Robust stabilization of nonlinear systems via stable kernel representations with L_2 -gain bounded uncertainty

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(Received 24 November 1993)

Abstract

The approach to robust stabilization of linear systems using normalized left coprime factorizations with $\mathcal{H}_\infty$ bounded uncertainty is generalized to nonlinear systems. A nonlinear perturbation model is derived, based on the concept of a stable kernel representation of nonlinear systems. The robust stabilization problem is then translated into a nonlinear disturbance feedforward $\mathcal{H}_\infty$ optimal control problem, whose solution depends on the solvability of a single Hamilton–Jacobi equation.

Keywords: Robust nonlinear control; Perturbation model; Kernel representation; Small-gain theorem; Nonlinear $\mathcal{H}_\infty$ control

1. Introduction

A wealth of literature is available on the problem of robustly stabilizing nonlinear uncertain systems. Here we propose a very particular approach, which directly generalizes the solution of the linear robust stabilization problem via normalized left coprime factorizations, as obtained in Glover and McFarlane [6] (see also [15]), to the nonlinear case. Essential ingredients in our approach are the stable kernel representation of nonlinear state space systems as introduced in [18, 17], the resulting nonlinear perturbation model, and the solution to a particular type of nonlinear $\mathcal{H}_\infty$ control problems. The theory is illustrated with a simple example admitting an explicit solution.

2. A nonlinear perturbation model

A very general perturbation model for linear systems is the *numerator–denominator* perturbation model, or *coprime factor uncertainty* model, as it is also known (see e.g. [23, 12, 24]). Let $G(s)$ be the transfer matrix of a linear system (i.e. $G(s)$ is a proper rational matrix). Left factorization of $G(s)$ over the *stable* proper rational

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matrices yields $G(s) = D^{-1}(s)N(s)$, with $D(s)$, $N(s)$ coprime stable proper rational matrices. The stable linear system

$$e = [N(s) \ : \ -D(s)] \begin{bmatrix} u \\ y \end{bmatrix} \quad (1)$$

(with “inputs” $\begin{bmatrix} u \\ y \end{bmatrix}$ and “outputs” e) will be called a *stable kernel representation* of $G(s)$, since by setting $e = 0$ in (1) one recovers the input–output map $y = G(s)u$. In the numerator–denominator perturbation model one considers the following class of perturbations:

$$\begin{aligned} N(s) &\mapsto N(s) + \Delta_N(s), \\ D(s) &\mapsto D(s) + \Delta_D(s), \end{aligned} \quad (2)$$

with $\Delta_N(s)$, $\Delta_D(s)$ stable proper rational matrices. (In applications one would normally include some extra weighting filters; however, they can be incorporated in the system transfer matrix $G(s)$, see [15, 13].) This results in the *perturbed* stable kernel representation

$$e_p = [N(s) \ : \ -D(s)] \begin{bmatrix} u \\ y \end{bmatrix} + [\Delta_N(s) \ : \ -\Delta_D(s)] \begin{bmatrix} u \\ y \end{bmatrix} \quad (3)$$

and the perturbed transfer matrix $G_p(s) = [D(s) + \Delta_D(s)]^{-1}[N(s) + \Delta_N(s)]$. Usually it is convenient to *normalize* the kernel representation (1) by starting with left coprime factors $D(s)$, $N(s)$ satisfying

$$N(s)N^T(-s) + D(s)D^T(-s) = I, \quad s \in \mathbb{C}. \quad (4)$$

A detailed treatment of the robust stabilization problem based on this normalized coprime factor uncertainty model is given in [6, 15], see also [24] for the unnormalized case.

Now let us consider smooth nonlinear systems

$$\Sigma: \begin{aligned} \dot{x} &= f(x) + g(x)u, \quad u \in \mathbb{R}^m, \\ y &= h(x), \quad y \in \mathbb{R}^p, \end{aligned} \quad (5)$$

where $x = (x_1, \dots, x_n)$ are local coordinates for an n -dimensional state space manifold M . Throughout we assume the existence of a distinguished equilibrium x_0 , i.e. $f(x_0) = 0$. Without loss of generality we assume $x_0 = 0$, and furthermore $h(0) = 0$.

Before defining a stable kernel representation for Σ and the resulting perturbation model we need some preliminaries. Let $\gamma > 0$. Σ is said to have L_2 -gain $\leq \gamma$ if there exists a nonnegative solution $V: M \rightarrow \mathbb{R}$ (a *storage function*) to the *dissipation inequality* [25],

$$V(x(t_1)) - V(x(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma^2 \|u(t)\|^2 - \|y(t)\|^2) dt, \quad V(0) = 0, \quad (6)$$

for all $t_1 \geq t_0$ and all $u \in L_2[t_0, t_1]$ (with $x(t_1)$ denoting the solution at time t_1 for initial condition $x(t_0)$ at time t_0). Σ is said to have L_2 -gain $< \gamma$ if there exists some $\tilde{\gamma} < \gamma$ such that Σ has L_2 -gain $\leq \tilde{\gamma}$. Throughout we will assume that if there exists a solution $V \geq 0$ to (6) then there also exists a differentiable solution $V \geq 0$ to (6), and we will restrict ourselves to these differentiable solutions.

Let Σ have L_2 -gain $\leq \gamma$. From [7, 8, 21] we recall that if additionally Σ is *zero-state observable* (i.e. $y(t) = 0$, $u(t) = 0$, $\forall t \geq 0$, implying $x(0) = 0$), then necessarily a solution $V \geq 0$ to (6) is *positive definite* ($V(x) > 0$, $x \neq 0$), and 0 is a locally asymptotically stable equilibrium of (5) with Lyapunov function V .

Next we consider the Hamilton–Jacobi–Bellman equation (corresponding to Σ with cost criterion $\int_{-\infty}^0 (\|u(t)\|^2 + \|y(t)\|^2) dt$)

$$W_x(x)f(x) + \frac{1}{2}W_x(x)g(x)g^T(x)W_x^T(x) - \frac{1}{2}h^T(x)h(x) = 0, \quad W(0) = 0, \quad (7)$$

where

$$W_x(x) = \left(\frac{\partial W}{\partial x_1}(x), \dots, \frac{\partial W}{\partial x_n}(x) \right).$$

The following theorem is a generalized version of [18, 17].

Theorem 2.1. Consider the nonlinear system (5), and assume it is zero-state detectable ($y(t) = 0$, $u(t) = 0$, $\forall t \geq 0$, implying $x(t) \rightarrow 0$, $t \rightarrow \infty$). Suppose there exists a smooth positive definite solution W to (7), and suppose there exists a smooth solution $k(x)$ to

$$W_x(x)k(x) = h^T(x). \quad (8)$$

Define the system Σ_s with inputs $\begin{bmatrix} u \\ y \end{bmatrix}$ and outputs e :

$$\begin{aligned} \dot{x} &= [f(x) - k(x)h(x)] + [g(x) : k(x)] \begin{bmatrix} u \\ y \end{bmatrix}, \\ \Sigma_s: \quad e &= h(x) - y. \end{aligned} \quad (9)$$

Then $f(x) - k(x)h(x)$ is locally asymptotically stable (w.r.t. the equilibrium $x = 0$) with Lyapunov function W , and globally asymptotically stable if W is proper (i.e. the sets $\{x \in M \mid 0 \leq W(x) \leq c\}$ are compact for every $c \geq 0$). Furthermore, Σ_s has L_2 -gain = 1. Setting $e = 0$ in Σ_s yields Σ , and Σ_s will be called a (nonlinear) stable kernel representation of Σ .

Proof (sketch, see [18, 17] for details). From (7) and (8) we obtain

$$W_x(x)[f(x) - k(x)h(x)] = -\frac{1}{2}W_x(x)g(x)g^T(x)W_x^T(x) - \frac{1}{2}h^T(x)h(x) \leq 0 \quad (10)$$

and (global) asymptotic stability follows from LaSalle's invariance principle. Similarly,

$$\begin{aligned} &W_x(x)([f(x) - k(x)h(x)] + g(x)u + k(x)y) \\ &= -\frac{1}{2}\|u - g^T(x)W_x^T(x)\|^2 - \frac{1}{2}\|e\|^2 + \frac{1}{2}\|u\|^2 + \frac{1}{2}\|y\|^2, \end{aligned} \quad (11)$$

proving L_2 -gain ≤ 1 by integration (see e.g. [21]). $\square$

Remark 2.2. If the linearized system $\dot{x} = (\partial f/\partial u)(0)x + g(0)u$, $y = (\partial h/\partial x)(0)x$ is anti-stabilizable, and if the imaginary eigenvalues of $(\partial f/\partial x)(0)$ are $(\partial h/\partial x)(0)$ -observable, then at least locally around 0 there exists a smooth non-negative solution $W \geq 0$ to (7) (see e.g. [11, 20]), which will be locally positive definite if the linearized system is observable.

Remark 2.3. Consider a star-shaped coordinate neighborhood of $x = 0$. Since $W_x(0) = 0$ and $h(x) = 0$ we can write (see e.g. [16])

$$W_x(x) = x^T M(x), \quad h(x) = C(x)x \quad (12)$$

for suitable matrices $M(x)$, $C(x)$, with entries depending smoothly on x . Assume that $M(x)$ is invertible for all x in the coordinate neighborhood of 0; then a solution $k(x)$ to (8) is given as [18]

$$k(x) = M^{-1}(x)C^T(x). \quad (13)$$

(See [9] for similar considerations in a different context.) Note furthermore that $M(0) = (\partial^2 W / \partial x^2)(0)$ (the Hessian matrix of W at 0). Thus, under the conditions of Remark 1, $M(0)$ will be positive definite, implying that $M(x)$ will be invertible for x near 0.

Remark 2.4 ([18]). If $-V$ is a negative definite solution to (7), then

$$\Sigma_{ri}: \begin{aligned} \dot{p} &= f(p) - g(p)g^T(p)V_p^T(p) - k(p)e, \quad p \in M, \\ \begin{bmatrix} u \\ y \end{bmatrix} &= \begin{bmatrix} -g^T(p)V_p^T(p) \\ h(p) - e \end{bmatrix} \end{aligned} \quad (14)$$

is a right inverse system to Σ_s , i.e., if $x(0) = p(0)$, then the input–output map of $\Sigma_s \circ \Sigma_{ri}$ is the identity mapping. Furthermore, $f(p) - g(p)g^T(p)V_p^T(p)$ is locally asymptotically stable (w.r.t. $p = 0$) with Lyapunov function V (and globally asymptotically stable if V is proper). Hence Σ_s has a stable right inverse, generalizing the linear notion of coprimeness.

Remark 2.5. For a linear system Σ , the stable kernel representation Σ_s reduces to the left *normalized* coprime factorization (1), (4).

Analogously to the linear case (cf. (3)), we will now consider *perturbed* nonlinear stable kernel representations

$$\begin{aligned} \dot{x} &= [f(x) - k(x)h(x)] + [g(x) \quad k(x)] \begin{bmatrix} u \\ y \end{bmatrix}, \\ e_p &= e + w, \end{aligned} \quad (15)$$

where w is the output of an arbitrary nonlinear state space system with input $\begin{bmatrix} u \\ y \end{bmatrix}$,

$$\begin{aligned} \Delta: \quad \dot{\varphi} &= \alpha(\varphi, u, y), \quad \alpha(0, 0, 0) = 0, \\ w &= \beta(\varphi, u, y), \quad \beta(0, 0, 0) = 0, \end{aligned} \quad (16)$$

having finite L_2 -gain. (More generally we could consider families of nonlinear input–output maps from $\begin{bmatrix} u \\ y \end{bmatrix}$ to w , parametrized by the set of initial conditions.) Setting $e_p = 0$ in (15) yields the *perturbed* system

$$\Sigma_p: \begin{aligned} \dot{x} &= f(x) + g(x)u + k(x)w, \\ y &= h(x) + w, \end{aligned} \quad (17)$$

with w the output of (16).

3. The robust stabilization problem

Consider the nonlinear system Σ given by (5), and its perturbed model Σ_p given by (17), (16). The *robust stabilization problem* is to find a controller

$$\begin{aligned} C: \quad \dot{\xi} &= l(\xi, y), \quad l(0, 0) = 0, \\ u &= m(\xi, y), \quad m(0, 0) = 0, \end{aligned} \quad (18)$$

with $\xi \in \mathbb{R}^c$ the controller state, such that the L_2 -gain of the closed-loop system (17), (18), from w to $z = \begin{bmatrix} u \\ y \end{bmatrix}$, is minimized, say equal to $\gamma^* \geq 0$.

By the small-gain theorem (see e.g. [3]) this will mean that the overall closed-loop system (16)–(18) will be L_2 -stable for all perturbations Δ with L_2 -gain strictly less than $1/\gamma^*$.

In state space terms, if there exist proper positive definite solutions $V_{\Sigma C}$, V_Δ to the dissipation inequalities

$$V_{\Sigma C}(x(t_1), \xi(t_1)) - V_{\Sigma C}(x(t_0), \xi(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma^2 \|w(t)\|^2 - \|z(t)\|^2) dt, \quad \gamma \geq \gamma^*,$$

$$V_\Delta(\varphi(t_1)) - V_\Delta(\varphi(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma_\Delta^2 \|z(t)\|^2 - \|w(t)\|^2) dt, \quad \gamma_\Delta < 1/\gamma,$$
(19)

then, assuming zero-state detectability [7], the overall closed-loop system (16)–(18) will be globally asymptotically stable with Lyapunov function $\gamma_\Delta V_{\Sigma C} + \gamma V_\Delta$, as can be readily checked from (19).

The problem of minimizing the L_2 -gain from w to $z = \begin{bmatrix} u \\ y \end{bmatrix}$ for (17) is a *standard* $\mathcal{H}_\infty$ optimal nonlinear control problem [19, 21, 10, 1, 22, 9]. Usually one first considers the *suboptimal* $\mathcal{H}_\infty$ problem of finding for given $\gamma > 0$ a controller C (if existing!) which makes the L_2 -gain from w to $z = \begin{bmatrix} u \\ y \end{bmatrix}$ less than or equal to γ . For the solution to the suboptimal $\mathcal{H}_\infty$ control problem we follow the approach of [2, 22]. For the *state feedback* suboptimal $\mathcal{H}_\infty$ control problem we consider the pseudo-Hamiltonian

$$K(x, p, u, w) = p^T [f(x) + g(x)u + k(x)w] - \frac{1}{2} \gamma^2 \|w\|^2 + \frac{1}{2} \|u\|^2 + \frac{1}{2} \|h(x) + w\|^2.$$
(20)

Solving $\partial K / \partial u = 0$, $\partial K / \partial w = 0$ leads to the saddle point $u^* = -g^T(x)p$, $w^* = (\gamma^2 - 1)^{-1} [h(x) + k^T(x)p]$. Substitution of u^* , w^* into K yields the Hamiltonian $H(x, p) = K(x, p, u^*, w^*)$ and the Hamilton-Jacobi-Isaacs equation $H(x, P_x^T(x)) = 0$ given as

$$P_x(x) [f(x) + (\gamma^2 - 1)^{-1} k(x)h(x)] + \frac{1}{2} \gamma^2 (\gamma^2 - 1)^{-1} h^T(x)h(x) + \frac{1}{2} P_x(x) [(\gamma^2 - 1)^{-1} k(x)k^T(x) - g(x)g^T(x)] P_x^T(x) = 0, \quad P(0) = 0.$$
(21)

If there exists a solution $P \geq 0$ to (21) then the suboptimal state feedback $\mathcal{H}_\infty$ control problem (for γ) is solvable by the state feedback

$$u = -g^T(x)P_x^T(x).$$
(22)

Following the certainty equivalence principle of [2] the solution to the *output* feedback suboptimal $\mathcal{H}_\infty$ control problem is, under appropriate conditions, given as

$$u = -g^T(\hat{x})P_x^T(\hat{x}),$$
(23)

with $\hat{x}(t)$ denoting the *worst-case estimate* of $x(t)$ given the measurements $y(\tau)$, $-\infty < \tau \leq t$, see [2, 22]. (Currently there is intense research activity about the precise conditions for the validity of the worst-case certainty equivalence principle, but we will not elaborate on this.) In general (see e.g. [22]), this will yield an infinite dimensional controller. In the present case, however, the situation is much simpler. Indeed, the suboptimal $\mathcal{H}_\infty$ control problem for (17) with $z = \begin{bmatrix} u \\ y \end{bmatrix}$ is an example of the so-called *disturbance feedforward problem*, discussed for the linear case in [5], and for the nonlinear case in [14]. In fact, by asymptotic stability

of $\dot{x} = f(x) - k(x)h(x)$ it follows that for a given control function $u(\tau)$, $-\infty < \tau \leq t$, the measurement record $y(\tau)$, $-\infty < \tau \leq t$, uniquely specifies the disturbance $\hat{w}(\tau) = y(\tau) - h(\hat{x}(\tau))$ and the state trajectory $\hat{x}(\tau)$. Indeed the state trajectory $\hat{x}(\cdot)$ is generated by the differential equations

$$\dot{\hat{x}}(\tau) = f(\hat{x}(\tau)) + g(\hat{x}(\tau))u(\tau) + k(\hat{x}(\tau))[y(\tau) - h(\hat{x}(\tau))], \quad x(-\infty) = 0. \quad (24)$$

Hence a controller solving the suboptimal $\mathcal{H}_\infty$ control problem for (17) is given as (substitute (23) into (24))

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}) - g(\hat{x})g^T(\hat{x})P_x^T(\hat{x}) + k(\hat{x})[y - h(\hat{x})], \\ u &= -g^T(\hat{x})P_x^T(\hat{x}). \end{aligned} \quad (25)$$

Based on the linear case, the same controller for the general nonlinear disturbance feedforward problem has been recently proposed in [14]. In this paper also a direct proof is provided showing that (25) solves the suboptimal $\mathcal{H}_\infty$ control problem *at least locally*, i.e. for initial states in a neighborhood of the origin and for disturbances $w(\cdot)$ which keep the state trajectories within this neighborhood. Summarizing, we have the following theorem.

Theorem 3.1. Suppose (cf. Theorem 2.1) that Σ_s given by (9) is a stable kernel representation of Σ such that $f(x) - k(x)h(x)$ is globally asymptotically stable. Suppose there exists a solution $P \geq 0$ to (21) for given $\gamma > 0$, and assume the certainty equivalence principle for the suboptimal $\mathcal{H}_\infty$ control problem for (17) holds. Then the controller (25) stabilizes the closed-loop system (16), (17), (25) for every perturbation system Δ as in (16), having L_2 -gain $< 1/\gamma$.

Remark 3.2. From the linear theory (cf. [6, 15]) and the local existence of solutions to (21) based on existence of solutions to the corresponding Riccati equation (cf. [20, 21]), it follows that the minimal γ^* , such that locally around 0 there exist solutions $P \geq 0$ to (21) for $\gamma > \gamma^*$, is given by

$$\gamma^* = [1 + \sigma_{\max}(XZ)]^{1/2}, \quad (26)$$

with X the Hessian matrix $(\partial^2 V / \partial x^2)(0)$ and Z the inverse Hessian matrix $[(\partial^2 W / \partial x^2)(0)]^{-1}$ of the solutions $W \geq 0$ and $-V \leq 0$ to (7).

Remark 3.3. A related approach to nonlinear robust stabilization will be found in [4].

Example 3.4. Let Σ be a lossless system, i.e. there exists $H: M \rightarrow \mathbb{R}$, $H(0) = 0$, $H(x) > 0$, $x \neq 0$, called the internal energy, such that $(d/dt)H = u^T y$ or, equivalently,

$$H_x(x)f(x) = 0, \quad H_x(x)g(x) = h^T(x). \quad (27)$$

Clearly, positive and negative definite solutions to (7) are given as H , and $-H$, respectively. Furthermore, $k(x)$ solving (8) is given as $g(x)$, and thus the perturbed system Σ_p is given as

$$\dot{x} = f(x) + g(x)[u + w], \quad y = g^T(x)H_x^T(x) + w. \quad (28)$$

The Hamilton–Jacobi–Isaacs equation (21) takes the form

$$\begin{aligned} P_x(x)[f(x) + (\gamma^2 - 1)^{-1}g(x)g^T(x)H_x^T(x)] + \frac{1}{2}[(\gamma^2 - 1)^{-1} - 1] \cdot P_x(x)g(x)g^T(x)P_x^T(x) \\ + \frac{1}{2}\gamma^2(\gamma^2 - 1)^{-1}H_x(x)g(x)g^T(x)H_x^T(x) = 0, \quad P(0) = 0, \end{aligned} \quad (29)$$

having the positive definite solution $P(x) = (\gamma^2/(\gamma^2 - 2))H(x)$ for $\gamma > \sqrt{2}$. It follows that the controller

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}) - \frac{\gamma^2}{\gamma^2 - 2}g(\hat{x})g^T(\hat{x})H_x^T(\hat{x}) + g(\hat{x})[y - g^T(\hat{x})H_x^T(\hat{x})], \\ u &= -\frac{\gamma^2}{\gamma^2 - 2}g^T(\hat{x})H_x^T(\hat{x}) \end{aligned} \quad (30)$$

robustly stabilizes Σ for every perturbation Δ with L_2 -gain $< 1/\gamma$. By Remark 3.2, γ^* is given by (26). Since $W = H$ and $V = -H$ we conclude that $\gamma^* = \sqrt{2}$, in accordance with the lower bound $\gamma > \sqrt{2}$ as derived above. From a physical point of view, if in (28) u 's denote external forces and y 's are the corresponding (disturbed) generalized velocities, then (30) corresponds to adding damping with regard to the *estimated* generalized velocities with a damping factor $\gamma^2(\gamma^2 - 2)^{-1}$, tending to ∞ for $\gamma \downarrow \gamma^* = \sqrt{2}$.

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