\nabla} \hat{p} - \frac{\rho}{\rho_o} g \mathbf{e}_z + \frac{\rho}{\rho_o} \Omega^2 \hat{r} \mathbf{e}_r + \nu \hat{\nabla}^2 \hat{\mathbf{u}}, \quad (1b)$$

$$\frac{D\hat{T}}{D\hat{t}} = \kappa \hat{\nabla}^2 \hat{T}. \quad (1c)$$

A temperature difference of 20 °C allowed us to retain the Boussinesq approximation in the governing equations,⁴⁴

TABLE I. Parameters used in the simulations. The dash in the table implies that the number from the previous column did not change.

		Case			
	Expression	NEU/DIR	HOM	LBU	SRO
<i>Dimensional parameters</i>					
Height	$2H$ (cm)	6	...	...	...
Radius	R (cm)	20	20	7	20
Final rotation rate	Ω (rad/s)	0.384	...	...	...
Rotation rate increment	$\Delta\Omega$ (rad/s)	0.279	0.279	0.279	0.069
Buoyancy frequency	N (rad/s)	0.97	0	0.97	0.97
Temperature drop	$2\Delta\hat{T}$ (°C)	20	0	20	20
Mean density	ρ_o (g/cm ³)	1	...	...	...
Gravitational acceleration	g (cm/s ²)	981	...	...	...
Kinematic viscosity	ν (cm/s ²)	0.01	...	...	...
Thermal diffusivity	κ (cm/s ²)	1.46×10^{-3}	...	...	...
Thermal expansion coefficient	α [(°C) ⁻¹]	2.86×10^{-4}	...	...	...
Ekman layer depth	$\hat{\delta}=(\nu/\Omega)^{1/2}$ (cm)	0.16	...	...	...
Rossby deformation radius	$R_d=NH/\Omega$ (cm)	7.58	0	7.58	7.58
Characteristic spin-up time	$\hat{t}_{\text{su}}=2H/(\nu\Omega)^{1/2}$ (s)	97	...	...	...
Characteristic velocity	$U=\Delta\Omega R$ (cm/s)	5.58	...	1.953	1.38
<i>Nondimensional parameters</i>					
Aspect ratio	$\Gamma=R/H$	6.67	6.67	2.33	6.67
Burger number	$\text{Bu}=2B/\Gamma$	0.379	0	1.083	0.379
Ekman number	$E=\nu/2\Omega H^2$	1.45×10^{-3}	...	...	...
Ekman layer depth	$\delta=\hat{\delta}/H$	5.38×10^{-2}	...	...	...
Froude number	$F=\Omega^2 H/2g$	2.30×10^{-4}	...	...	...
Prandtl number	$\sigma=\nu/\kappa$	6.85	...	...	...
Reynolds number	$\text{Re}_{\delta}=U\hat{\delta}/\nu$	90	90	32	22
Reynolds number rotational	$\text{Re}_{\Omega}=UR/\nu$	11 160	11 160	1367	2760
Rossby number	$\epsilon=\Delta\Omega/\Omega$	0.727	0.727	0.727	0.1797
Spin-up time	$\tau_{\text{su}}=\Omega\hat{t}_{\text{su}}/2\pi$	5.9	5.9	5.9	6.85
Stratification parameter	$B=N/2\Omega$	1.26	0	1.26	1.26

where any variations in the physical properties of fluid (water) with temperature were considered negligible. In all cases, the sidewall of the cylinder is thermally insulated.

In the equations above, $\hat{p}$ and $\hat{T}$ are the total pressure and temperature functions, respectively, while ρ_o is the mean density of the fluid. The unit vector $\mathbf{e}_r$ points in the positive $\hat{r}$ -direction. The kinematic viscosity and thermal diffusion coefficients are ν and κ , respectively, and the material derivative operator is defined as $D/D\hat{t}=\partial/\partial\hat{t}+\hat{\mathbf{u}}\cdot\hat{\nabla}$, where $\hat{\nabla}$ is

FIG. 1. Schematic of the fluid system. At $\hat{t}=0$, the cylinder is instantly accelerated from the initial rotation rate $\Omega_i=\Omega(1-\epsilon)$ to a new rotation rate Ω . The inset shows the formation of the corner region during the first stage of spin-up at $\tau\approx 6$.

the vector differential operator. We assume that the noninertial reference frame does not accelerate as a whole and its angular acceleration is zero (constant rotation rate). We do not make any *a priori* assumptions about the magnitude of the centrifugal acceleration term in Eq. (1b).

We consider the case of a stable linear background density stratification characterized by the buoyancy frequency squared,

$$N^2 = -\frac{g}{\rho_o} \frac{d\rho_b}{dz} = \text{constant} > 0, \quad (2)$$

so that the background density profile is

$$\rho_b(\hat{z}) = \rho_o \left(1 - \frac{N^2}{g} \hat{z} \right). \quad (3)$$

In the case of a thermal stratification, the relationship between the fluid temperature and density is given by the equation of state

$$\rho = \rho_o \{1 - \alpha(\hat{T} - \hat{T}_o)\}, \quad (4)$$

where α is the coefficient of thermal expansion and $\hat{T}_o = \text{constant}$ is the reference temperature. At $\tau=0$, the horizontal surfaces are set to different temperatures so that the total temperature difference $2\Delta\hat{T}$ occurs over the height of the cylinder $2H$. The square of the buoyancy frequency (2) becomes

$$N^2 = \alpha g \frac{\Delta\hat{T}}{H} > 0, \quad (5)$$

while the background temperature profile is

$$\hat{T}_b(\hat{z}) = \hat{T}_o + \frac{\Delta\hat{T}}{H}\hat{z} = \hat{T}_o + \frac{N^2}{\alpha g}\hat{z}, \quad (6)$$

so that $\hat{T}_o + 2\Delta\hat{T}$ and $\hat{T}_o$ are the initial temperature values at the top and bottom boundaries respectively. In this study, we focus on the case of an open fluid domain (free upper surface), assuming the stress-free condition at the top (effects of surface tension and deformation are neglected). The total pressure and temperature functions are defined as $\hat{p} = \hat{P} - \rho_o g \hat{z} + \rho_o (\Omega \hat{r})^2 / 2$ and $\hat{T} = \hat{T}_o + \hat{T}_H$, where $\hat{P}$ and $\hat{T}_H$ are the reduced pressure and temperature functions, respectively. If we scale the space variables $\hat{r}, \hat{z}$ with the characteristic length H , the velocity components with the characteristic velocity $\Delta\Omega H = \epsilon\Omega H$, the pressure $\hat{P}$ with $2\Omega^2 H^2 \epsilon \rho_o$, the temperature $\hat{T}_H$ with $2\Delta\hat{T}$, and the time $\hat{t}$ with $(2\Omega)^{-1}$, the governing equations in dimensionless form can be written as

$$\nabla \cdot \mathbf{u} = 0, \quad (7a)$$

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \frac{\epsilon}{2} \mathbf{u} \cdot \nabla \mathbf{u} \\ = -\nabla P + \frac{4B^2}{\epsilon} T_H \mathbf{e}_z - \mathbf{e}_z \times \mathbf{u} - \frac{8FB^2}{\epsilon} T_H r \mathbf{e}_r + E \nabla^2 \mathbf{u}, \end{aligned} \quad (7b)$$

$$\frac{\partial T_H}{\partial t} + \frac{\epsilon}{2} \mathbf{u} \cdot \nabla T_H = \frac{E}{\sigma} \nabla^2 T_H. \quad (7c)$$

The initial and boundary conditions become

$$u_r = u_z = 0, u_\theta = -r, T_H = \frac{z}{2} \quad \text{at} \quad \tau = 0, \quad (8a)$$

$$u_r = u_\theta = u_z = 0 \quad \text{at} \quad z = 0, \quad (8b)$$

$$\frac{\partial u_r}{\partial z} = \frac{\partial u_\theta}{\partial z} = u_z = 0, \quad \text{at} \quad z = 2, \quad (8c)$$

$$u_r = u_\theta = u_z = 0, \frac{\partial T_H}{\partial r} = 0, \quad \text{at} \quad r = \Gamma. \quad (8d)$$

The top and bottom walls are either maintained at constant temperatures (Dirichlet boundary conditions)

$$T_H = 0, \quad \text{at} \quad z = 0, \quad (9a)$$

$$T_H = 1, \quad \text{at} \quad z = 2, \quad (9b)$$

or are insulated (Neumann boundary conditions),

$$\frac{\partial T_H}{\partial z} = 0 \quad \text{at} \quad z = 0, 2. \quad (10)$$

In the equations above, $\epsilon = \Delta\Omega/\Omega$ is the Rossby number, $E = \nu/2\Omega H^2$ is the Ekman number, $B = N/2\Omega$ is the stratification parameter, $\sigma = \nu/\kappa$ is the Prandtl number, $\Gamma = R/H$ is the aspect ratio, and $F = \Omega^2 H/2g$ is the rotational Froude number. The nondimensional rotation period is defined as $\tau = \Omega \hat{t}/2\pi = t/4\pi$, which will be used to present the results.

We characterized the azimuthal perturbations using the energy equation. In the system of governing Eq. (6), we decompose all variables into axisymmetric and nonaxisymmetric parts. The axisymmetric part represents the mean flow (averaged quantities on the azimuth), while the nonaxisymmetric part corresponds to the flow perturbations.⁴⁵ Thus, any quantity in Eq. (6) can be expressed as

$$q(r, \theta, z) = \bar{q}(r, z) + q'(r, \theta, z), \quad (11)$$

where

$$\bar{q}(r, z) = \frac{1}{2\pi} \int_0^{2\pi} q(r, \theta, z) d\theta. \quad (12)$$

Substituting Eq. (11) into the momentum Eq. (7b), taking the dot product with $\mathbf{u}'$, and integrating over the entire domain V yield the energy equation for the azimuthal disturbances

$$\begin{aligned} \frac{de}{dt} &= \frac{d}{dt} \int_V \frac{1}{2} |\mathbf{u}'|^2 dV \\ &= -\frac{\epsilon}{2} \int_V \mathbf{u}' \cdot (\mathbf{u}' \cdot \nabla \bar{\mathbf{u}}) dV + \frac{4B^2}{\epsilon} \int_V T_H u'_z dV \\ &\quad - \frac{8FB^2}{\epsilon} \int_V T_H r u'_r dV - E \int_V |\nabla \mathbf{u}'|^2 dV = \sum_{i=1}^4 H_i. \end{aligned} \quad (13)$$

The left-hand-side of Eq. (13) represents the kinetic energy growth rate of the azimuthal disturbance due to (H_1) shear of the mean axisymmetric flow (barotropic production), (H_2) conversion of gravitational potential energy (baroclinic production), (H_3) conversion of centrifugal potential energy, and (H_4) viscous dissipation, which is always a sink for the kinetic energy of disturbances.

One global measure we have used to characterize the various solutions obtained is the kinetic energy in the n th Fourier mode of the solution,

$$E_n = \frac{1}{2} \int_{r=0}^{\Gamma} \int_{z=0}^2 \mathbf{u}_n \cdot \mathbf{u}_n^* r dr dz, \quad (14)$$

where $\mathbf{u}_n$ is the n th Fourier mode of the velocity field and $\mathbf{u}_n^*$ is its complex conjugate.

B. Numerical method

The governing equations (7a)–(7c), (8a)–(8d), (9a), (9b), and (10) are discretized on a staggered grid with the velocities at the faces and all the scalars in the center of the computational cell; the resulting system of equations is solved by a fractional-step method. The discretization of both viscous and advective terms is performed by second-order-accurate central finite-difference approximations. The elliptic equation necessary to enforce incompressibility is solved directly using trigonometric expansions in the azimuthal direction and the tensor-product method⁴⁶ for the other two directions. Temporal evolution is via a third-order Runge–Kutta scheme, which calculates the nonlinear terms explicitly and the viscous terms implicitly. The stability limit due to the explicit treatment of the convective terms is $\text{CFL} < \sqrt{3}$, where CFL is the Courant, Friedrichs, and Lewy number. A useful feature of this scheme is the possibility to advance in time by a variable time step without reducing the accuracy or introducing interpolations. The three-dimensional simulations listed in Table I were conducted using uniform grids in the azimuthal direction and nonuniform grids in both the radial and axial directions with clustering at the walls. We have varied δt in all the simulations in this paper such that the local $\text{CFL} \leq 1.5$, where $\text{CFL} = (|u_r|/\delta r + |u_\theta|/(r\delta\theta) + |u_z|/\delta z)\delta t$, with the velocity components averaged at the center of each computational cell. Then, the smallest such determined local δt is used for time advancement. At least ten grid points were placed inside the bottom Ekman and sidewall boundary layers, respectively. For most of the runs, $n_\theta \times n_r \times n_z = 96 \times 351 \times 151$ grid points were used in the azimuthal radial and axial directions, respectively. One run was made with $n_\theta \times n_r \times n_z = 192 \times 601 \times 251$ in order to verify the grid-independence results during the axisymmetric stage of the flow and to test the adequacy of the coarser grid in resolving all the relevant flow scales.

Various test problems were also used for the verification of the numerical code. In particular, for the spin-up in a cylindrical container of height $2H$, Fig. 2 shows a comparison between experimental results for the azimuthal velocity $\hat{u}_\theta$ at mid-depth H and the corresponding numerical results of our code. The numerical results were in excellent agreement with the experimental results of Ref. 21 for salinity stratification and Ref. 31 for thermal stratification. We have also performed similar checks as in Ref. 45 who studied a different but related problem of baroclinic instabilities in the presence of rotation and stratification.

III. RESULTS

A. Axisymmetric spin-up

In this section, we consider three different sets of parameters where the flow remained axisymmetric throughout the spin-up process. We consider highly stratified flow large-Burger number (LBU), strong rotational effects (small Rossby number) with thermal stratification SRO, and homogeneous flow HOM with parametric values listed in Table I. In all cases, the sidewall and top/bottom walls of the cylinder are thermally insulated.

FIG. 2. Comparison of stratified spin-up experiments with numerical simulations for the azimuthal velocity $\hat{u}_\theta$. (a) Salinity stratification: the fluid depth is $2H=20$ cm, tank radius is $R=46$ cm, and the buoyancy frequency is $N=1.35$ s⁻¹. The spin-up sequence is $\Omega=0.2 \rightarrow 0.4$ s⁻¹ ($\epsilon=0.5$) at $\hat{t}=270$ s⁻¹. The vertical position is $\hat{z}=H$. The symbols (+) are the results of laboratory measurements in Ref. 21 and the dashed lines (—) are our numerical simulations. (b) Thermal stratification—parameters: $2H=6$ cm, $R=9.5$ cm, $N=0.97$ s⁻¹, $\Omega=0.384$ s⁻¹, $St^{-1}(=NH/2\Omega_i R)=0.49$, $E(=\nu/2\Omega_i H^2)=7.24 \times 10^{-4}$, $\epsilon(=\Delta\Omega/\Omega_i)=0.222$, and $\Omega_i=0.314$ s⁻¹. The vertical and radial locations are at the mid-depth $\hat{z}=H$ and $\hat{r}/R=0.64$. The circles (O) correspond to the laser-Doppler measurements in Ref. 31 and the solid lines (—) to our numerical simulations.

Figure 3 depicts the evolution of the temperature field for LBU and SRO (see movies 3 available online). The formation of the corner regions at early states of flow development is shown at $\tau \approx 1$. After the corner regions form, the fluid flushes back toward the center of the tank and the stratification near the axis $r=0$ decreases. For LBU, the state at $\tau=85$ shows a three-layer structure, i.e., a stratified layer bounded above and below by mixed fluid. The final state of solid-body rotation is delayed in SRO when compared to LBU.

Figure 4 shows the evolution of vertical vorticity for LBU at three levels $z=0.01, 0.25$, and 1.0 . The aspect ratio of the cylinder is almost three times smaller in this case ($\Gamma=2.33$) compared to SRO and HOM ($\Gamma=6.67$). The vorticity distribution at early times ($\tau < 10$) consists of a system of annular bands of vorticity which changes sign from one location to another. The system never breaks the axial symmetry, slowly advancing toward the state of a new solid-body rotation. The vorticity pattern at $\tau=85$ for LBU is not shown because of the miniscule values of ω_z at that time.

The evolution of axial vorticity ω_z for SRO shown in Fig. 5 and movies 5 is similar to the LBU case but with one important distinction: at around $\tau=20$, the flow tends to break the symmetry at the lower z -levels ($z < 1$). A wavy

FIG. 3. Isotherms on the planes $\theta=0-\pi$. At $\tau=0$, there are 32 linearly spaced contour levels in the range $T_H \in [0, 1]$. (a) LBU: the region shown is $r \in [0, 2.33]$ and $z \in [0, 2]$ (not to scale). (b) SRO: the region shown is $r \in [0, 6.67]$ and $z \in [0, 2]$ (not to scale). Movies 3, available in the online version, show the spatiotemporal characteristics for the isotherms on the planes $\theta=0-\pi$ over several rotation periods at a rate of 10 frames/s, with each frame being one rotation period apart (enhanced online).

[URL: <http://dx.doi.org/10.1063/1.3505025.1>]
 [URL: <http://dx.doi.org/10.1063/1.3505025.2>]

FIG. 4. LBU: axial vorticity ω_z ; there are seven positive (solid/black) and seven negative (white/dashed) linearly spaced contour levels in the range $\omega_z \in [-1, 1]$. The region shown is $r \in [0, 2.33]$ and $z \in [0, 2]$.

FIG. 5. SRO: axial vorticity ω_z . There are seven positive (solid/black) and seven negative (white/dashed) linearly spaced contour levels in the range $\omega_z \in [-1, 1]$. The region shown is $r \in [0, 6.67]$ and $z \in [0, 2]$. Movies 5, available in the online version, show the spatiotemporal characteristics for the vertical vorticity (enhanced online).
 [URL: <http://dx.doi.org/10.1063/1.3505025.3>]
 [URL: <http://dx.doi.org/10.1063/1.3505025.4>]
 [URL: <http://dx.doi.org/10.1063/1.3505025.5>]

perturbation with multiple inflection points propagates along the temperature front at some distance from the outer side-wall (see movies 5 available online). Its amplitude grows in time, but there is no cascade toward large-scale vortices that would permanently break the symmetry of the flow. After $\tau \sim 40-45$, the spin-up flow “relaminarizes,” and the vorticity isolines acquire a circular shape (Fig. 5, $\tau=60$).

The spin-up simulations for a homogeneous fluid (HOM) for the set of parameters listed in Table I did not develop small-scale instabilities near the outer wall nor large-scale eddy formation at late times on any horizontal plane despite that a large amount of horizontal shear $\Delta\Omega$ was initially transferred to the fluid. Clearly, for other values of the parameter set, we could expect that eddies or turbulence could occur for a homogeneous fluid.

B. Nonaxisymmetric spin-up with insulating boundary conditions: Case NEU

The evolution of the temperature field on the planes $\theta=0-\pi$ over several rotation periods is shown in Fig. 6 for $(\Gamma, \epsilon, \text{Bu}) = (6.67, 0.727, 0.379)$ with the top and bottom walls insulated. During spin-up, the isotherms around the corner regions deform due to the Ekman transport in the bottom boundary layer toward the outer sidewall. A similar transport in the upper part of the fluid domain is absent because of the imposed free-surface (stress-free) boundary condition. Pockets of cold and warm fluid form isotherms with rounded configurations inside the fully developed corner regions depicted in Fig. 6 at $\tau=7$. The downwelling motion in the central part of the cylinder coupled with the insulating boundary conditions for temperature forces the isotherms around $r=0$ to be displaced vertically ($\tau \approx 7$). The maximum height and radius of the corner regions are approximately $h_{\text{NEU}} \approx 1.68$ and $L_{\text{NEU}} \approx 4$, and these are consistent with the theoretical estimates $h=2/3\text{Bu}(=1.74)$ and $L=2\Gamma/3(=4.44)$ in Ref. 20; the time to formation is $\tau \approx 7.7$ rotation periods. For homogeneous fluids, the spin-up time is τ_{su}

FIG. 6. NEU: isotherms on the planes $\theta=0-\pi$; at $\tau=0$, there are 15 linearly spaced contour levels in the range $T_H \in [0, 1]$ showing the formation of the corner regions and the motion after the vortex core is formed. The region shown is $r \in [0, 6.67]$ and $z \in [0, 2]$. Movie 6, available in the online version, shows the spatiotemporal characteristics for the isotherms on the planes $\theta=0, \pi$ over 145 rotation periods at a rate of 10 frames/s, with each frame being one rotation period apart (enhanced online).
[URL: <http://dx.doi.org/10.1063/1.3505025.6>]

$= (2E)^{-1/2} / \pi \approx 6$, whereas Ref. 20 found that the corner regions mature until a time $\tau_{su} = 1.3 / (\Gamma \sqrt{2EN}) \approx 9.8$ rotation periods.

The collapse of the corner regions is accompanied by the generation of internal waves revealed in the undulation of the isotherms ($\tau=34$). At late times, the temperature gradient has decreased, and the state at $\tau=85$ shows the development of a stratified layer bounded above and below by relatively well-mixed fluid (see movie 6 in the online version of the paper).

The formation and development of the Ekman bottom boundary layer flow at $(r, \theta) = (5.5, 0)$ are depicted in Fig. 7(a) where the horizontal solid line marks the thickness of the Ekman layer δ . The vertical profiles of the radial velocity u_r show an increased fluid transport in the meridional plane at early spin-up times ($\tau \leq 1$) followed by a gradual decrease in both the Ekman transport and amplitude of u_r . The value of u_r decreases near $z=0$ at about 18–20 rotation periods as the entrainment of the interior fluid into the bottom Ekman layer shuts down. Figures 7(b)–7(d) illustrate the vertical profiles of temperature $T_H(z)$ on the plane $\theta=0$ for $r=0.1, 4.5$, and 6.5 . The dashed lines in the figures depict the background temperature profile at time $\tau=0$. Near the axis of rotation ($r=0.1$), the temperature profiles for $\tau>0$ lie beneath the initial temperature contour, implying that during spin-up the inner part of the cylinder experiences a strong downwelling event. The largest deviation from the initial linear profile occurs after about 18–20 rotation periods. The temperature continuously varies along the top and bottom boundaries of the cylinder due to the thermally insulated boundaries. The Ekman boundary layer homogenizes in the corner regions as well as near the center. At very late times ($\tau \sim 120$), the shape of the isotherms deviates significantly from its initial linear profile, indicating that significant mixing has occurred in the system, i.e., the vertical profile for T_H plotted at different radial locations are very similar.

Several snapshots of the adjustment of the thermally stratified flow from one state of rotation to another showing

FIG. 7. NEU: vertical profiles of the radial velocity u_r and temperature T_H at different radial locations and $\theta=0$. (a) Radial velocity u_r at times $\tau=1$ (O), 3 (□), 7 (▽), and 20 (△). Note that the solid line indicates the depth of the Ekman layer. [(b)–(d)] Temperature profiles T_H at $\tau=0$ (—), 3 (O), 20 (△), 45 (◇), and 120 (×).

the formation of three-dimensional vortex structures are presented in Fig. 8, but the spatiotemporal nature of this evolution is better appreciated from movie 8 (available in the online version). The temperature profile on the left is shown on the $\theta=0-\pi$ -plane. The vortex structure colored by temperature and identified by the Q -criterion in Ref. 47 is shown in the middle of the figure, with a plan view of the same vortex on the right. The $Q>0$ identifies vortices as flow regions where the second invariant of $\nabla \mathbf{u}$ is positive. In an incompressible flow, Q is a local measure of the excess of the rotation rate relative to the strain rate, i.e., if $\mathbf{S}$ and $\mathbf{\Omega}$ are the symmetric and antisymmetric components of $\nabla \mathbf{u}$, then the second invariant can be written as $Q = \frac{1}{2} (\|\mathbf{\Omega}\|^2 - \|\mathbf{S}\|^2)$.⁴⁸ The growth of the isotherms around the corner halts at $\tau \approx 7$ and the front becomes stationary. During this growth, the vortex core remains axisymmetric, but small disturbances appear to grow on and under the surface as shown in Fig. 8(b). These small-scale structures within the corner regions correspond to the pockets of relatively cold and warm fluid depicted in Fig. 6(b).

The vortex structure correlates well with the vertical vorticity ω_z shown at different horizontal planes in Fig. 9. At $\tau=0$, the vertical vorticity is equal to -2 in the entire flow domain as it is in solid-body rotation. When $\tau<5$, the vorticity distribution is axisymmetric and forms a system of smooth concentric rings. The vorticity is positive near the sidewall and retains negative values near the center. The distribution of vorticity in the radial direction is not monotonic after $\tau \sim 1$. As time progresses ($\tau=2-5$), the vorticity rings of opposite signs alternate with each other at some distance from the sidewall (see also Fig. 8 and movies 9). Around τ

FIG. 8. (Color) NEU: temperature and vortex structure identified by the isosurfaces of $Q=0$ colored by temperature. The figures on the left correspond to $T_H \in [0,1]$ and on the middle/right to the vortex structure. The region shown is $r \in [0,6.67]$ and $z \in [0,2]$. Movie 8, available in the online version, shows the spatiotemporal characteristics for the isotherms and vortex structure. The animations run over 100 rotation periods at a rate of 4 frames/s, with each frame being one rotation period apart (enhanced online). [URL: <http://dx.doi.org/10.1063/1.3505025.7>]

$=6-7$, the rings start disintegrating into small-scale patches of positive and negative vorticities shown in Fig. 9(a) at $\tau=7$. This process is more vigorous at the lower vertical levels than at the top. The flow retains a solid-body rotation (concentric vorticity rings) near the axis of rotation until $\tau \sim 10$. The vorticity distribution acquires a shape of spiral bands around $\tau=15$ followed by the development of large-scale eddies at about $\tau=20$. The vorticity distribution is very coherent at various vertical levels (see Fig. 9, $\tau=34$, and movies 9) including the lowest level $z=0.01$, which is well inside the bottom Ekman layer ($\delta=0.054$). After $\tau \sim 20$, the highly distorted (asymmetric) flow near the sidewall forms three distinctive cyclonic eddies (at $\tau=22$), which undergo further merging forming two large cyclones by $\tau=25$. At later times, the vertical coherence of the flow pattern is lost, and the columnar eddies disintegrate into a system of smaller vorticities.

A comparison of the rate of growth of the first four instability modes ($n=1, \dots, 4$) to the rate of decay of the mean current ($n=0$) is shown in Fig. 10. The energy curve of the mean current decreases monotonically in time until it is overcome by the fastest growing modes $n=1, 2$ at about $\tau \approx 40$. All other modes stay below the mode $n=0$ at all times, although experiencing oscillations at late times.

FIG. 9. NEU: axial vorticity ω_z ; there are seven positive (solid/black) and seven negative (white/dashed) linearly spaced contour levels in the range $\omega_z \in [-1,1]$. Movies 9, available in the online version, show the spatiotemporal characteristics for the axial vorticity at different z -levels (enhanced online). [URL: <http://dx.doi.org/10.1063/1.3505025.8>] [URL: <http://dx.doi.org/10.1063/1.3505025.9>] [URL: <http://dx.doi.org/10.1063/1.3505025.10>]

The time evolution of each term of the perturbation energy Eq. (13) allows us to determine the contribution to the instability of the barotropic (H_1), baroclinic (H_2), centrifugal (H_3), and viscous dissipation (H_4) terms. The flow was not initially perturbed, and therefore the small noise associated

FIG. 10. NEU: time evolution of different energy modes E_n : $n=0$ (—); $n=1$ (○); $n=2$ (---); $n=3$ (-.-); and $n=4$ (◇).

FIG. 11. NEU: (a) time evolution of (a) H_i -terms in the rate of change of kinetic energy of azimuthal perturbations, barotropic term H_1 (—), baroclinic term H_2 (---), centrifugal term H_3 (- - -), and viscous dissipation term H_4 (····); (b) the rate of change of kinetic energy $de/d\tau$; and (c) the kinetic energy $e(\tau)$.

with the round-off errors of the numerical discretization triggered the three-dimensional instabilities. Figures 11(a) and 11(b) illustrate the time history of each individual term H_i of the rate of change of kinetic energy and $de/d\tau$, respectively. The kinetic energy term $e(\tau)$ in Fig. 11(c) demonstrates a small peak at time $\tau \approx 7$ reaching a maximum at $\tau = 30$. The centrifugal term (H_3) remains negligible over the entire simulation time interval and does not influence the flow dynamics, which is governed by the interplay among the barotropic (H_1), baroclinic (H_2), and viscous dissipation (H_4) terms. At early spin-up times, the barotropic term, which is positive, serves as a source of azimuthal instabilities through the mean current shear. However, at later times, it acquires both positive and negative values and even experiences undulations about the zero level. During some time intervals, the perturbations are amplified by absorbing the energy from the mean current. The baroclinic (gravity) term reaches peak values at $\tau \approx 20$. The release of the potential energy is accompanied by the oscillations of both barotropic and baroclinic terms (the local maxima of the baroclinic oscillations are correlated with the local minima of barotropic oscillations). According to Fig. 11(c), the energetic stage of the flow evolution ends by $\tau = 135$, and the final state consists of irregular eddies shown in Fig. 8(d) and movie 8.

FIG. 12. DIR: isotherms on the planes $\theta = 0 - \pi$; at $\tau = 0$, there are 32 linearly spaced contour levels in the range $T_H \in [0, 1]$, showing the formation of the corner regions and the motion after the vortex core is formed. The region shown is $r \in [0, 6.67]$ and $z \in [0, 2]$. Movie 12, available in the online version, shows the spatiotemporal characteristics for the isotherms on the planes $\theta = 0, \pi$ over 145 rotation periods at a rate of 10 frames/s, with each frame being one rotation period apart (enhanced online).

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C. Nonaxisymmetric spin-up with specified temperature boundary conditions: Case DIR

The evolution of the temperature field on the planes $\theta = 0 - \pi$ over several rotation periods is shown in Fig. 12 and movie 12 for $(\Gamma, \epsilon, Bu) = (6.67, 0.727, 0.379)$. These are the same parameters as in the NEU but with specified constant temperatures $T_H = 1, 0$ at the top and bottom walls, respectively. Initially, there are no radial temperature gradients and the isotherms are horizontal. At the outset of spin-up, the isotherms around the corner regions deform due to the Ekman transport in the bottom boundary layer toward the outer sidewall. Since the bottom boundary acts as a source of dense fluid by the constant temperature boundary condition, a significant contribution to the temperature anomalies is due to the vertical advection of temperature in the corner regions, which drastically enhances the horizontal temperature gradient.

The downwelling motion in the central part of the cylinder coupled with the specified constant temperatures boundary conditions force the isotherms around $r = 0$ to be severely compressed in the vertical direction ($\tau \approx 7$). The stratification near the bottom at $r = 0$ increases from its initial state due to downwelling. As the fluid from the corner regions flushes back toward $r = 0$, the highly stratified core acts as a restituting force that delays the development of instabilities. The generation of internal waves is only evident at $\tau = 85$, and only at late times the temperature gradient recovers its initial linear profile (see movie 12 available online).

The formation and development of the Ekman bottom boundary layer flow at $(r, \theta) = (5.5, 0)$ are depicted in Fig. 13(a). The horizontal solid line marks the thickness of the Ekman layer δ . The vertical profiles of the radial velocity u_r show an increased fluid transport in the meridional plane at early spin-up times ($\tau \leq 1$) followed by a gradual decrease in both the Ekman transport and amplitude of u_r . The value of u_r becomes negligible within the Ekman layer at about 18–20 rotation periods. Figures 13(b)–13(d) show the verti-

FIG. 13. DIR: vertical profiles of the radial velocity u_r and temperature T_H at different radial locations and $\theta=0$. (a) Radial velocity u_r at times $\tau=1$ ($\circ$), 3 ($\square$), 7 (∇), and 20 ($\triangle$). Note that the solid line indicates the depth of the Ekman layer. [(b)–(d)] Temperature profiles T_H at $\tau=0$ (—), 3 ($\circ$), 20 ($\triangle$), 45 ($\diamond$), and 120 ($\times$).

cal profiles of temperature $T_H(z)$ on the plane $\theta=0$ for $r=0.1, 4.5$, and 6.5 . The dashed lines in the figures depict the background temperature profile at time $\tau=0$. Near the axis of rotation ($r=0.1$), the temperature profiles for $\tau>0$ lie beneath the initial temperature contour due to downwelling. The largest deviation from the initial linear profile occurs after about 18–20 rotation periods. At very late times ($\tau>85$), the initial temperature gradient is almost recovered.

Snapshots of the formation of three-dimensional vortex structures are presented in Fig. 14. Movie 14 (available in the online version) shows the evolution of the spatiotemporal nature of the spin-up and corresponds to the same figure. The temperature profile on the left is shown on the $\theta=0-\pi$ -plane. The vortex structure is colored by temperature and identified by the isosurfaces of $Q=0$ and shown in the middle of the figure; a plan view of the same vortex structure is depicted on the right. During the growth of the corner regions ($\tau<7$), the vortex core remains axisymmetric with small disturbances under the surface. Clearly, the amount of small-scale structures within the corner regions is less than in NEU as can be seen by comparing Figs. 14(b) and 8(b), showing the perturbations on the boundary between the vortex core and the bottom wall (top view). The contour levels of axial vorticity ω_z in Fig. 15 ($\tau=7$) also illustrate the Ekman layer instabilities generated on this boundary. The wobbling of the inner core, associated with the formation of spiral bands at $\tau=15$, amplifies in time until a dumb-bell shape structure composed of two anticyclones manifests itself around $\tau=30$ (see Figs. 14 and 15, $\tau=34$, and movies 14 and 15). The vorticity patterns at different z -levels show two cyclonic eddies that accompany anticyclones at $z=0.01$, but

FIG. 14. (Color) DIR: temperature and vortex structure identified by the isosurfaces of $Q=0$ colored by temperature. The figures on the left correspond to $T_H \in [0, 1]$ and on the middle/right to the vortex structure. The region shown is $r \in [0, 6.67]$ and $z \in [0, 2]$. Movie 14, available in the online version, shows the spatiotemporal characteristics for the isotherms and vortex structure. The animations run over 100 rotation periods at a rate of 4 frames/s, with each frame being one rotation period apart (enhanced online). [URL: <http://dx.doi.org/10.1063/1.3505025.12>]

on the upper levels $z=1, 1.9$, these eddies are only seen at late times. The instability mechanism seems to be effective with the appearance of the elliptic vortex structure combined with the baroclinic waves, enhancing the wobbling of the flow.

The rate of growth of the first four instability modes ($n=1, \dots, 4$) in Fig. 16 is consistent with the evolution of the vortex structure and ω_z shown in Figs. 14 and 15 and movies 14 and 15. The monotonic rate of decay of the mean current ($n=0$) is also shown in the figure. The initial growth of the modes is suppressed at early times and achieves a local minimum at around $\tau \sim 30$, particularly $n=4$. At about $\tau \approx 100$, the energy of the mean current is overcome by the fastest growing mode $n=1$. Modes $n=2, 3$, and 4 remain below the mode $n=0$ at all times.

The rate of change of kinetic energy and the kinetic energy of azimuthal perturbations are depicted in Fig. 17. Figures 17(a) and 17(b) illustrate the time history of each individual term H_i of the rate of change of kinetic energy and de/dr , respectively. The kinetic energy term shown in Fig. 17(c) demonstrates a small peak at time $\tau \approx 7$ becomes negligible at around $\tau=30$ (the flow becomes almost axisymmetric), followed by a continuous growth until it reaches a global maximum around $\tau \approx 90-100$ continued by a monotonic

FIG. 15. DIR: axial vorticity ω_z ; there are seven positive (solid/black) and seven negative (white/dashed) linearly spaced contour levels in the range $\omega_z \in [-1, 1]$. The region shown is $r \in [0, 6.67]$ and $z \in [0, 2]$. Movies 15, available in the online version, show the spatiotemporal characteristics for the axial vorticity at different z -levels (enhanced online).

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 [URL: <http://dx.doi.org/10.1063/1.3505025.14>]
 [URL: <http://dx.doi.org/10.1063/1.3505025.15>]

decay until it becomes negligible at late times $\tau > 180$. It is evident from Fig. 17 that the centrifugal term (H_2) remains negligible over the entire simulation.

At early spin-up times, both the barotropic (H_1) and baroclinic (H_2) terms, which are positive, are a source of azimuthal instabilities through the mean current shear. At late times, the barotropic term acquires both positive and negative values and experiences undulations about the zero level. The baroclinic (gravity) term reaches a local maximum at $\tau = 7$ and a global maximum at $\tau \approx 85$. At around $\tau = 7$, the flow is forced to an axisymmetric state due to the stabilizing role of the stratified core, but notice that the vortex structure has already lost its axisymmetry near the bottom wall, and it only

FIG. 16. DIR: time evolution of different energy modes E_n : $n=0$ (—); $n=1$ (○); $n=2$ (---); $n=3$ (-.-); and $n=4$ (◇).

shows an axisymmetric character well above the bottom boundary. The vortex core seems to lose its stability through a resonant mechanism generated at the interface of its surface and the bottom wall. Starting at around $\tau \approx 85$, the viscous dissipation is no longer able to counteract the amount of potential energy being released. The flow of potential energy is accompanied by the oscillations in both barotropic and baroclinic terms (the maxima of the baroclinic oscillations are also correlated with the minima of barotropic oscillations). The energetic stage of the flow evolution spans for a long time in DIR, and only by the time $\tau \approx 180$, all energy terms essentially reduce to zero. At this time, the flow shows an irregular system of weak eddies (see movie 12).

FIG. 17. DIR: time evolution of (a) H_i -terms in the rate of change of kinetic energy of azimuthal perturbations, barotropic term H_1 (—), baroclinic term H_2 (---), centrifugal term H_3 (—), and viscous dissipation term H_4 (-.-); (b) the rate of change of kinetic energy $de/d\tau$; and (c) the kinetic energy $e(\tau)$.

FIG. 18. Isotherms (top) and isobars (bottom) on the planes $\theta=0-\pi$ at $\tau \approx 7$. Initially ($\tau=0$), there are 15 linearly spaced contour levels in the range $T_H \in [0, 1]$. The region shown is $r \in [0, 6.67]$ and $z \in [0, 2]$. The symbol $\bullet$ indicates a reference point for comparing the angle formed by the isotherms and isobars at that particular location.

D. Comparison between NEU and DIR cases

In order to explain why NEU develops eddies earlier than DIR, let us examine the baroclinic contribution to vorticity, which is proportional to $\nabla T_H \times \nabla p$. This vector is also proportional to the angle between surfaces of constant temperature and surfaces of constant pressure. Figure 18 depicts the isotherms and isobars for NEU and DIR at $\tau=7$ when the corner regions have matured. The dots shown in Fig. 18 are located around the boundary between the corner regions and vortex core. Clearly, the isotherms and the isobars around the vortex core are nearly orthogonal in NEU, whereas in DIR, the isotherms and isobars are almost parallel. The contribution to the vorticity due to the vertical shears is more pronounced in NEU than in DIR, and the growth of eddies in DIR is delayed due to the relative strong stratification around $r=0$, which acts as a restoring mechanism. Previous studies have implicitly assumed that the destabilizing effect is mainly baroclinic (the shear is mainly vertical). However, as shown in Figs. 11 and 17, both vertical and horizontal shears contribute to the instability. The ratio of the barotropic to the baroclinic terms H_1/H_2 is proportional to $h/L = \Omega/N$ at the time when the corner regions mature.^{18,20} The numerical value of this ratio for NEU at around $\tau \approx 10$ is $H_1/H_2 = 0.177/0.542 \approx \tan(20^\circ)$, whereas $\Omega/N \approx \tan(21^\circ)$ (see Fig. 11). In DIR, the initial disturbance is caused by the combined effect of vertical and horizontal shears where $H_1/H_2 = 0.051/0.132 \approx \tan(22^\circ)$ with $\Omega/N \approx \tan(21^\circ)$ at $\tau \approx 10$, but the subsequent growth of instabilities (around $\tau \approx 40$) is purely baroclinic, as shown in Fig. 17. These results imply that the nonaxisymmetric instability of the spin-up flow in a circular cylinder is strongly influenced by the imposed temperature boundary conditions on top/bottom walls of the cylinder.

IV. DISCUSSION AND CONCLUSIONS

We have conducted three-dimensional time-dependent numerical simulations of nonlinear spin-up of a thermally stratified fluid in a large aspect ratio circular cylinder taking into account both fixed temperature and thermally insulated boundary conditions. The evolution of stratified spin-up has

been characterized by two distinct stages. In the first stage, we found the existence of the bottom Ekman layer, which pumps the stratified fluid from the interior into the corner regions that remained axisymmetric. The instantaneous vertical density profiles near the outer sidewall and the core of the cylinder demonstrate the regions of strong upwelling and downwelling respectively. Because of the vigorous stirring inside the corner regions, the fluid tends to homogenize there.

In our simulations, the formation time of the axisymmetric corner regions is $\tau_{su} \approx 7.7$ rotation periods, the homogeneous spin-up time is $\tau_{su} = (2E)^{-1/2} / \pi \approx 5.9$ and the estimate of Ref. 20 $\tau_{su} = 1.3 / (\Gamma \sqrt{2EN}) \approx 9.8$ rotation periods. The maximum height and radius of the corner regions are consistent with the theoretical estimates $h=2/3Bu$ and $L=2\Gamma/3$ of Ref. 20.

In the second stage, the flow becomes unstable depending on the relative values of Γ , ϵ and Bu , but the subsequent development of the instability strongly depends on the specified temperature boundary conditions on the top and bottom walls of the cylinder (the sidewall is always thermally insulated). When the end-walls are insulated (case NEU), the axisymmetry is lost via a baroclinic instability of the corner regions and subsequent propagation of instability toward the inner region of the cylinder. When the temperatures at the top and bottom walls are specified (case DIR), the spin-up flow loses its axisymmetry through the wobbling of the inner core in a manner that is reminiscent of an elliptical instability.

The numerical results for HOM suggest that in order to develop eddies at late spin-up times (considering that our simulations are in the transient Ekman boundary layer regime), a baroclinic instability of the corner region is necessary. The azimuthal instability in this regime seems to be sensitive to the specific values of the Burger and Rossby numbers as evidenced by the results from LBU and SRO. Notice that the value of the Rossby number in SRO (see Table I) is still big enough to be considered in the nonlinear regime of stratified spin-up. Nonaxisymmetric instabilities were observed in Ref. 22 for similar values of Burger and Rossby numbers ($Bu=0.38$ and $\epsilon=0.18$) but a diffusivity value (Schmidt number) two orders of magnitude larger than in our numerical simulations. The instabilities observed in the experiments are probably enhanced by diffusion effects since local baroclinic instabilities are dependent on the diffusion coefficient.⁴⁹ It would be ideal to consider flows with large σ (≈ 700 for salt in water) as in these laboratory experiments. Unfortunately, the typical value of E/σ is too small for numerical resolution of density diffusion with affordable computations even with modern supercomputers. Some researchers^{10,20,45} argue that a value of $\sigma \approx 7$ may be used to provide an artificial diffusion coefficient whose contribution is significant only in the numerical smoothing of the interface of the corner regions and the central core and has negligible influence in the bulk dynamics. The agreement between our numerical simulations ($\sigma=20$) with the experiments in Ref. 21 ($\sigma=700$ corresponding to salt in water) suggests that for the early stages of flow development (prior to the symmetry breakup), the flow dynam-

ics has a weak dependence on σ for the range of parametric values selected in this investigation.

Furthermore, the parametric study of Ref. 18 suggests that for short containers ($\Gamma > 1$), the flow instability is markedly different than that of tall containers ($\Gamma < 1$). The experimental results reported that the centerline of the central core was distorted into a helix, leading to the breakup into different lenses. We also notice that the stratification at the core in the early stages of spin-up becomes significantly smaller than the initial stratification (particularly for small containers $\Gamma \geq 1$). Therefore, we can expect the aspect ratio Γ , the Ekman number E , and Prandtl number Pr to play an important role in onset of instabilities and subsequent formation of large columnar eddies.

In conclusion, this numerical study has shown that thermally stratified spin-up flows may develop nonaxisymmetric instabilities that lead to the formation of large-scale columnar vortices in high Rossby number spin-up flow at late times. To the best of our knowledge, previous studies where the stratification was created by temperature have reported axisymmetric flow patterns only.^{25–27,29–31} Furthermore, we are not aware of any three-dimensional numerical simulation of stratified spin-up in which columnar eddies have been observed.

By characterizing the azimuthal instabilities without introducing any simplification, we were able to determine to what extent an insulating boundary condition changes the time-dependent emergence of the instability. The computation of the time evolution of each term of the perturbation energy equation allowed us to determine the contribution to the instability of the barotropic and baroclinic terms. Hence, we were able to identify the mechanisms responsible for the subsequent growth of eddies at the late stages of flow development when using Neumann or Dirichlet boundary conditions in a numerical setup. The results from our numerical simulations are in excellent agreement with previous laboratory experiments of stratified spin-up.

Finally, there are many aspects of nonlinear spin-up flow that require more exploration, and this study provides the framework for further investigations. One of them is the study in parameter space (Γ, Bu, Pr) for fixed ϵ of incremental spin-up in cylindrical and annular geometries with flat and sloping bottoms, which will be the subject of forthcoming papers.

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