The Connectivity of the Basis Graph of a Branching Greedoid

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ABSTRACT

A result of Korte and Lovász states that the basis graph of every 2-connected greedoid is connected. We prove that the basis graph of every 3-connected branching greedoid is $(\delta - 1)$ -connected, where δ is the minimum in-degree (disregarding the root) of the underlying rooted directed (multi)graph. We also give examples showing that this result is (in some sense) best possible.

1. INTRODUCTION

A result of Korte and Lovász [2] states that the basis graph of every 2-connected greedoid is connected. This also implies that every spanning tree of a 2-connected rooted graph can be changed into a given second one by successively exchanging a single leaf in every step. As part of a general study on the diameter of basis graphs for different classes of greedoids, Ziegler [3] showed that the number of exchanges of leaves in the above procedure can be bounded above by $r^2 - r + 1$, where r is the rank of the greedoid. A natural problem is whether there are a number of completely different (independent) ways to change one spanning tree into another. This problem asks for information on the connectivity of the basis graph. In this paper we prove that the basis graph of a 3-connected branching greedoid is $(\delta - 1)$ -connected, where δ is the minimum in-degree (disregarding the root) of the underlying rooted directed (multi)graph. We give examples to show that this lower bound cannot be improved.

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2. TERMINOLOGY

We refer to [1] for terminology and results on greedoids, but repeat the definitions for the notions we use in this paper.

A *greedoid* on a finite set E is a pair $(E, \mathcal{F})$, where $\mathcal{F}$ is a nonempty collection of subsets of E satisfying:

- (1) $\emptyset \in \mathcal{F}$.
- (2) For every nonempty $X \in \mathcal{F}$ there is an $x \in X$ such that $X \setminus \{x\} \in \mathcal{F}$.
- (3) For $X, Y \in \mathcal{F}$ such that $|X| > |Y|$, there is an $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{F}$.

The sets in $\mathcal{F}$ are called *feasible*. Thus a greedoid is a matroid if and only if every subset of a feasible set is again feasible. The maximal feasible sets of a greedoid are called its *bases*. Like in the case of matroids, (3) implies that all bases of a greedoid have the same cardinality, which we call the *rank* of the greedoid. Two bases X and Y of a greedoid $(E, \mathcal{F})$ are called *adjacent* if $|X \cap Y| = |X| - 1$ and $X \cap Y \in \mathcal{F}$, that is, if X and Y differ in exactly one element and their intersection is feasible. This gives rise to the *basis graph* $G(\Delta)$ of a greedoid Δ on the set of bases of Δ , whose edges represent pairs of adjacent bases. If $v, u_1, u_2, \dots, u_n$ are distinct vertices of $G(\Delta)$, then a *fan* between v and $\{u_1, u_2, \dots, u_n\}$, denoted by $v - \{u_1, u_2, \dots, u_n\}$, is a set of n paths having only v in common and connecting v to each vertex of $\{u_1, u_2, \dots, u_n\}$.

With $D = (V, E, r)$ we denote a finite rooted directed multigraph with vertex set V , arc set E , and a specified vertex $r \in V$, which we call the *root* of D . We define the *in-degree* of a vertex $v \in V$ as the number of nonparallel in-arcs of v . Let $\mathcal{F}$ be the collection of arc sets of trees of D that contain r and are directed away from r (each vertex $v \neq r$ of such a tree has one arc of the tree coming in). Such trees are called *branchings* or *out-trees*. Then $\Delta = (E, \mathcal{F})$ is a greedoid, which we call the *branching greedoid* (of D). The *minimum degree* of D , denoted by $\delta(D)$ or δ , is the minimum in-degree taken over all $v \in V \setminus \{r\}$. If every vertex of D is at distance at most 1 from r , we say that D is *starlike*. Two or more (directed) paths are called (*internally*) *disjoint* if they have at most their end vertices in common. Let k be an integer and $k \geq 2$. Assuming that D is not starlike and that D admits a spanning out-tree ($\mathcal{F}$ contains a basis with cardinality $|V| - 1$) we say that D and, equivalently, Δ is *k-connected* if no vertex v of D can be separated from r by removing less than k vertices different from r and v , or equivalently, if there are k disjoint directed paths from r to any other vertex not adjacent to r by an arc directed away from r . If D is starlike, we define the connectivity of Δ equal to $|V| - 1$ (the rank of Δ). It is easy to check that this definition coincides with the definition of the connectivity of a general greedoid given in [3, Definition 2.3']. If X is a basis of a branching greedoid Δ and v is a vertex of the corresponding out-tree with out-degree zero, then v is called a *leaf* of X ; if uv is an arc of this out-tree, then uv is called a *leaf arc* of X .

3. RESULTS

Theorem 1. Let Δ be a 3-connected branching greedoid of a rooted directed multigraph $D = (V, E, r)$. Then the basis graph $G(\Delta)$ of Δ is $(\delta - 1)$ -connected.

To prove Theorem 1 we use the following lemmas, the first of which is [2, Corollary 3.3], and the second of which follows immediately from the definition of δ .

Lemma 2. The basis graph of a 2-connected branching greedoid is connected.

Lemma 3. Let Δ be a 2-connected branching greedoid of $D = (V, E, r)$ and let $v \in V \setminus \{r\}$. Then at least δ bases of Δ contain v as a leaf.

Proof. The branching greedoid Δ_v of $D - v$ is connected and hence has a basis X corresponding to a spanning out-tree of $D - v$. Since there are at least δ arcs $e_1, e_2, \dots, e_\delta$ of D ending at v , $X \cup \{e_1\}, X \cup \{e_2\}, \dots, X \cup \{e_\delta\}$ are δ bases of Δ that all contain v as a leaf. ■

Lemma 4. Let Δ be a 3-connected branching greedoid of $D = (V, E, r)$ and let $v \in V \setminus \{r\}$. Then the set $B_v := \{X \mid X \text{ is a basis of } \Delta \text{ with leaf } v\}$ induces a $(\delta - 1)$ -connected subgraph of $G(\Delta)$.

Proof. Let $e_1, e_2, \dots, e_k$ be the arcs of D ending at v and starting at $v_1, v_2, \dots, v_k$. Clearly $k \geq \delta$ and the branching greedoid Δ_v of $D - v$ is 2-connected. By Lemma 2, $G(\Delta_v)$ is connected. Now the following observations are obvious.

- (4) For any basis X of $G(\Delta_v)$, the set of bases $\{X \cup \{e_1\}, X \cup \{e_2\}, \dots, X \cup \{e_k\}\} \subseteq B_v$ induces a complete graph in $G(\Delta)$.
- (5) If, for two bases X and Y of $G(\Delta_v)$, the edge XY corresponds to an exchange of two leaf arcs ending at a common leaf w of X and Y , then $X \cup \{e_j\}$ is adjacent to $Y \cup \{e_j\}$ in $G(\Delta)$ whenever $w \neq v_j$ ($j = 1, \dots, k$).

Hence the subgraph H of $G(\Delta)$ induced by B_v arises from $G(\Delta_v)$ by replacing each vertex by a k -clique, and each edge by $k - 1$ independent edges according to (4) and (5). It is clear that, for any set $S \subseteq V(H)$ with $|S| < \delta - 1 \leq k - 1$, $H - S$ is connected. Hence H is $(\delta - 1)$ -connected. ■

Lemma 5. Let $D = (V, E, r)$ be 3-connected and let X be a basis of the branching greedoid Δ of D . Then for any $v \in V \setminus \{r\}$ there are $\delta - 1$ bases $X_1, X_2, \dots, X_{\delta-1}$ of Δ such that X_i ($i = 1, 2, \dots, \delta - 1$) contains v as a leaf and there is a fan $X - \{X_1, X_2, \dots, X_{\delta-1}\}$ in $G(\Delta)$.

Proof. We distinguish two cases.

Case 1. v is a leaf of X .

Let e be the leaf arc of X ending at v . There are at least $\delta - 1$ arcs $e_1, e_2, \dots, e_{\delta-1}$ ending at v and different from e . Now $X_i = (X \cup \{e_i\}) \setminus \{e\}$ ($i = 1, 2, \dots, \delta - 1$) is a basis of Δ containing v as a leaf, and X is adjacent to X_i ($i = 1, 2, \dots, \delta - 1$). Hence there is a fan $X - \{X_1, X_2, \dots, X_{\delta-1}\}$.

Case 2. v is not a leaf of X .

Let e be a leaf arc of X ending at u . Consider the branching greedoid Δ_u of $D - u$. Clearly, Δ_u is 2-connected, and $X \setminus \{e\}$ is a basis of Δ_u . Since v is a vertex of $D - u$, by Lemma 3 there are at least $\delta - 1$ bases $X'_1, X'_2, \dots, X'_{\delta-1}$ of Δ_u with common leaf v . Let $X_i := X'_i \cup \{f\}$ for some arc f ending at u ($i = 1, 2, \dots, \delta - 1$). By Lemma 4, $X, X_1, X_2, \dots, X_{\delta-1}$ are vertices of a $(\delta - 1)$ -connected subgraph of $G(\Delta)$, implying that $G(\Delta)$ contains a fan $X - \{X_1, X_2, \dots, X_{\delta-1}\}$. ■

Proof of Theorem 1. Let X and Y be two bases of Δ . It is sufficient to prove that there are at least $\delta - 1$ disjoint paths in $G(\Delta)$ connecting X and Y .

Denote by $L(X)$ and $L(Y)$ the set of leafs of X and Y , respectively.

If $L(X) \cap L(Y) \neq \emptyset$, the result follows from Lemma 4.

Suppose $L(X) \cap L(Y) = \emptyset$ and let $v \in L(Y)$. By Lemma 5, there is a fan $X - (X_1, X_2, \dots, X_{\delta-1})$ in $G(\Delta)$ such that v is a leaf of X_i ($i = 1, 2, \dots, \delta - 1$).

By Lemma 4, there are $\delta - 1$ disjoint paths from X_i to Y ($i = 1, 2, \dots, \delta - 1$).

Letting i run through $1, 2, \dots, \delta - 1$, we obtain a collection of paths connecting X and Y from which we can choose $\delta - 1$ disjoint paths connecting X and Y . This can be shown as follows. If we delete a set of $t \leq \delta - 2$ vertices from $G(\Delta)$ not containing X or Y , then at least one path connecting X and X_i remains and at least one of the paths connecting X_i and Y remains.

This completes the proof of Theorem 1. ■

The following result is an immediate consequence of Theorem 1.

Corollary 6. Let Δ be a 3-connected branching greedoid and suppose, for some $n_1, n_2 \in \mathbb{N}$ with $n_1 < n_2$, there are bases X_1 and X_2 of Δ with n_1 and n_2 leaves, respectively. Then, for any $n \in \mathbb{N}$ with $n_1 < n < n_2$, there are at least $\delta - 1$ bases of Δ with n leaves.

If X_1 and X_2 in Corollary 6 have a common leaf v , then all $\delta - 1$ bases can be chosen to have v as a leaf. This follows from Lemma 4.

4. CONCLUDING REMARKS

Let $\mathcal{H}$ be the class of strongly δ -connected directed graphs that contain a directed Hamilton path (a directed path containing all vertices of the graph). Let $\mathcal{G}$ be the class of rooted directed graphs obtained from members of $\mathcal{H}$ in the following way. To every $H \in \mathcal{H}$ add a root r and another new vertex v . Also add arcs from r to all vertices of H , and arcs ending at v and starting from precisely δ vertices of H including the end vertex of a directed Hamilton path of H .

Let $G \in \mathcal{G}$ be obtained from $H \in \mathcal{H}$ in this way. It is clear that G is non-starlike with $\delta(G) = \delta$ and that G contains a directed Hamilton path P from r to v . Let X be the set of arcs of P . Then X is a basis of the branching greedoid Δ of G and there are precisely $\delta - 1$ bases of Δ adjacent to X (they only differ in the arc ending at v). Hence the basis graph $G(\Delta)$ of Δ is (at most) $(\delta - 1)$ -connected, showing that we cannot replace “ $(\delta - 1)$ -connected” in Theorem 1 by “ t -connected” for $t \geq \delta$. In this sense, Theorem 1 is best possible.

If r is a unique cut vertex of $D = (V, E, r)$, we can do better by considering the subgraphs of D induced by r and the vertices of each of the components of $D - r$. Let $D_1, D_2, \dots, D_m$ denote these subgraphs. Then it is easy to see that $G(\Delta)$ is isomorphic to the (Cartesian) product graph of $G(\Delta_1), G(\Delta_2), \dots, G(\Delta_m)$, where $\Delta, \Delta_1, \dots, \Delta_m$ are the branching greedoids of $D, D_1, \dots, D_m$, respectively. This implies that $G(\Delta)$ is at least $(k_1 + k_2 + \dots + k_m)$ -connected if $G(\Delta_i)$ is at least k_i -connected ($i = 1, 2, \dots, m$).

Finally, let v be a vertex of D at distance 1 from the root r . If v has only one in-arc, we can contract this arc (r, v) to obtain a new digraph D' . It is easy to see that the basis graph of the branching greedoid of D is isomorphic to that of D' . Therefore, we may always assume that every vertex of D at distance 1 from the root r has at least two in-arcs.

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