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A note on scoring rules that respect majority in choice and elimination[☆]

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Abstract

In a recent paper [Mathematical Social Sciences 43 (2002) 151], M.R. Sanver investigates scoring rules for social choice problems with n voters and m alternatives. He proves that unless $n \in \{2, 3, 4, 6, 8\}$ a scoring rule cannot simultaneously respect majority in choice and majority in elimination. In this short technical note, we first point out a serious flaw in Sanver's proof. Then we provide a complete proof for a corrected version of Sanver's statement: Unless $n \in \{2, 3, 4, 5, 6, 8, 10, 12\}$ a scoring rule cannot simultaneously respect majority in choice and majority in elimination.

Keywords: Scoring rules; Respecting majority

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1. Introduction

We consider a set $N = \{1,2, \ldots ,n\}$ of $n \geq 2$ *voters* and a set A of $m \geq 3$ *alternatives*. A bijection p from A to $\{1, \ldots ,m\}$ yields a *preference* ordering of the m alternatives. Sloppily speaking, the alternative x with $p(x) = 1$ is the least desirable alternative, and the alternative y with $p(y) = m$ is the most desirable alternative according to preference p. Every voter $i \in N$ has a preference ordering p_i of A. These preferences are collected in a preference profile vector $p = (p_1, \ldots ,p_n)$. A *social choice problem* is an ordered

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triplet (N,A,p). A *social choice rule* F assigns to any social choice problem (N,A,p) a non-empty subset $F(N,A,p) \subseteq A$ of alternatives. In this paper, we will consider the following two natural properties of social choice rules:

- A social choice rule F *respects majority in choice* if and only if for any (N,A,p) and any $x \in A$ with $|\{i \in N\colon p_i(x) = m\}| > n/2$, we always have $x \in F(N,A,p)$. In other words, an alternative that is ranked best by the majority of voters should always be chosen.
- A social choice rule F *respects majority in elimination* if and only if for any (N,A,p) and any $x \in A$ with $|\{i \in N\colon p_i(x) = 1\}| > n/2$, we always have $x \not\in F(N,A,p)$. In other words, an alternative that is ranked worst by the majority of voters should never be chosen.

An interesting special case of social choice functions are the *score functions* that are based on a so-called *score vector* $s = (s_1, \ldots, s_m)$ of real numbers with $s_i \geq s_{i+1}$ for $1 \leq i \leq m-1$, and $s_1 > s_m$. Then the score assigned by voter i to alternative x equals $\sigma_i(x;p) = s_{m-p_i(x)+1}$. The overall score assigned to alternative x equals $\sigma(x;p) = \Sigma_{i \in N}\sigma_i(x;p)$. The score function F_s chooses the alternatives with maximum overall scores, that is

$$F_s(N,A,p) \; = \; \{x \in A\colon \sigma(x;p) \geq \sigma(y;p) \text{ for all } y \in A\}.$$

In a *standard* scoring rule, the score vector s may only depend on m. In a *generalized* scoring rule, the score vector s may depend on n and m. Typical examples of standard scoring rules are the *plurality rule* where $s = (1,0,0,0, \ldots ,0)$ and the *antiplurality rule* where $s = (1,1, \ldots ,1,0)$. Every score vector can be normalized by a linear transformation such that it is of the form $(1, \ldots ,0)$ with $s_1 = 1$, $s_m = 0$, and $0 \leq s_j \leq 1$ for $j = 2, \ldots ,m-1$. Throughout the paper, we will only consider such normalized score vectors. For more information on scoring rules and social choice, the reader is referred to Moulin (1983, 1988).

The relationships between standard scoring rules and majority conditions has been extensively treated in the literature. It is well known since Condorcet that scoring rules may leave out an alternative that gets a majority of votes against any opponent in the pairwise comparisons. Another ‘‘majority-like’’ condition states that a Condorcet loser (that is, an alternative beaten by any other alternative in pairwise comparisons) should not be elected. The Borda count is the only standard scoring rule that never selects a Condorcet loser. This statement was already known to Nanson (1882), and modern proofs appear in Fishburn and Gehrlein (1976) and Smith (1973).

The majority in choice condition has already been proposed by Smith (1973) and by Richelson (1978, 1980). Both authors remark that the plurality rule satisfies this condition, whereas the Borda count and the antiplurality rule do not satisfy this condition. In a paper in French, Lepelley (1992) proves that the plurality rule is the only (standard) scoring rule which satisfies majority in election. In another paper in French, Lepelley and Merlin (1998) introduced the concept of majority in elimination. They prove that a standard scoring rule satisfies majority in elimination if and only if

Table 1
Summary of our results. An entry ‘ + ’ means that for these values of n and m there exists a scoring rule that simultaneously respects majority in choice and elimination. An entry ‘ − ’ means that no such scoring rule exists

n	2	3	4	5	6	7	8	9	10	11	12	13	14	...
$m=3$	+	+	+	+	+	−	+	−	+	−	+	−	−	...
$m=4$	+	+	+	+	+	−	+	−	+	−	−	−	−	...
$m=5$	+	+	+	+	+	−	+	−	−	−	−	−	−	...
$m=6$	+	+	+	+	+	−	+	−	−	−	−	−	−	...
$m=7$	+	+	+	+	+	−	+	−	−	−	−	−	−	...
$m=7$	+	+	+	+	+	−	+	−	−	−	−	−	−	...
$m=8$	+	+	+	+	+	−	+	−	−	−	−	−	−	...
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...

$\Sigma_{i=1}^{m} s_i \geq m/2$. As an immediate consequence, no standard scoring rule can simultaneously respect majority in choice and elimination (as the plurality rule does not satisfy this condition). Lepelley and Vidu (2000) analyzed similar issues in the case where the preferences are single-peaked.

A recent paper by Sanver (2002) investigates in detail the question whether generalized scoring rules can simultaneously respect majority in choice and elimination. Sanver arrives at the following statement: ‘‘*A generalized scoring rule F_s cannot simultaneously respect majority in choice and elimination, except for $n \in \{2, 3, 4, 6, 8\}$*’’. In Section 2, we will point out a serious flaw in Sanvers’s argument that makes his proof invalid. In fact even Sanver’s statement is incorrect, since (as we will show in this note) there do exist scoring rules for $n \in \{5, 10, 12\}$ that simultaneously respect majority in choice and elimination. In Section 3, we will prove the following corrected version of Sanver’s statement; see Table 1 for an illustration.

Theorem 1.1. *There exists some generalized scoring rule F_s that simultaneously respects majority in choice and elimination, if and only if one of the following cases holds*:

(i) $n \in \{2, 3, 4, 5, 6, 8\}$ *and* $m \geq 3$
(ii) $n = 10$ *and* $m \in \{3,4\}$
(iii) $n = 12$ *and* $m = 3$

2. Discussion of the arguments of Sanver

We remind the reader that throughout this paper we deal with normalized score vectors that are of the form $(1, \ldots, 0)$ with $s_1 = 1$, $s_m = 0$, and $0 \leq s_j \leq 1$ for $j = 2, \ldots, m-1$. The following proposition is a slightly rewritten statement of Sanver (2002).

Proposition 2.1. ((Sanver, 2002)) *A normalized generalized scoring rule F_s respects majority in choice if and only if*

(i) $s_2 \leq 4/(n+2)$ *when n is even*
(ii) $s_2 \leq 2/(n+1)$ *when n is odd.*

Indeed, an alternative x that is ranked best by a majority of the voters has overall score at least $(n+2)s_1/2$ if n is even, and overall score at least $(n+1)s_1/2$ if n is odd. Any other alternative y has overall score at most $(n-2)s_1/2 + s_2(n+2)/2$ if n is even, and at most $(n-1)s_1/2 + s_2(n+1)/2$ if n is odd. The stated inequalities (i) and (ii) are necessary and sufficient to have the overall score of x greater or equal to the overall score of y.

Sanver's proof is centered around Proposition 2.2 stated below. Note that the statement of Proposition 2.2 is absolutely true, but also absolutely vain: For $i = m$, the left-hand side in the stated inequalities equals $s_1/(1-s_m) = 1$, whereas their right-hand side is strictly less than 1. Hence, the required conditions are fulfilled a priori by the normalization of the score vector.

Proposition 2.2. ((Sanver, 2002)) *A normalized generalized scoring rule F_s respects majority in elimination only if*

(i) $s_{m-i+1}/(1-s_i) > (n-2)/(n+2)$ *for some* $2 \leq i \leq m$ *when* $n > 2$ *is even*
(ii) $s_{m-i+1}/(1-s_i) > (n-1)/(n+1)$ *for some* $2 \leq i \leq m$ *when n is odd.*

Sanver claims that the necessary conditions in these two propositions are generally inconsistent. This clearly cannot be true, since the necessary conditions (i) and (ii) in Proposition 2.2 are satisfied by *any* normalized score vector. The main flaw in Sanver's argumention, however, is that he ignores the case $i = m$ in these necessary conditions (i) and (ii). As a consequence, Sanver comes to the (wrong) conclusion that a generalized scoring rule cannot simultaneously respect majority in choice and elimination, unless $n \in \{2, 3, 4, 6, 8\}$. According to Theorem 1.1, there are counter examples to this statement for $n = 5$, $n = 10$, and $n = 12$.

In the last section of his paper, Sanver observes that no standard scoring rule can simultaneously respect majority in choice and elimination. This result remains correct, since it is an (almost) immediate consequence of Proposition 2.1 stated above. Moreover, this result is also implicit in the work of Lepelley and Merlin (1998).

3. The proof of the main result

This section is devoted to a proof of Theorem 1.1. Exactly as in Section 2, we will restrict our attention to normalized score vectors with $s_1 = 1$ and $s_m = 0$. First, let us settle the trivial case with $n = 2$ voters: In this case, any alternative that is ranked best by the majority of voters has score 2, and any alternative that is ranked worst by the majority of voters has score 0. Clearly, majority in choice and elimination are respected.

Section 3.1 deals with the cases of odd $n \geq 3$, and Section 3.2 deals with the cases of even $n \geq 4$.

3.1. The cases with an odd number of voters

This subsection deals with the case of an odd number n of voters. We first derive the positive results for $n = 3$ and $n = 5$ in Lemma 3.1. Then Lemma 3.2 gives the negative result for odd $n \geq 7$.

Lemma 3.1. *For $n \in \{3,5\}$ and for any $m \geq 3$, there exists a generalized scoring rule F_s that simultaneously respects majority in choice and elimination.*

Proof. We use the score vector $s = (1,2/(n+1),\ldots,2/(n+1),0)$. By Proposition 2.1, the resulting scoring rule respects majority in choice.

Let us prove that this scoring rule also respects majority in elimination. We start with the case $n = 3$. If a majority of at least two voters ranks alternative $x \in A$ worst, then the overall score $\sigma(x;p)$ is at most $s_1 + 2s_m = 1$. One of the other $m - 1$ alternatives is *never* ranked worst. The overall score of this alternative is at least $3s_{m-1} = 3/2 > \sigma(x;p)$. Hence, F_s indeed eliminates x.

We turn to the case $n = 5$. If a majority of at least three voters ranks alternative $x \in A$ worst, then the overall score $\sigma(x;p)$ is at most $2s_1 + 3s_m = 2$. We distinguish two subcases. First, assume that some other alternative y is ranked best by at least two voters. Then $\sigma(y;p) \geq 2s_1 + 2s_m + s_{m-1} = 7/3 > \sigma(x;p)$. Secondly, assume that all other alternatives are ranked best by at most one voter. Then there exists an alternative y that is once ranked best and never ranked worst. This yields $\sigma(y;p) \geq s_1 + 4s_{m-1} = 7/3 > \sigma(x;p)$. In both subcases, the rule F_s correctly eliminates x. $\square$

Lemma 3.2. *For any odd $n \geq 7$ and for any $m \geq 3$, there does not exist a generalized scoring rule F_s that simultaneously respects majority in choice and elimination.*

Proof. The proof is done by contradiction. Suppose that there exists a normalized score vector s with $s_1 = 1$ and $s_m = 0$, such that the corresponding scoring rule F_s respects majority in choice and elimination. Let $n = 2k - 1$ with $k \geq 4$. Proposition 2.1 yields that $s_2 \leq 2/(n+1) = 1/k$.

Consider a preference profile p with the following properties. There is one alternative $x \in A$ that is ranked worst by k voters and ranked best by the other $k - 1$ voters. Hence, $\sigma(x;p) = k - 1$. The remaining $m - 1$ alternatives are ranked as follows.

(a) If $k \leq m - 1$ holds, then each of the remaining $m - 1$ alternatives is ranked best by at most one voter, and it is ranked second or worse by all the other voters.
(b) If $k \geq m$ holds, then each of the remaining $m - 1$ alternatives is ranked best by $\lceil k/(m-1) \rceil$ or $\lfloor k/(m-1) \rfloor$ voters, it is ranked worst by at least one voter, and it is ranked second or worse by the other voters.

Case (a) is easily completed: For any alternative $y \neq x$, its overall score $\sigma(y;p)$ is at most $s_1 + (n-1)s_2 \leq 1 + (2k-2)/k < k-1$. Hence, $\sigma(x;p) > \sigma(y;p)$ holds for all such y, and the scoring rule F_s does not respect majority in elimination. In Case (b), the overall score of any alternative $y \neq x$ satisfies

$$\begin{aligned}\sigma(y;p) &\leq \left\lceil \frac{k}{m-1} \right\rceil s_1 + s_m + \left(n-1-\left\lceil \frac{k}{m-1} \right\rceil\right) s_2 \\ &\leq \left\lceil \frac{k}{m-1} \right\rceil \left(1-\frac{1}{k}\right) + (2k-2)\frac{1}{k} \quad \leq \quad \left\lceil \frac{k}{2} \right\rceil (k-1)\frac{1}{k} + (k-1)\frac{2}{k} \\ &= \frac{k-1}{k}\left(\left\lceil \frac{k}{2} \right\rceil + 2\right) \leq k-1.\end{aligned}$$

In this chain of inequalities, we have used that $s_2 \leq 1/k$, that $m \geq 3$, and that $k \geq 4$. Since $\sigma(y;p) \leq k-1 = \sigma(x;p)$ holds, also in Case (b) the scoring rule F_s does not respect majority in elimination. □

3.2. The cases with an even number of voters

This subsection deals with the case of an even number n of voters. We first derive the positive results for $n \in \{4, 6, 8, 10, 12\}$ in Lemma 3.3. Then Lemma 3.4 gives the negative results for $n = 10$ and $n = 12$, and Lemma 3.5 gives the negative results for even $n \geq 14$.

Lemma 3.3. *There exists a generalized scoring rule F_s that simultaneously respects majority in choice and elimination, if one of the following cases holds*:

(i) $n \in \{4, 6, 8\}$ *and* $m \geq 3$
(ii) $n = 10$ *and* $m \in \{3, 4\}$
(iii) $n = 12$ *and* $m = 3$

Proof. Let $n = 2k-2$ for some integer k. We use the score vector $s = (1, 2/k, \ldots, 2/k, 0)$. By Proposition 2.1 the resulting scoring rule respects majority in choice, as $s_2 = 4/(n+2)$.

We will show that this scoring rule F_s also respects majority in elimination for the ranges of m stated in (i)–(iii). If a majority of voters ranks alternative $x \in A$ worst, then the overall score $\sigma(x;p)$ of this alternative is at most $(k-2)s_1 + k\,s_m = k-2$.

In case (i) we have $k \in \{3,4,5\}$. We distinguish two subcases. First, assume that some alternative $y \neq x$ is ranked best by at least two voters. Then $\sigma(y;p) \geq 2s_1 + (k-2)s_{m-1} + (k-2)s_m = 2 + 2(k-2)/k$, and this yields $\sigma(y;p) > \sigma(x;p)$. Secondly, assume that all alternatives $y \neq x$ are ranked best by at most one voter. Then some alternative y is once ranked best and never ranked worst. This yields $\sigma(y;p) \geq s_1 + (2k-3)s_{m-1} = 1 + 2(2k-3)/k > \sigma(x;p)$. In both subcases, the rule F_s correctly eliminates x.

In the cases (ii) and (iii) we have $k \in \{6, 7\}$. The sum of the scores of the $m-1$ alternatives $y \neq x$ is at least

$$k\,s_1+\sum_{i=2}^{m-1}(2k-2)\,s_i+(k-2)\,s_m \;=\; k+(m-2)(2k-2)\frac{2}{k}.$$

For $(k,m)\in\{(6,\ 3),(6,\ 4),(7,\ 3)\}$, this lower bound on the sum of $m-1$ scores is strictly greater than $(m-1)(k-2)$. One of these $m-1$ scores must be at least the average value, and hence strictly larger than $k-2$. Consequently, in all these cases F_s correctly eliminates the alternative x. □

Lemma 3.4. *There does not exist a generalized scoring rule F_s that simultaneously respects majority in choice and elimination, if one of the following cases holds*:

(i) $n=10$ *and* $m\geq 5$
(ii) $n=12$ *and* $m\geq 4$

Proof. The proofs are done by contradiction. Suppose that there exists such a scoring rule F_s that respects majority in choice and elimination. Then Proposition 2.1 yields that $s_2\leq 1/3$ if $n=10$ and that $s_2\leq 2/7$ if $n=12$.

For $n=10$ we consider the following preference profile p. Alternative x is ranked worst by 6 voters and ranked best by 4 voters; hence $\sigma(x;p)=4$. Alternative y is ranked best by 3 voters, second by 3 voters, and worst by 4 voters; hence $\sigma(y;p)=3+3s_2\leq 4$. For $j=1,2,3$ alternative z_j is ranked best by 1 voter, and second or worse by 9 voters; hence $\sigma(z_i;p)\leq 1+9s_2\leq 4$. The remaining $m-5$ alternatives are all ranked second or worse by all 10 voters; hence their score is at most $10s_2<4$. The scoring rule fails to eliminate alternative x.

For $n=12$ we consider the following preference profile p. Alternative x is ranked worst by 7 voters and ranked best by 5 voters; hence $\sigma(x;p)=5$. Alternative y is ranked best by 1 voter, second by 10 voters, and worst by 1 voter; hence $\sigma(y;p)=1+10s_2<5$. For $j=1,2$ alternative z_j is ranked best by 3 voters, worst by 2 voters, and second or worse by 7 voters; hence $\sigma(z_i;p)\leq 3+7s_2\leq 5$. The remaining $m-4$ alternatives are all ranked second or worse by all 12 voters; hence their score is at most $12s_2<5$. The scoring rule fails to eliminate alternative x. □

Lemma 3.5. *For any even $n\geq 14$ and for any $m\geq 3$, there does not exist a generalized scoring rule F_s that simultaneously respects majority in choice and elimination.*

Proof. Once again, the proof is done by contradiction. Suppose that there exists such a scoring rule F_s that respects majority in choice and elimination. Let $n=2k-2$ for some integer $k\geq 8$. Then Proposition 2.1 yields that $s_2\leq 2/k$.

Consider the following preference profile p. Alternative x is ranked worst by k voters and ranked best by $k-2$ voters; hence $\sigma(x;p)=k-2$. The remaining $m-1$ alternatives are ranked as follows.

(a) If $m=3$, then alternative y is ranked best by $\lceil k/2\rceil$ voters, ranked second by $k-1$ voters, and ranked worst by the remaining voters. Alternative z is ranked best by

$\lfloor k/2 \rfloor$ voters, ranked second by $k-1$ voters, and ranked worst by the remaining voters.

(b) If $m \geq 4$ holds, then each of the remaining $m-1$ alternatives is ranked best by $\lceil k/(m-1) \rceil$ or $\lfloor k/(m-1) \rfloor$ voters, and it is ranked second or worse by the other voters.

In Case (a), $\sigma(y;p)$ and $\sigma(z;p)$ are at most $\lceil k/2 \rceil s_1 + (k-1)s_2 \leq \lceil k/2 \rceil + 2(k-1)/k < \sigma(x;p)$. Hence, the scoring rule F_s does not eliminate alternative x. In Case (b), we have for the overall score of any alternative $y \neq x$ that

$$\begin{aligned} \sigma(y;p) &\leq \left\lceil \frac{k}{m-1} \right\rceil s_1 + \left(n - \left\lceil \frac{k}{m-1} \right\rceil \right) s_2 \\ &\leq \left\lceil \frac{k}{m-1} \right\rceil \left(1 - \frac{2}{k} \right) + (2k-2)\frac{2}{k} \leq \left\lceil \frac{k}{3} \right\rceil \frac{k-2}{k} + \frac{4}{k}(k-1). \end{aligned}$$

For $k=8$, this final upper bound equals $23/4 < k-2$. For $k \geq 9$, this final upper bound is at most $\lceil k/3 \rceil + 4 \leq k-2$. Hence, in either case $\sigma(y;p) \leq k-2 = \sigma(x;p)$ holds for all alternatives $y \in A$. The scoring rule F_s does not respect majority in elimination. $\square$

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References

Fishburn, P.C., Gehrlein, W.V., 1976. Borda's rule, positional voting, and Condorcet's simple majority principle. Public Choice 28, 79–88.

Lepelley, D., 1992. Une charactérization du vote à la majorité simple. RAIRO Operations Research 26, 361–365.

Lepelley, D., Merlin, V., 1998. Choix social positionnel et principe majoritaire. Annales d'économie et statistiques 51, 29–48.

Lepelley, D., Vidu, L., 2000. Règles positionnelles itératives, principe majoritaire et préférences unimodales. RAIRO Operations Research 34, 347–362.

Moulin, H., 1983. In: The Strategy of Social Choice. North-Holland, Amsterdam.

Moulin, H., 1988. In: Axioms of Cooperative Decision Making. Cambridge University Press, Cambridge.

Nanson, E.J., 1882. Methods of election. Transactions and Proceedings of the Royal Society of Victoria 18, 197–240.

Richelson, J.T., 1978. A comparative analysis of social choice functions II. Behavioral Sciences 35, 38–44.

Richelson, J.T., 1980. Running-off empty: Run-off points systems. Public Choice 35, 457–468.

Sanver, M.R., 2002. Scoring rules cannot respect majority in choice and elimination simultaneously. Mathematical Social Sciences 43, 151–155.

Smith, J.H., 1973. Aggregation of preferences with a variable electorate. Econometrica 41, 1027–1041.