Bipartite Regular Graphs with Fixed Diameter

H. J. Broersma and F. Göbel

Faculty of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

For given nonnegative integers k and D , we consider the problem of determining $n_0(k, D)$, the smallest number n for which there exists a k -regular bipartite graph on n vertices with diameter D . We solve the problem for all pairs (k, D) with $D \not\equiv 2 \pmod{4}$ and $D \not\equiv 3 \pmod{4}$, for all pairs (k, D) with k even or k prime and $D \not\equiv 3 \pmod{4}$, for all pairs with $D \leq 9$ or $k \leq 4$, and for a few other pairs. In the remaining cases, we obtain lower and upper bounds for $n_0(k, D)$. © 1995 John Wiley & Sons, Inc.

1. INTRODUCTION

We use Bondy and Murty [2] for terminology and notation not defined here and consider simple graphs only.

For given nonnegative integers k and D , we consider the problem of determining $n_0(k, D)$, the smallest number n for which there exists a k -regular bipartite graph on n vertices with diameter D .

Bermond et al. [1] considered the following related problem: Given Δ, D , find the largest number n for which a graph on n vertices exists with diameter D and degrees at most Δ . Delorme [4] considered the analogous problem for bipartite graphs, whereas Fellows et al. [5, 6] considered planar graphs. Our research was triggered by Heydemann's question [7] whether a 4-regular graph on 14 vertices exists with diameter 2. This question was answered affirmatively by Broersma and Jagers [3].

We solve the problem [of determining $n_0(k, D)$] for all pairs (k, D) with $D \not\equiv 2 \pmod{4}$ and $D \not\equiv 3 \pmod{4}$, for all pairs (k, D) with k even or k prime and $D \not\equiv 3 \pmod{4}$, for all pairs with $D \leq 9$ or $k \leq 4$, and for a few other pairs. In the remaining cases, we obtain lower and upper bounds for $n_0(k, D)$.

A (k, D) -graph is a k -regular bipartite graph with diameter D . Note that a (k, D) -graph exists if and only if $k = 0, D = 0$, or $k = 1, D = 1$, or $k \geq 2, D \geq 2$. In the sequel, we assume that $k \geq 2$ and $D \geq 2$.

2. NOTATION AND PRELIMINARY RESULTS

Let G be a (k, D) -graph and let v be a vertex of G that has distance D to some other vertex of G . Then, we denote the set of vertices at distance i from v in G by V_i and the cardinality of V_i by a_i ($i = 0, 1, \dots, D$). Obviously, $a_0 = 1$ and $a_1 = k$. For $i = 1, 2, \dots, D$, let b_i be the number of edges of G joining the vertices of V_{i-1} and V_i . (Note that all V_i are independent sets in G since G is bipartite.) Obviously, $b_1 = k$ and $b_2 = k(k - 1)$.

The above notation will be used throughout this paper.

We start with some necessary conditions for (k, D) -graphs:

Lemma 1. *If G is a (k, D) -graph, then*

- (a) $a_{i-1} + a_{i+1} \geq k$ ($i = 1, \dots, D - 1$),
- (b) $b_{i-1} + b_i = ka_{i-1}$ ($i = 2, \dots, D$),
- (c) $b_i \leq a_{i-1}a_i$ ($i = 1, \dots, D$),
- (d) $\sum_{i=0}^D (-1)^i a_i = 0$.

Proof.

- (a) Since the k neighbors of each vertex in V_i are in $V_{i-1} \cup V_{i+1}$, $a_{i-1} + a_{i+1} \geq k$ ($i = 1, \dots, D - 1$).
- (b) Since all vertices in V_{i-1} have degree k and are mutually nonadjacent, the definition of b_{i-1} and b_i implies that $b_{i-1} + b_i = ka_{i-1}$ ($i = 2, \dots, D$).

- (c) It is obvious from the definitions of b_i and a_i that $b_i \leq a_{i-1}a_i$ ($i = 1, \dots, D$).
- (d) Since G is regular and bipartite with classes $V_0 \cup V_2 \cup \dots$ and $V_1 \cup V_3 \cup \dots$, $\sum_{i=0}^D (-1)^i a_i = 0$. ■

Corollary 2.

- (a) All b_i are divisible by k .
- (b) $a_i \geq 2$ ($i = 1, \dots, D-1$).

Proof.

- (a) Since $b_1 = k$, and by Lemma 1(b), $b_i = ka_{i-1} - b_{i-1}$, all b_i are divisible by k .
- (b) If $a_i = 1$ for some $i \in \{1, \dots, D-1\}$, then by Lemma 1(b), $b_i + b_{i+1} = k$, contradicting (a) since all b_i are positive. Hence, $a_i \geq 2$ ($i = 1, \dots, D-1$). ■

We denote b_i/k by f_i and note that all f_i are integers by Corollary 2(b). Moreover, we note that $a_0 = f_1$, $a_D = f_D$, and by Lemma 1(b), $a_i = f_i + f_{i+1}$ ($i = 1, \dots, D-1$).

The notation f_i will be used throughout this paper.

Lemma 3.

- (a) $f_i + f_{i+1} + f_{i+2} + f_{i+3} \geq k$ ($i = 1, \dots, D-3$; $D \geq 4$).
- (b) If k is prime, then $f_i + f_{i+1} + f_{i+2} + f_{i+3} \geq k+1$ ($i = 1, \dots, D-3$; $D \geq 4$).
- (c) $f_i + f_{i+1} + f_{i+2} \geq \lceil 2\sqrt{k} \rceil - 1$ ($i = 1, \dots, D-2$; $D \geq 3$).

Proof.

- (a) The inequality follows immediately from Lemma 1(a).
- (b) Suppose that $f_i + f_{i+1} + f_{i+2} + f_{i+3} = k$ for some $i \in \{1, \dots, D-3\}$. Then, $a_i = f_i + f_{i+1} < k$. Since k is prime, this implies that a_i is relative prime to k . By Lemma 1(c), $ka_{i+1} \leq a_i a_{i+1}$ and $ka_{i+2} \leq a_{i+1} a_{i+2}$, so that $ka_{i+1} \leq a_{i+1}(a_i + a_{i+2}) = a_{i+1}k$. This implies that $ka_{i+1} = a_{i+1}k$, or, equivalently, $a_{i+1} = [(ka_{i+1})/a_i]$. Since a_i is relative prime to k , we conclude that a_i divides f_{i+1} , which is absurd since $0 < f_{i+1} < a_i$. Hence, $f_i + f_{i+1} + f_{i+2} + f_{i+3} \geq k+1$.
- (c) Suppose that $f_i + f_{i+1} + f_{i+2} \leq \lceil 2\sqrt{k} \rceil - 2$. Then, $f_i + f_{i+2} \leq \lceil 2\sqrt{k} \rceil - 3$. We first show that

$$(1) (f_i + f_{i+2} - k)^2 - 4f_i f_{i+2} > 0.$$

Clearly, $(f_i + f_{i+2} - k)^2 - 4f_i f_{i+2} = k^2 - 2k(f_i + f_{i+2}) + (f_i - f_{i+2})^2 \geq k(k - 2\lceil 2\sqrt{k} \rceil + 6)$. From $(\sqrt{k} - 2)^2 \geq 0$, we obtain $k - 4\sqrt{k} + 4 \geq 0$ and $k + 6 \geq 4\sqrt{k} + 2 = 2(2\sqrt{k} + 1) > 2\lceil 2\sqrt{k} \rceil$. Combining the above observations, we obtain (1).

By Lemma 1(c), $ka_{i+1} \leq (f_i + f_{i+1})(f_{i+1} + f_{i+2})$ or, equivalently, $f_{i+1}^2 + (f_i + f_{i+2} - k)f_{i+1} + f_i f_{i+2} \geq 0$; hence [using (1)] either

$$(2) k - (f_i + f_{i+2}) + \sqrt{(f_i + f_{i+2} - k)^2 - 4f_i f_{i+2}}/2$$

or

$$(3) [(k - (f_i + f_{i+2}) - \sqrt{(f_i + f_{i+2} - k)^2 - 4f_i f_{i+2}})/2].$$

First suppose that (2) holds. Then, $4\sqrt{k} - 2 > 2\lceil 2\sqrt{k} \rceil - 4 \geq 2(f_i + f_{i+1} + f_{i+2}) \geq k + f_i + f_{i+2} + \sqrt{(f_i + f_{i+2} - k)^2 - 4f_i f_{i+2}} > k + 2$; hence, $(\sqrt{k} - 2)^2 = k - 4\sqrt{k} + 4 < 0$, a contradiction.

Next suppose that (3) holds. Since $f_{i+1} \geq 1$, from (3) we obtain $\sqrt{(f_i + f_{i+2} - k)^2 - 4f_i f_{i+2}} \leq k - (f_i + f_{i+2}) - 2$, so that $(f_i + f_{i+2} - k)^2 - 4f_i f_{i+2} \leq (k - (f_i + f_{i+2}))^2 - 4(k - (f_i + f_{i+2})) + 4$ or, equivalently, $(f_i + 1)(f_{i+2} + 1) \geq k$. On the other hand, $f_i + f_{i+2} \leq \lceil 2\sqrt{k} \rceil - 3$ implies that $(f_i + 1) + (f_{i+2} + 1) \leq \lceil 2\sqrt{k} \rceil - 1 < 2\sqrt{k}$; hence, $(f_i + 1)(f_{i+2} + 1) < k$, a contradiction.

We conclude that $f_i + f_{i+1} + f_{i+2} \geq \lceil 2\sqrt{k} \rceil - 1$. ■

We use the above results in the next section to determine lower bounds for $n_0(k, D)$. To determine upper bounds for $n_0(k, D)$ in the sequel, we often use the following lemma:

Lemma 4. Let $\alpha_0, \alpha_1, \dots, \alpha_D$ and $\beta_1, \dots, \beta_D$ be sequences of positive integers satisfying

- (a) $\alpha_0 = 1, \alpha_D = 1, \beta_1 = k, \beta_D = k$ with $k \geq 2$,
- (b) $\beta_i + \beta_{i+1} = k\alpha_i$ ($i = 1, \dots, D-1$),
- (c) $\alpha_i \leq \beta_i \leq (k-1)\alpha_i$ ($i = 1, \dots, D-1$),
- (d) $\beta_i \leq \alpha_{i-1}\alpha_i$ ($i = 1, \dots, D$).

Then, a (k, D) -graph on $\sum_{i=0}^D \alpha_i$ vertices exists.

Proof. Let $V_0, V_1, \dots, V_D$ be pairwise disjoint vertex sets with $|V_i| = \alpha_i$. We will define a graph G with vertex set $\cup_{i=0}^D V_i$ as follows:

For each edge e of G , we will choose a number i such that e is incident with a vertex in V_{i-1} and a vertex in V_i . This implies that G is bipartite.

The indegree of a vertex in V_i is defined as the number of its neighbors in V_{i-1} , and the outdegree, as the number of its neighbors in V_{i+1} . We will now prove the following claim by induction on i .

Claim. Between V_0 and V_1 , V_1 and V_2 , $\dots$, V_{i-1} and V_i , the edges of G can be placed such that the degrees (of the vertices) in $V_0, V_1, \dots, V_{i-1}$ are k and the indegrees for the vertices in V_j differ by at most 1 ($j = 1, \dots, i$); hence, the outdegrees also differ by at most 1 ($j = 0, \dots, i-1$).

Proof of Claim. For $i = 1$, the claim is true: By (c), $\alpha_1 \leq \beta_1 = k$, and by (d), $k = \beta_1 \leq \alpha_0\alpha_1 = \alpha_1$; hence, $\alpha_1 = k$. It is indeed possible to place the edges as required.

Suppose that the claim is true for some i with $1 \leq i < D$. Between V_{i-1} and V_i , there are β_i edges [this follows from (b)]; hence, the indegrees in V_i can be determined as follows:

Write $\beta_i = q\alpha_i + r$ with $0 \leq r < \alpha_i$. Then, the indegrees in V_i must be r times $q + 1$ and $\alpha_i - r$ times q . Now we can make all degrees in V_i equal to k , and the outdegrees in V_i will be r times $k - q - 1$ and $\alpha_i - r$ times $k - q$. The sum of these outdegrees is $r(k - q - 1) + (\alpha_i - r)(k - q) = (k - q)\alpha_i - r = k\alpha_i - \beta_i = \beta_{i+1}$ [by (b)]. It is easy to choose the end vertices of these edges in V_{i+1} [using (c) and (d)] in such a way that the indegrees in V_{i+1} differ by at most 1. This proves the claim.

It is obvious that the diameter of the resulting graph is at least D . To prove that it is at most D , we note that all indegrees and all outdegrees are positive (with trivial exceptions for V_0 and V_D). This follows from (c) and the above construction. Now any two vertices of G are connected by a path of length at most D , either through V_0 or through V_D . This completes the proof. ■

We remark that the conditions in Lemma 4 are redundant. For example, from (a), (c), and (d), it follows that all α_i and β_i are positive. However, it was not our objective to give an economic set of conditions. All conditions, superfluous or not, are easy to check where we apply the lemma.

3. LOWER BOUNDS ON $n_0(k, D)$

Using Lemma 3, we obtain the following lower bounds on $n_0(k, D)$ in case $D \geq 4$. For $D < 4$, we obtain exact values for $n_0(k, D)$ in Section 5.

Theorem 5.

(a) If $D \geq 4$, then

$$n_0(k, D) \geq \frac{k}{2} \cdot D + \begin{cases} 2k & \text{if } D \equiv 0 \pmod{4} \\ \frac{3}{2}k + 2 & \text{if } D \equiv 1 \pmod{4} \\ k + 4 & \text{if } D \equiv 2 \pmod{4} \\ \frac{k}{2} + 2(\lceil 2\sqrt{k} \rceil - 1) & \text{if } D \equiv 3 \pmod{4}. \end{cases}$$

(b) If $D \geq 10$, $D \equiv 2 \pmod{4}$, and k is odd, then

$$n_0(k, D) \geq \frac{k}{2} \cdot D + k + 6.$$

Proof.

(a) Let $D \geq 4$. Suppose that G is a (k, D) -graph and let $f_1, f_2, \dots, f_D$ be defined as in Section 2. Then, $n_0(k, D) = 2 \sum_{i=1}^D f_i$, and $f_1 = 1, f_2 = k - 1$, and $f_{D-1} + f_D \geq k$. Let $D = 4q + r$ with $0 \leq r \leq 3$.

If $r = 0$, using Lemma 3(a), we obtain $\sum_{i=3}^{D-2} f_i \geq (q - 1)k$; hence, $n_0(k, D) \geq 2(q - 1)k + 4k = (k/2) \cdot D + 2k$.

If $r = 1$, we similarly obtain (using $f_i \geq 1$) $n_0(k, D) \geq 2(q - 1)k + 4k + 2 = (k/2) \cdot D + (3/2)k + 2$, and if $r = 2$, we obtain $n_0(k, D) \geq 2(q - 1)k + 4k + 4 = (k/2) \cdot D + k + 4$.

If $r = 3$, using Lemma 3(a) and (c), we obtain $\sum_{i=3}^{D-5} f_i \geq (q - 1)k$ and $f_{D-4} + f_{D-3} + f_{D-2} \geq \lceil 2\sqrt{k} \rceil - 1$; hence, $n_0(k, D) \geq 2(q - 1)k + 2(\lceil 2\sqrt{k} \rceil - 1) + 4k = (k/2) \cdot D + (k/2) + 2(\lceil 2\sqrt{k} \rceil - 1)$.

(b) Suppose that $f_{D-2} = f_{D-3} = 1$ and $f_{D-7} + f_{D-6} + f_{D-5} + f_{D-4} = k$. Then, as in the proof of Lemma 3(b) using Lemma 1(c), we obtain from $k f_{D-6} \leq (f_{D-7} + f_{D-6})(f_{D-6} + f_{D-5})$ and $k f_{D-5} \leq (f_{D-6} + f_{D-5})(f_{D-5} + f_{D-4})$ that $f_{D-6} f_{D-4} = f_{D-7} f_{D-5}$. Similarly, $k f_{D-4} \leq (f_{D-5} + f_{D-4})(f_{D-4} + f_{D-3})$ implies that $(f_{D-7} + f_{D-6}) f_{D-4} \leq (f_{D-5} + f_{D-4}) f_{D-3}$. Together with $f_{D-6} f_{D-4} = f_{D-7} f_{D-5}$, this implies that $(f_{D-4} + f_{D-5}) f_{D-7} \leq (f_{D-4} + f_{D-5}) f_{D-3} \leq f_{D-4} + f_{D-5}$, so that $f_{D-7} = 1, f_{D-5} = f_{D-6} f_{D-4}$, and $k = f_{D-7} + f_{D-6} + f_{D-5} + f_{D-4} = (1 + f_{D-6})(1 + f_{D-4})$. Now from $k = k f_{D-3} \leq (f_{D-4} + f_{D-3})(f_{D-3} + f_{D-2}) = (f_{D-4} + 1)2$, we obtain $f_{D-6} = 1$ and $f_{D-5} = f_{D-4} = (k/2) - 1$. In particular, we obtain that k is even. This means that for odd k we can increase the lower bound for $n_0(k, D)$ in Theorem 5(a) by 2. ■

Theorem 6. If $D \geq 4$ and k is prime, then

$$n_0(k, D) \geq \frac{k+1}{2} \cdot D + \begin{cases} 2k - 2 & \text{if } D \equiv 0 \pmod{4} \\ \frac{3}{2}k - \frac{1}{2} & \text{if } D \equiv 1 \pmod{4} \\ k + 1 & \text{if } D \equiv 2 \pmod{4} \\ \frac{k}{2} + 2(\lceil 2\sqrt{k} \rceil - 1) - \frac{7}{2} & \text{if } D \equiv 3 \pmod{4}. \end{cases}$$

Proof. Using Lemma 3(b) instead of Lemma 3(a), it is easy to prove Theorem 6 using the same arguments as in the proof of Theorem 5(a). We leave the details to the reader. ■

4. UPPER BOUNDS ON $n_0(k, D)$

Using Lemma 4 and suitable constructions, we obtain the following upper bounds on $n_0(k, D)$ in the case $D \geq 8$

and $k \geq 4$. For $D < 8$ or $k < 4$, we obtain exact values for $n_0(k, D)$ in Section 5. ■

Theorem 7. Let u, v, w, x , and k be positive integers such that $k = u + v + w + x$ and $uw = vx$. If $D = 4q + r \geq 8$ with $r \in \{0, 1, 2, 3\}$, then

$$n_0(k, D) \leq \frac{k}{2} \cdot D + \begin{cases} 2k & \text{if } r = 0 \\ \frac{3}{2}k + 2u & \text{if } r = 1 \\ k + 2(u + v) & \text{if } r = 2 \\ \frac{k}{2} + 2(u + v + w) & \text{if } r = 3. \end{cases}$$

Proof. First suppose that $D = 8$. Consider the f -sequence and the corresponding a -sequence below:

$$\begin{aligned} f: & 1, (k-1), u, v, w, x, (k-1), 1 \\ a: & 1, k, (k+u-1), (u+v), (v+w), (w+x), (x+k-1), k, 1. \end{aligned}$$

It is clear that the a -sequence and the b -sequence corresponding to the f -sequence (recall that $b_i = kf_i$, $i = 1, 2, \dots, D$) satisfy conditions (a) and (b) of Lemma 4. To check (c) of Lemma 4, note that for $i \geq 1$, $a_i \leq (k-1) + f_i \leq kf_i = b_i$, and for $3 \leq i \leq D-2$, $b_i = kf_i \leq kf_i + (k - (f_i + f_{i+1}))f_{i+1} = kf_i + kf_{i+1} - (f_i + f_{i+1})f_{i+1} \leq (k-1)(f_i + f_{i+1}) = (k-1)a_i$. The remaining cases ($i = 1, 2, D-1$) are easy to check. Condition (d) of Lemma 4 can be checked as follows: $b_4 = kv = uv + v^2 + vw + vx = uv + v^2 + vw + uw = (u+v)(v+w) = a_3a_4$; b_5 runs similarly, and the remaining cases are even simpler.

Hence, there exists a (k, D) -graph corresponding to the indicated f -sequence. Repeating the subsequence u, v, w, x , adding an extra u , an extra u, v , or an extra u, v, w at the end of the last subsequence u, v, w, x in the case $r = 1, 2$, or 3 , respectively, and using similar arguments as above, one easily shows that there exists a (k, D) -graph with $D = 4q + r \geq 8$ on n vertices, where $n = 2 \sum_{i=1}^D f_i$. Using $k = u + v + w + x$, we get $2(\sum_{i=1}^{4q-2} f_i + f_{D-1} + f_D) = 2(2k + (q-1)k) = (k/2) \cdot D + (2 - (r/2))k$. From this we obtain

$$n_0(k, D) \leq \frac{k}{2} \cdot D + \begin{cases} 2k & \text{if } r = 0 \\ \frac{3}{2}k + 2u & \text{if } r = 1 \\ k + 2(u + v) & \text{if } r = 2 \\ \frac{k}{2} + 2(u + v + w) & \text{if } r = 3. \end{cases}$$

Corollary 8. If $D \geq 8$, $k \geq 4$, and k is even, then

$$n_0(k, D) \leq \frac{k}{2} \cdot D + \begin{cases} 2k & \text{if } D \equiv 0 \pmod{4} \\ \frac{3}{2}k + 2 & \text{if } D \equiv 1 \pmod{4} \\ k + 4 & \text{if } D \equiv 2 \pmod{4} \\ \frac{3}{2}k + 2 & \text{if } D \equiv 3 \pmod{4}. \end{cases}$$

Proof. Apply Theorem 7 with $u = v = 1$ and $w = x = (k/2) - 1$. ■

Corollary 9. If $D \geq 8$, $k \geq 5$, and k is a nonprime odd number, then

$$n_0(k, D) \leq \frac{k}{2} \cdot D + \begin{cases} 2k & \text{if } D \equiv 0 \pmod{4} \\ \frac{3}{2}k + 2 & \text{if } D \equiv 1 \pmod{4} \\ k + 2d & \text{if } D \equiv 2 \pmod{4} \\ \frac{k}{2} + 2(d' + d'' - 1) & \text{if } D \equiv 3 \pmod{4}, \end{cases}$$

where d is the smallest odd divisor ≥ 3 of k , and d' and d'' are divisors of k such that $k = d'd''$ and $d' + d''$ is minimum.

Proof. Apply Theorem 7 with $u = 1$, $v = d - 1$, $w = (d-1)[(k/d) - 1]$, and $x = (k/d) - 1$ if $D \not\equiv 3 \pmod{4}$, and with $u = d' - 1$, $v = 1$, $w = d'' - 1$, and $x = (d' - 1)(d'' - 1)$ if $D \equiv 3 \pmod{4}$. ■

For prime numbers k , it is not possible to find a partition satisfying the hypothesis of Theorem 7, in accordance with the lower bounds on $n_0(k, D)$ in Theorem 6.

Theorem 10. If $D \geq 8$, $k \geq 5$, and k is prime, then

$$n_0(k, D) \leq \frac{k+1}{2} \cdot D + \begin{cases} 2k-2 & \text{if } D \equiv 0 \pmod{4} \\ \frac{3}{2}k - \frac{1}{2} & \text{if } D \equiv 1 \pmod{4} \\ k+1 & \text{if } D \equiv 2 \pmod{4} \\ \frac{5}{2}k - \frac{7}{2} & \text{if } D \equiv 3 \pmod{4}. \end{cases}$$

Proof. The proof is similar to the proof of Theorem 7 and Corollary 8, starting with the f -sequence indicated below for $D = 8$:

$$f: 1, (k-1), 1, 1, \frac{k-1}{2}, \frac{k-1}{2}, (k-1), 1.$$

We leave the details to the reader. ■

Note that the upper bounds in Theorem 10 hold for any odd $k \geq 5$. In the case $D \equiv 2 \pmod{4}$, these bounds could be better than those in Corollary 9.

For fixed k , the upper bounds found so far make a relatively large jump when going from $D \equiv 2 \pmod{4}$ to $D \equiv 3 \pmod{4}$. This can be smoothed out by giving a better upper bound in the case $D \equiv 3 \pmod{4}$.

Theorem 11. *Let $k \geq 4$ and $D = 4q + 7$ with $q \geq 1$. Then,*

$$n_0(k, D) \leq \frac{k + e(k)}{2} \cdot D + \frac{k}{2} + 4\lceil \sqrt{k} \rceil - 2 - \frac{7}{2} e(k),$$

where $e(k)$

$$= \begin{cases} 0 & \text{if } k \text{ is a square,} \\ 1 & \text{if } k \text{ is not a square and } k \geq \lceil \sqrt{k} \rceil(\lceil \sqrt{k} \rceil - 1), \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Consider the following f -sequence: $1, (k-1), (\lceil \sqrt{k} \rceil - 1), 1, (\lceil \sqrt{k} \rceil - 1), (k + e(k) - 2\lceil \sqrt{k} \rceil + 1), \dots, (\lceil \sqrt{k} \rceil - 1), 1, (\lceil \sqrt{k} \rceil - 1), (k-1), 1$, where the subsequence f_3, f_4, f_5, f_6 is repeated q times.

If k is a square [so $e(k) = 0$], using Theorem 7 with $u = w = \sqrt{k} - 1$ and $v = 1$, $x = k + 1 - 2\sqrt{k}$, we obtain the result. If k is not a square and $k \geq \lceil \sqrt{k} \rceil(\lceil \sqrt{k} \rceil - 1)$ [so $e(k) = 1$], we are in a similar situation as in Theorem 10 (the sum of the elements $f_3, \dots, f_6$ is $k + 1$). Using Lemma 4, it is not difficult to complete the proof for this case. In the last case, using $k \geq (\lceil \sqrt{k} \rceil - 1)^2 + 1$ and Lemma 4, it is again not difficult to complete the proof. We leave the details to the reader. ■

5. EXACT VALUES OF $n_0(k, D)$

In this section, we first determine $n_0(k, D)$ for $D = 2, \dots, 7$ and $k \geq 2$.

Theorem 12. *Let $k \geq 2$.*

- (a) $n_0(k, 2) = 2k$,
- (b) $n_0(k, 3) = 2k + 2$,
- (c) $n_0(k, 4) = 4k$,

- (d) $n_0(k, 5) = 4k + 2$,
- (e) $n_0(k, 6) = 4k + 4$,
- (f) $n_0(k, 7) = 4k + 2\lceil 2\sqrt{k} \rceil - 2 (k \geq 3)$.

Proof.

- (a) The only $(k, 2)$ -graph is $K_{k,k}$.
- (b) Let V_i, a_i ($i = 0, 1, 2, 3$), and b_i ($i = 1, 2, 3$) be defined as in Section 2. Lemma 1(d) gives $a_0 + a_2 = a_1 + a_3$. Now, $a_3 \geq 1$; hence, $n_0(k, 3) \geq 2(a_1 + a_3) \geq 2k + 2$. The value $2k + 2$ is realized by the graph " $K_{k+1,k+1}$ minus a perfect matching."
- (c) From Theorems 5 and 6, we obtain $n_0(k, 4) \geq 4k$. This lower bound can easily be attained. When $k = 2$, we simply take C_8 ; when $k \geq 3$, let G be a $(k-1, 3)$ -graph on $2k$ vertices [cf. (b)]. Then, $K_2 \times G$ satisfies all conditions.
- (d) From Theorems 5 and 6, we obtain $n_0(k, 5) \geq 4k + 2$. The sequences $(a_0, \dots, a_5) = (1, k, k, k, k, 1)$ and $(b_1, \dots, b_5) = (k, k^2 - k, k, k^2 - k, k)$ satisfy the conditions of Lemma 4; hence, the value $4k + 2$ can be realized.
- (e) From Theorems 5 and 6, we obtain $n_0(k, 6) \geq 4k + 4$. The sequences $(a_0, \dots, a_6) = (1, k, k, 2, k, k, 1)$ and $(b_1, \dots, b_6) = (k, k^2 - k, k, k, k^2 - k, k)$ satisfy the conditions of Lemma 4; hence, the value $4k + 4$ can be realized.
- (f) From Theorems 5 and 6, we obtain $n_0(k, 7) \geq 4k + 2\lceil 2\sqrt{k} \rceil - 2$. The following constructions show that equality holds if $k \geq 3$. If $k > \lceil \sqrt{k} \rceil(\lceil \sqrt{k} \rceil - 1)$, consider the following a -sequence: $1, k, (k + \lceil \sqrt{k} \rceil - 2), \lceil \sqrt{k} \rceil, \lceil \sqrt{k} \rceil, (k + \lceil \sqrt{k} \rceil - 2), k, 1$, and b -sequence: $k, (k-1)k, (\lceil \sqrt{k} \rceil - 1)k, k, (\lceil \sqrt{k} \rceil - 1)k, (k-1)k, k$. If $k \leq \lceil \sqrt{k} \rceil(\lceil \sqrt{k} \rceil - 1)$, then consider the following a -sequence: $1, k, (k + \lceil \sqrt{k} \rceil - 2), \lceil \sqrt{k} \rceil, \lceil \sqrt{k} \rceil - 1, (k + \lceil \sqrt{k} \rceil - 3), k, 1$, and b -sequence: $k, (k-1)k, (\lceil \sqrt{k} \rceil - 1)k, k, (\lceil \sqrt{k} \rceil - 2)k, (k-1)k, k$. Except for $k = 2$, these sequences satisfy the conditions of Lemma 4. ■

Next, we determine $n_0(k, D)$ for $k = 2, 3, 4$.

Theorem 13.

- (a) $n_0(2, D) = 2D$ ($D \geq 2$),
- (b) $n_0(3, D) = 2D + 4$ ($D \geq 4$),
- (c) $n_0(4, D) = 2D + 8$ ($D \geq 4$).

Proof.

- (a) The cycles are the only connected 2-regular graphs.

- (b) For $D \in \{4, 5, 6, 7\}$, see Theorem 12. Let $D \geq 8$. If D is even, then $n_0 = 2(a_1 + a_3 + \dots + a_{D-1}) \geq 2 \cdot (3 + [(D-4)/2] \cdot 2 + 3)$ since $a_3, a_5, \dots, a_{D-3} \geq 2$. If D is odd, then $n_0 = 2(a_0 + a_2 + \dots + a_{D-1}) \geq 2 \cdot (1 + 3 + [(D-5)/2] \cdot 2 + 3)$ for similar reasons. Hence, $n_0(3, D) \geq 2D + 4$. Now choose $a_0 = a_D = 1$, $a_1 = a_2 = k$, $a_3 = \dots = a_{D-3} = 2$, $a_{D-2} = a_{D-1} = k$, and apply Lemma 4.
- (c) The proof is similar to the proof of (b). We leave the details to the reader. ■

From here on, we let $D \geq 8$ and $k \geq 5$.

Theorem 14.

- (a) If $D \equiv 0 \pmod{4}$, then

$$n_0(k, D) = \begin{cases} \frac{k}{2} \cdot D + 2k & \text{if } k \text{ is nonprime} \\ \frac{k+1}{2} \cdot D + 2k - 2 & \text{if } k \text{ is prime.} \end{cases}$$

- (b) If $D \equiv 1 \pmod{4}$, then

$$n_0(k, D)$$

$$= \begin{cases} \frac{k}{2} \cdot D + \frac{3}{2}k + 2 & \text{if } k \text{ is nonprime} \\ \frac{k+1}{2} \cdot D + \frac{3}{2}k - \frac{1}{2} & \text{if } k \text{ is prime.} \end{cases}$$

Proof. It turns out that for $D \equiv 0$ or $1 \pmod{4}$ the lower bounds of Theorems 5(a) and 6 coincide with the upper bounds of Corollaries 8 and 9 and Theorem 10. ■

For the other residue classes of $D \pmod{4}$, the situation is less satisfactory.

Theorem 15. If $D \equiv 2 \pmod{4}$, then

$$n_0(k, D) = \begin{cases} \frac{k}{2} \cdot D + k + 4 & \text{if } k \text{ is even} \\ \frac{k}{2} \cdot D + k + 6 & \text{if } k \text{ is odd and } 3 \nmid k \\ \frac{k+1}{2} \cdot D + k + 1 & \text{if } k \text{ is prime.} \end{cases}$$

Proof. For $D \equiv 2 \pmod{4}$, and the values of k indicated above, the lower bounds of Theorems 5(a) and (b) and 6 coincide with the upper bounds of Corollaries 8 and 9 and Theorem 10. ■

Theorem 16. If $D \equiv 3 \pmod{4}$, then

$$n_0(k, D) = \begin{cases} \frac{k}{2} \cdot D + \frac{k}{2} + 4\sqrt{k} - 2 & \text{if } k \text{ is a square} \\ 3D + 11 & \text{if } k = 6 \\ 4D + 10 & \text{if } k = 7 \\ 4D + 14 & \text{if } k = 8 \\ 5D + 17 & \text{if } k = 10. \end{cases}$$

Proof. When k is a square, the lower bound of Theorem 5(a) coincides with the upper bound of Theorem 11. For $k = 6, 8, 10$, compare Theorem 5(a) and Corollary 8. For $k = 7$, compare Theorems 6 and 11. ■

Remark. An isolated case for which we have determined $n_0(k, D)$ is $k = 5, D = 11$: The sequence $(a_0, \dots, a_{11}) = (1, 5, 5, 2, 3, 4, 4, 3, 2, 5, 5, 1)$ shows that $n_0(5, 11) \leq 40$, whereas from Theorem 6, we have $n_0(5, 11) \geq 40$. The minimal cases for which we have not determined $n_0(k, D)$ are $(k, D) = (5, 15), (11, 11), (25, 10)$.

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