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Properties of recoverable region and semi-global stabilization in recoverable region for linear systems subject to constraints[☆]

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Abstract

This paper investigates linear systems subject to input and state constraints. It is shown that the recoverable region (which is the largest domain of attraction that is theoretically achievable) can be semiglobally stabilized by *continuous* nonlinear feedbacks while satisfying the constraints. Moreover, when trying to compute the recoverable region, a reduction technique shows that we only need to compute the recoverable region for a system of lower dimension which generally leads to a considerable simplification in the computational effort.

Keywords: Linear systems; State constraints; Input constraints; Stabilization; Recoverable region

1. Introduction

In this paper, we revisit the problem of stabilization of general linear time-invariant systems subject to input and state constraints. Over the past years there has been rather strong interest (see for instance, [Bernstein & Michel, 1995; Kapila & Grigoriadis, 2002; Saberi & Stoorvogel, 1999; Tarbouriech & Garcia, 1997]) in this problem, possibly due to a wide recognition of the inherent constraints on the input and state in most practical control systems. A result due to Sontag and Sussmann (1990) shows that, for linear stabilizable systems, only systems which have no open-loop poles with positive real parts can be globally asymptotically stabilized by a bounded feedback. However, global stabilization in general requires a nonlinear controller as was established first by Fuller (1969) and more recently by Sussmann and Yang (1991). Later, it is shown in Lin and Saberi (1993) that systems which are globally stabilizable by nonlinear control laws are semi-globally stabilizable by *linear* control laws. It is easily established that for systems having open-loop poles with positive real parts, global or semi-global stabilization with constrained input is impossible.

More recently, the global and semi-global stabilization results for input constraints are extended to linear systems with state and input constraints in Saberi, Han, and Stoorvogel (2002), where global and semi-global stabilization are defined relative to the admissible set. The admissible set is defined as the set of initial conditions that do not violate the constraints at time 0. It turns out that invariant zeros, infinite zeros and right-invertibility properties play a crucial role. In Saberi et al. (2002) these invariant zeros and infinite zeros are labeled as constraint invariant zeros and constraint infinite zeros. For systems with right invertible constraints, it is shown that the necessary conditions for global and semi-global stabilization are that the system is stabilizable and the constrained invariant zeros are in the closed left-half plane. Moreover, for global stabilization one needs an additional condition that the constrained infinite zeros are of order less than or equal to one. For constraints

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that are right invertible and at most weakly non-minimum phase, it is possible to achieve semi-global stabilization by a linear control law; however, in general one has to use nonlinear control laws for global stabilization. For the case of non-right invertible constraints, the complete development of necessary and sufficient conditions for semi-global, global stabilization, and output regulation turns out to be a very complex and challenging problem that is yet to be resolved. If a system has at least one of the constraint invariant zeros in the open right-half plane, a so-called non-minimum phase constraint, then neither semi-global nor global stabilization in the admissible set is possible.

The notion of *recoverable region* (*set*), sometimes called domain of null controllability or null controllable region, is closely related to the stabilization of linear systems subject to constraints. Generally speaking, for a system with constraints, an initial state is said to be *recoverable* if it can be driven to zero by some control without violating the constraints on the state or input. The set of all recoverable initial conditions denoted by $\mathscr{R}_C$ is said to be the recoverable region. The recoverable region is closely related to the stabilization problem for it represents the maximum achievable constrained domain of attraction, which is defined as the set of all initial condition which can be made to converge to the equilibrium point without violating constraints. As such, the goal of stabilization is to design a feedback such that the constrained domain of attraction of the equilibrium point of the closed loop system is equal or close to the recoverable region. The earliest literature in this respect can be traced back to the 1960s. For the case of input constraints, J. L. LeMay in 1964 first studied the conditions for characterizing the *maximal region of recoverability* and the *maximal region of reachability* (LeMay, 1964). LeMay also derived a method for calculation of recoverable regions based on optimal control techniques. It is known that for any state in the recoverable region there exists a time-optimal control law that drives the state to zero. This fact builds a direct connection between the characterization of the recoverable region and time-optimal control. There exists a vast literature in the 1960s and 1970s that were devoted to time-optimal control, among them we mention Flügge-Lotz (1968), Fulks (1970), Lee and Markus (1967), Pontryagin, Boltyanskii, Gamkrelidze, and Mischenko (1962). The book (Ryan, 1982) presented a set of very detailed results of time-optimal control of systems with input constraints whose number of unstable eigenvalues is between 1 and 4. It also provided some results for explicit characterization of the recoverable region, including

- Systems with one or two unstable real eigenvalues;
- Systems with two unstable complex eigenvalues;
- Systems with three unstable eigenvalues which are proportional: (λ, 2λ, 3λ), where $\lambda > 0$;
- Some systems with four unstable poles can be reduced to systems with lower order unstable dynamics.

Note that the above crucially depends on the fact that in the case of only input constraints the recoverable region is completely determined by the unstable dynamics. More recently in 1995, Stephan et al. extended some of LeMay's results to systems with input and state constraints (Stephan, Bodson, & Lehoczky, 1995,1998). They examined computational issues of the recoverable regions for planar systems with state and input constraints.

There are two lines of research in the literature on stabilization problems in the presence of non-minimum phase constraints. A traditional line employs the construction of invariant sets. A common denominator in the stream of literature taking this approach is the idea of seeking a control law that does not violate the constraints posed on actuators and at the same time makes a subset of the admissible set invariant. Subsets of the admissible set which can be made invariant in this way are called positive invariant sets. Two candidate positively invariant sets widely used in the literature are *ellipsoidal sets* and *polyhedral sets*. Ellipsoidal sets are classical in control theory. It has been shown in Hu, Lin, and Shamash (2001a) for the case of input constraints that we can only approximate the recoverable set arbitrarily well by a finite number of ellipsoidal sets but computationally this is very demanding. More recently, polyhedral sets have received great attention, see for example Bitsoris (1988), Blanchini and Miani (1996), Blanchini (1998), Cwikel and Gutman (1986), Vassilaki, Hennet, and Bitsoris (1988) among others. In principle polyhedral sets are not intrinsically conservative but this might require an exponential growth in the number of edges with the related exponential growth in the required numerical effort. For a detailed perspective in this line of research, the reader should consult the excellent review in Blanchini (1999). Further information in this regard can be found in two survey papers (Dontchev & Lempio, 1992; Gayek, 1991).

The second line of research takes a fundamental view of global and semi-global stabilization relative to the recoverable region. In global stabilization problem one would seek a stabilizing feedback law that achieves a constrained domain of attraction for the equilibrium point of the closed loop system that is equal to the recoverable region. The semi-global stabilization problem deals with the issue of designing a family of stabilizing feedback laws such that, for any a priori given set, a member among the family of stabilizing feedback laws achieves a constrained domain of attraction for the equilibrium point of the closed loop system that contains the given set. The literature on this line of research, with the exception of Saberi et al. (2002), has only focused on input constraints. Moreover, no results are yet available for global stabilization in the presence of non-minimum phase constraints. Note that, in the case of input constraints only, the presence of non-minimum phase constraints is equivalent to existence of open right-half plane poles (i.e., exponentially unstable open-loop systems). For semi-global stabilization problem, Choi (1999) showed that for exponentially unstable *discrete-time* linear systems subject to input constraints any compact subset of the maximal recoverable region can be exponentially stabilized via a periodic linear variable structure controller. However, Choi (2001) showed that in general linear feedback can not achieve global stabilization for discrete-time unstable systems. Also, Hu, Lin, and Qiu (2001b) studied the possibility of semi-global stabilization of continuous-time systems with two unstable open-loop poles. It should be emphasized that all the mentioned works deal only with the case when the constraints are posed on the inputs. Recently, in a more general setting including input and state constraints, Saberi et al. (2002) have provided solvability conditions for semi-global stabilization in the admissible set of systems subject to right-invertible and non-minimum phase constraints.

In this paper, we focus on two issues. The first issue is properties and computational issues for the recoverable region. Our goal is to provide a reduction in computation and removal of some of the computational complexity involved in obtaining the recoverable set. The second issue is semi-global stabilization via continuous state feedbacks in the recoverable region. In the special case that the constraints are right invertible, these questions were addressed in Saberi et al. (2002).

This paper is organized as follows. After the introduction we present some preliminary results in Section 2. In Section 3 we discuss the issues related to computing the recoverable region and present a reduction technique which allows us to reduce the computational effort by developing an explicit relationship between the recoverable region of the full system and the recoverable region of a subsystem of lower order. In Section 4 we establish that for any compact set contained in the interior of the recoverable region, there exists a continuous controller that stabilizes the system and contains the chosen compact set in its domain of attraction while satisfying the constraints.

Notation: For any set $\mathscr{C} \subset \mathbb{R}^n$, $\operatorname{int}\mathscr{C}$ denotes the interior of set $\mathscr{C}$, $\partial\mathscr{C}$ the boundary of set $\mathscr{C}$, and $\overline{\mathscr{C}}$ the closure of set $\mathscr{C}$.

2. Preliminaries

This section provides the fundamentals for our development. We start with a description of our system model and its constraints. Then we introduce some basic notions that we are interested in. After that we recall a taxonomy of constraints related to the constrained system Σ in (1). This taxonomy provides us some basic terminology for the rest of the paper.

Consider the time-invariant linear system

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu, \\ z = C_z x + D_z u, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $z \in \mathbb{R}^p$ is the constrained output which is subject to the constraint $z(t) \in \mathscr{S}$ for all $t \geqslant 0$, where $\mathscr{S}$ is a given subset of $\mathbb{R}^p$. Note that the case of input constraints is included as a special case in this general setup by letting $C_z = 0$ and $D_z = I$ in the constrained output equation. However, one should note the difference between input saturation and input constraints: a saturation can be overloaded, whereas, a constraint should never be violated.

We make a general assumption on the constraint set $\mathscr{S}$ and the structure of the constrained output.

Assumption 1. *The set $\mathscr{S}$ is compact, convex and contains* 0 *as an interior point. Moreover, we assume $C_z^T D_z = 0$ and*

$$\mathscr{S} = (\mathscr{S} \cap \operatorname{im} C_z) + (\mathscr{S} \cap \operatorname{im} D_z). \tag{2}$$

This assumption is satisfied in many cases. In fact, it is a general reflection of the separability of input constraints and state constraints. If the initial state of the system is arbitrary then, given the constraint on the output, constraint violation can never be avoided. For this reason, we need to define an admissible set of initial conditions.

Definition 2. Given the system Σ in (1) and a constraint set $\mathscr{S}$ satisfying Assumption 1, the set

$$\mathscr{A}(\Sigma, \mathscr{S}) := \{x \in \mathbb{R}^n | C_z x \in \mathscr{S}\},$$

is said to be the admissible set of initial conditions.

Remark 3. In view of Assumption 1, we observe that $C_z x + D_z u \in \mathscr{S}$ implies $C_z x \in \mathscr{S}$. Therefore, if the state is not in the admissible set then constraint violation is unavoidable.

Remark 4. If $\mathscr{S}$ is a polytope described by all $z \in \mathbb{R}^p$ for which $Rz \leqslant q$ with R and q a given matrix and vector, respectively, then clearly the admissible set $\mathscr{A}(\Sigma, \mathscr{S})$ is given by all $x \in \mathbb{R}^n$ for which $RC_z x \leqslant q$. Note that in connection with polytopes, the inequality "$\leqslant$" is always interpreted componentwise.

Definition 5. Given system Σ in (1) together with a constraint set $\mathscr{S}$ satisfying Assumption 1. The recoverable region $\mathscr{R}_C(\Sigma, \mathscr{S})$ of this system is the set of all initial states $x(0) \in \mathscr{A}(\Sigma, \mathscr{S})$ for which there exists a control input u such that $x(t) \to 0$ as $t \to \infty$ while $z(t) \in \mathscr{S}$ for all $t \geqslant 0$.

2.1. Taxonomy of constraints

We review briefly, the taxonomy of constraints for the system Σ, given by (1), which has emerged from the study of stabilization of such systems (Saberi et al., 2002). It is known that structural properties of this system play important roles in the solvability of certain constrained stabilization problems. Specifically, right invertibility, the location of invariant zeros, and the order of infinite zeros of the quadruple (A, B, C_z, D_z) dictate the solvability conditions for some constrained stabilization problems. The taxonomy of the constraints is based on these structural properties.

The first category in the taxonomy of constraints is based on whether the system Σ is right invertible or not.

Definition 6. The constraints are said to be right invertible constraints if the system Σ is right invertible and non-right invertible constraints if the system Σ is non-right invertible.

The second category of constraints is based on the location of the invariant zeros of the system Σ, which are labeled the *constraint invariant zeros* of the plant. In the following definition, $\mathbb{C}^-$, $\mathbb{C}^0$, and $\mathbb{C}^+$ denote respectively the set of complex numbers with negative real part, zero real part, and positive real part.

Definition 7. The constraints are said to be

- *minimum phase constraints* if all the constraint invariant zeros are in $\mathbb{C}^-$.
- *at most weakly non-minimum phase constraints* if all the constraint invariant zeros are in $\mathbb{C}^- \cup \mathbb{C}^0$.
- *strongly non-minimum phase constraints* if one or more of the constraint invariant zeros are in $\mathbb{C}^+$.

The third categorization is based on the order of the infinite zeros of the system Σ, which are labeled as the constraint infinite zeros of the plant.

Definition 8. The constraints are said to be *type one constraints* if the order of all constraint infinite zeros is less than or equal to one.

3. Properties and computational issues of the recoverable region

This section is devoted to some characterizations of the recoverable region $\mathscr{R}_C(\Sigma,\mathscr{S})$ of system Σ as defined in Definition 5. The first set of properties of the recoverable region $\mathscr{R}_C(\Sigma,\mathscr{S})$ are more or less well known. They are compiled in the following lemma for easy reference.

Lemma 9. *Consider system Σ in (1) and a compact, convex constraint set $\mathscr{S}$ containing 0 in the interior. The recoverable region $\mathscr{R}_C(\Sigma,\mathscr{S})$ for this system has the following properties*:

(i) *If (A,B) is controllable, then for any initial $x(0)\in\mathscr{R}_C(\Sigma,\mathscr{S})$ there exists $T>0$ and an input signal u such that $x(T)=0$ while $z(t)\in\mathscr{S}$ for all $t\in[0,T]$.*
(ii) *The set $\mathscr{R}_C(\Sigma,\mathscr{S})$ is convex and contains the origin as an interior point.*
(iii) *If (A,B) is stabilizable, then the set $\mathscr{R}_C(\Sigma,\mathscr{S})$ is open in case we have only input constraints, i.e. $C_z=0$, but in general this need not be true.*
(iv) *The set $\mathscr{R}_C(\Sigma,\mathscr{S})$ is bounded if all the invariant zeros of the system (1) are in the open right half plane, the system is left invertible and the constraints are of type one.*

Proof. See Appendix B. □

Remark 10. Note that item (i) of the above lemma states that infinite-time recoverability is equivalent to finite-time recoverability.

Remark 11. As is clear from the example in Section 5, the recoverable region is in general not a polytope. Of course, like any set, it can be arbitrarily well approximated by a polytope.

Remark 12. Assume that $C_z=0$ and $D_z=I$ in (1), i.e. the system is only subject to input constraints and without state constraints. Then, under a suitable coordinate system in the state space, the plant can be split into two subsystems:

$$\Sigma_s\colon \dot{x}_s = A_s x_s + B_s u,$$

$$\Sigma_u\colon \dot{x}_u = A_u x_u + B_u u,$$

where the eigenvalues of A_s are in the closed left-half plane (at most critically unstable) and those of A_u are in the open right-half plane (antistable). Then it was already established by LeMay (1964) that

(i) $\mathscr{R}_C(\Sigma_u,\mathscr{S})$ is bounded;
(ii) $x\in\mathscr{R}_C(\Sigma,\mathscr{S})$ if and only if $x_u\in\mathscr{R}_C(\Sigma_u,\mathscr{S})$.

The fact stated above tells us that, without state constraints, the recoverable region is completely determined by the exponentially unstable part of the system. On the other hand, for the case of state constraints, this decomposition is no longer possible. Later in this section we show that, in general, a different type of order reduction is possible of which the above is actually a special case.

Next, we present our first reduction result for the set $\mathscr{R}_C(\Sigma,\mathscr{S})$. In order to do so first we represent Σ in a special coordinate basis (scb). A brief review of scb is presented in Appendix A and we obtain the system (A.1) which is in the scb form.

We can extract a subsystem from the full system in scb consisting of the state variables x_a and x_b, input variable ζ consisting of z_0 and z_d and output $\bar{z}$:

$$\Sigma_1\colon \begin{cases} \dot{x}_a = A_a x_a + K_{ab}C_b x_b + K_{a2}\zeta, \\ \dot{x}_b = (A_b + K_{bb}C_b)x_b + K_{b2}\zeta, \\ \bar{z} = \begin{pmatrix} C_b \\ 0 \end{pmatrix} x_b + \begin{pmatrix} 0 \\ I \end{pmatrix}\zeta. \end{cases} \tag{3}$$

The state dimension of this system equals n_a+n_b. Obviously ζ is not an input for the original system. However, for the moment we view ζ as the input to this subsystem while $\bar{z}$ is a constrained output for this subsystem.

A transformation of the system into scb clearly affects the constraint set and we obtain a new constraint set $\mathscr{S}_z = T_z^{-1}\mathscr{S}$. Thus, the constraint on $\bar{z}$ becomes

$$\bar{z}(t) \in \mathscr{S}_z \quad \text{for all } t \geqslant 0.$$

Let $\mathscr{R}_C(\Sigma_1, \mathscr{S}_z)$ be the recoverable region of subsystem Σ_1 with the constraint set $\mathscr{S}_z$. The following theorem shows the relationship between the recoverable region of the full system Σ and the recoverable region of the subsystem Σ_1.

Theorem 13. *Consider the plant Σ as given by* (1) *and a constraint set $\mathscr{S}$ satisfying Assumption* 1. *Assume that we have extracted the subsystem Σ_1 in* (3) from Σ *as described above. Then the closure of the set $\mathscr{R}_C(\Sigma, \mathscr{S})$ is equal to*

$$T_x \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \, x_1 \in \overline{\mathscr{R}_C(\Sigma_1, \mathscr{S}_z)} \right\} \cap \mathscr{A}(\Sigma, \mathscr{S}), \tag{4}$$

where x_2 is of compatible dimension, i.e. $x_2 \in \mathbb{R}^{n_c+n_d}$.

In (4), x_1 is the state of Σ_1 consisting of x_a and x_b while x_2 is the rest of the state variables of the system Σ which in scb is composed of x_c and x_d.

Remark 14. If $\mathscr{R}_C(\Sigma_1, \mathscr{S}_z)$ is approximated by a polytope consisting of all $x \in \mathbb{R}^{n_a+n_b}$ for which $R_1 x \leqslant q_1$ while $\mathscr{S}$ is described by all $z \in \mathbb{R}^p$ for which $Rz \leqslant q$ then $\mathscr{R}_C(\Sigma, \mathscr{S})$ is approximated by the set of all $x \in \mathbb{R}^n$ for which

$$\begin{pmatrix} RC_z \\ (R_1 \;\; 0)T_x^{-1} \end{pmatrix} x \leqslant \begin{pmatrix} q \\ q_1 \end{pmatrix}.$$

By improving the approximation of $\mathscr{R}_C(\Sigma_1, \mathscr{S}_z)$ we can approximate $\mathscr{R}_C(\Sigma, \mathscr{S})$ arbitrarily well in this way.

The decomposition of the recoverable region as presented in Theorem 13 is therefore very important from a computational point of view. Although it does not capture which boundary points of the recoverable set actually are part of the recoverable set itself, by approximating or exactly computing the set $\mathscr{R}_C(\Sigma_1, \mathscr{S}_z)$, we immediately obtain with arbitrary accuracy the set $\mathscr{R}_C(\Sigma, \mathscr{S})$.

As pointed out by Stephan et al. (1998), numerical computation of recoverable regions suffers from dimension growth. Papers such as (Yfoulis, Muir, & Wellstead, 2002) try to improve the gridding methods but the exponential growth with dimension is not avoided. In this sense, any reduction of dimension in the computation of the recoverable region is crucial for improvement of computation efficiency. The above method allows us to obtain the recoverable set for the system Σ from the recoverable set of a lower-dimensional system in a transparent way. Note that the transformation in scb and the computation of the transformation matrices (in particular T_x) has been implemented in Matlab and works very well on numerous examples.

Proof. It is obvious that $\mathscr{R}_C(\Sigma, \mathscr{S})$ is contained in $\mathscr{A}(\Sigma, \mathscr{S})$. Moreover, assuming that in the first subsystem we have ζ as a free input we clearly enlarge the recoverable set. The reverse inclusion follows from the proof of Theorem 18 since there we prove that for any compact set contained in the interior of (4), we can find a controller which contains this compact set in its constrained domain of attraction. □

Let us next have a different look at the structure of the system Σ which will provide some interesting results for special cases. To do so we need to define another subsystem. Consider the remaining dynamics in the system Σ besides the subsystem Σ_1. We consider the system in the special coordinate basis and we get the following description for the dynamics which together with Σ_1 describe the full system:

$$\Sigma_2\colon \begin{cases} \dot{x}_c = A_c x_c + K_{c2}\zeta + B_c u_c + K_{cb} z_b, \\ \dot{x}_d = A_d x_d + K_{d2}\zeta + B_d u_d + K_{db} z_b, \\ \zeta = \begin{pmatrix} 0 \\ C_d \end{pmatrix} x_d + \begin{pmatrix} I \\ 0 \end{pmatrix} u_0. \end{cases} \tag{5}$$

Note that Σ_2 is only affected by Σ_1 via the signal z_b. When we set $z_b = 0$ then we decouple Σ_2 from Σ_1 and when we also ignore the constraints on z_b by setting

$$\mathscr{S}_2 := \left\{ \zeta \in \mathbb{R}^{n_2} \,\middle|\, \exists z_b \text{ such that } \begin{pmatrix} z_b \\ \zeta \end{pmatrix} \in \mathscr{S}_z \right\},$$

and view $\mathscr{S}_2$ as the constraint set for Σ_2, we obtain an independent system Σ_2. In this way we define the recoverable region $\mathscr{R}_C(\Sigma_2, \mathscr{S}_2)$ for the second subsystem.

The following lemma establishes the recoverable set of the second subsystem Σ_2 and shows conditions under which we can completely characterize the recoverable set of the original set from the recoverable set of the subsystem Σ_1. Theorem 13 did not capture which boundary points belong to the recoverable set and the following theorem does this explicitly for a special case.

Theorem 15. *Consider the plant Σ as given by* (1) *and a constraint set $\mathscr{S}$ satisfying Assumption* 1. *Assume that system Σ has been decomposed into two subsystems in scb as described by* (3) *and* (5). *Then we have the following properties*:

(i) *It holds that $\overline{\mathscr{R}_C(\Sigma_2, \mathscr{S}_2)} = \mathscr{A}(\Sigma_2, \mathscr{S}_2)$.*
(ii) *If the constraints are right invertible, then $\mathscr{S}_2 = \mathscr{S}_z$.*
(iii) *If the constraints are right invertible and of type one, then $\mathscr{R}_C(\Sigma_2, \mathscr{S}_2) = \mathscr{A}(\Sigma_2, \mathscr{S}_2)$, and*

$$\mathscr{R}_C(\Sigma, \mathscr{S}) = T_x[\mathscr{R}_C(\Sigma_1, \mathscr{S}_z) \times \mathscr{A}(\Sigma_2, \mathscr{S}_2)]. \tag{6}$$

Proof. The first property is evident from the fact that the system Σ_2 has a special structure as constructed within the scb. It is strongly controllable which yields that we can make $\bar{z}$ follow any trajectory with arbitrary accuracy and therefore any initial state that is admissible at time 0 can be steered to zero without violating any constraints. For details we refer to the semi-global stabilization result in Saberi et al. (2002). The last properties also follow from this paper.

We have achieved a reduction from computing the recoverable region for the system Σ to the computation of the recoverable region for the subsystem Σ_1. As noted before a reduction in system order is crucial in making the computation of the recoverable region feasible. The question remains whether we can achieve further reductions. In scb the matrix A_a is in fact a block diagonal matrix. With this one more step refining, subsystem Σ_1 becomes:

$$\Sigma_1 : \begin{cases} \dot{x}_a^{-0} = A_a^{-0} x_a^{-0} + K_{ab}^{-0} C_b x_b + K_{a2}^{-0} \zeta, \\ \dot{x}_a^{+} = A_a^{+} x_a^{+} + K_{ab}^{+} C_b x_b + K_{a2}^{+} \zeta, \\ \dot{x}_b = (A_b + K_{bb} C_b) x_b + K_{b2} \zeta, \\ \bar{z} = \begin{pmatrix} C_b \\ 0 \end{pmatrix} x_b + \begin{pmatrix} 0 \\ I \end{pmatrix} \zeta. \end{cases} \tag{7}$$

Note that the eigenvalues of A_a^{-0} and A_a^{+} are in the closed left-half plane and open right-half plane respectively. We extract a subsystem from Σ_1 given by:

$$\bar{\Sigma}_1 : \begin{cases} \dot{x}_a^{+} = A_a^{+} x_a^{+} + K_{ab}^{+} C_b x_b + K_{a2}^{+} \zeta, \\ \dot{x}_b = (A_b + K_{bb} C_b) x_b + K_{b2} \zeta, \\ \bar{z} = \begin{pmatrix} C_b \\ 0 \end{pmatrix} x_b + \begin{pmatrix} 0 \\ I \end{pmatrix} \zeta, \end{cases} \tag{8}$$

with state dimension $n_a^{+} + n_b$. We can relate the recoverable region of Σ_1 to the recoverable region of $\bar{\Sigma}_1$ and then, using Theorem 13, we can relate the recoverable region of Σ to the recoverable region of $\bar{\Sigma}_1$.

Theorem 16. *Consider the plant Σ as given by (1) and a constraint set $\mathscr{S}$ satisfying Assumption 1. Define Σ_1 by (3) and $\bar{\Sigma}_1$ by (8). We have*

$$\mathscr{R}_C(\Sigma_1, \mathscr{S}_z) = \mathbb{R}^{n_a^{-0}} \times \mathscr{R}_C(\bar{\Sigma}_1, \mathscr{S}_z) \tag{9}$$

and the closure of $\mathscr{R}_C(\Sigma, \mathscr{S})$ is given by

$$T_x \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \;\middle|\; x_2 \in \overline{\mathscr{R}_C(\bar{\Sigma}_1, \mathscr{S}_z)} \right\} \cap \mathscr{A}(\Sigma, \mathscr{S}), \tag{10}$$

where x_1 and x_3 are of compatible dimension, i.e. $x_1 \in \mathbb{R}^{n_a^{-0}}$ and $x_3 \in \mathbb{R}^{n_c + n_d}$.

Using the decompositions from the scb we have in the above that x_1 is equal to x_a^{-0}, x_2 denotes the variables of $\bar{\Sigma}_1$ consisting of x_a^{+} and x_b while x_3 is composed of x_c and x_d.

Proof. See Appendix C. □

Remark 17. Again, as with Theorem 13, the above theorem does not capture the boundary points of the recoverable set. However, if $\mathscr{R}_C(\bar{\Sigma}_1, \mathscr{S}_z)$ is approximated by a polytope $R_2 x \leqslant q_2$ while $\mathscr{S}$ is described by $Rz \leqslant q$ then $\mathscr{R}_C(\Sigma, \mathscr{S})$ is approximated by

$$\begin{pmatrix} RC_z \\ (\,0 \quad R_2 \quad 0\,) T_x^{-1} \end{pmatrix} x \leqslant \begin{pmatrix} q \\ q_2 \end{pmatrix}.$$

By improving the approximation of $\mathscr{R}_C(\bar{\Sigma}_1, \mathscr{S}_z)$ we can approximate $\mathscr{R}_C(\Sigma, \mathscr{S})$ arbitrarily well in this way.

Since the transformation into scb and the associated computation of T_x is already implemented in Matlab, the remaining problem is the computation or approximation of $\mathscr{R}_C(\bar{\Sigma}_1, \mathscr{S}_z)$.

If the constraint is right invertible and at most weakly non-minimum phase then $\bar{\Sigma}_1$ is actually an empty (zero-dimensional) system and we have

$$\mathscr{R}_C(\Sigma_1, \mathscr{S}_z) = \mathbb{R}^{n_a},$$

and the closure of $\mathscr{R}_C(\Sigma, \mathscr{S})$ is equal to the admissible set. If this subsystem $\bar{\Sigma}_1$ has dimension two or less the tools from the book by Ryan (1982) can be used. Otherwise, gridding tools are needed as mentioned in Remark 14.

Note that the reduction of the computation of the recoverable region from Σ to the computation of the recoverable region for the lower order system $\bar{\Sigma}_1$ actually yields the result in Remark 12 as a special case.

4. Semi-globally stabilization in the recoverable region

The first objective of this paper was the reduction in the computation of the recoverable region as outlined in the previous section. The second objective of this paper is to show the possibility of stabilization without violating the constraints for any compact subset $\mathscr{K}$ contained in the interior of $\mathscr{R}_C(\Sigma, \mathscr{S})$ by a continuous feedback. Regarding the existence of Lipschitz continuous controllers, our main result is summarized in the following theorem.

Theorem 18. *Given the linear time-invariant system Σ in (1) with a constraint set $\mathscr{S}$ satisfying Assumption 1. Assume that (A, B) is stabilizable. Then, for any compact subset $\mathscr{K}$ contained in the interior of $\mathscr{R}_C(\Sigma, \mathscr{S})$, there exists a Lipschitz-continuous (in general nonlinear) feedback $u = f(x)$ such that the zero equilibrium point of the closed-loop system is asymptotically stable with a domain of attraction containing $\mathscr{K}$ and moreover, $z(t) \in \mathscr{S}$ for all $t \geqslant 0$ when $x(0) \in \mathscr{K}$.*

Moreover, for all initial conditions inside $\mathscr{K}$ the state converges to the origin exponentially fast.

Remark 19. Note that although this theorem is a pure existence result, this paper will also establish that we only need to design a controller for a subsystem which can have considerably lower dimension and in this way it does reduce the complexity of computational tools, available for actually designing controllers, which grow exponentially with the dimension.

In the previous section we connected the recoverable set of the original system Σ to that of the reduced system $\bar{\Sigma}_1$ through the intermediate system Σ_1. We will use these three layers to look also into the design of controllers. We first look in the next subsection at the design of controllers for systems of the form $\bar{\Sigma}_1$. This will turn out to be the most involved design step. In the second subsection we will extend a controller for this subsystem to a controller for the original system Σ. Note that we assume in proof that the system has no invariant zeros on the imaginary axis. This is without loss of generality since changing A to $A_\kappa = A + \kappa I$ with κ arbitrary small would remove the zeros on the imaginary axis. Clearly a controller for this system with a certain constrained domain of attraction would, when applied to the original system, always yield a larger domain of attraction and according to the following lemma, the recoverable set of this new system would be only marginally smaller than the recoverable set of the original system.

Lemma 20. *Consider system Σ in (1) and a convex, compact constraint set $\mathscr{S}$ containing 0 in the interior. Let Σ^κ be the system obtained from Σ by replacing A by $A + \kappa I$ with $\kappa > 0$. Assume (A, B) stabilizable. Then for any compact subset $\mathscr{K} \subset \operatorname{int} \mathscr{R}_C(\Sigma, \mathscr{S})$ there exists $\kappa^* > 0$ such that*

$$\mathscr{K} \subset \mathscr{R}_C(\Sigma^\kappa, \mathscr{S}) \subset \mathscr{R}_C(\Sigma, \mathscr{S}),$$

for any $\kappa \in [0, \kappa^*]$.

Proof. See Appendix D. □

4.1. Proof of Theorem 18 for the subsystem $\bar{\Sigma}_1$

As we mentioned before, the subsystem $\bar{\Sigma}_1$ in (8) is the core of the original system Σ, which causes most of the design difficulties under constraints. Therefore, we first prove Theorem 18 for systems which are left-invertible, have relative degree zero, and have only antistable invariant zeros. Obviously, subsystem $\bar{\Sigma}_1$ is one of such systems. To simplify notation, we assume the system is in the following form

$$\Sigma_0 : \begin{cases} \dot{\xi} = A_0 \xi + B_0 \zeta, \\ \bar{z} = \begin{pmatrix} C_0 \\ 0 \end{pmatrix} \xi + \begin{pmatrix} 0 \\ I \end{pmatrix} \zeta, \end{cases} \tag{11}$$

with constraint $\bar{z}(t) \in \mathscr{S}_z$ for all $t \geqslant 0$, where the un-break observable eigenvalues of (C_0, A_0), i.e. the invariant zeros, are in the open right-half plane (antistable) and (A_0, B_0) is stabilizable.

Consider the set $\mathscr{R}_C(\Sigma_0, \mathscr{S}_z)$. Our first objective is to choose the input in such a way that we stay inside this set. If this is possible then we call the set positive invariant. In order to do this we can try to choose at each boundary point of the set, an input such that the derivative of the state points inside or tangent to the set and then expand this feedback to the full set. We will show that this basic idea works although we need to spend quite some effort on avoiding technical difficulties:

- We need that the set $\mathscr{R}_C(\Sigma_0, \mathscr{S}_z)$ is bounded and closed, since the suggested design is based on designing the feedback on the boundary.
- If the derivative does not point inside but tangent to the set then we are not guaranteed that the state stays in the set.
- Our aim is to achieve asymptotic stability and the above idea only looks at achieving positive invariance and this is clearly not the same.
- The feedback that we choose in this way might not even be continuous and therefore we are not sure that the closed loop system has a (unique) solution.

The first technical issue mentioned above can actually be resolved due to the extra structure of the system (11):

Lemma 21. *Given the stabilizable linear system Σ_0 in (11) whose invariant zeros are antistable and a convex, compact constraint set $\mathscr{S}_z$ containing 0 in the interior. The recoverable region $\mathscr{R}_C(\Sigma_0, \mathscr{S}_z)$ for this system has the following properties:*

(i) *The set $\mathscr{R}_C(\Sigma_0, \mathscr{S}_z)$ is bounded.*
(ii) *For any initial condition $\xi_0 \in \partial \mathscr{R}_C(\Sigma_0, \mathscr{S}_z)$, there exists an input u such that the state of the system remains in $\overline{\mathscr{R}_C(\Sigma_0, \mathscr{S}_z)}$, while the constraint $\bar{z}(t) \in \mathscr{S}_z$ is satisfied for all $t \geqslant 0$.*

Proof. See Appendix E. □

By property (ii) of the above lemma, it seems feasible to find a feedback such that the compact set $\mathscr{R}_C(\Sigma_0, \mathscr{S}_z)$ becomes invariant. However, in order to avoid the technical difficulties mentioned before, it turns out that it is desirable to start working with an auxiliary system:

$$\Sigma_0^\kappa : \begin{cases} \dot{\xi} = A_\kappa \xi + B_0 \tilde{\zeta}, \\ \tilde{z} = \begin{pmatrix} C_0 \\ 0 \end{pmatrix} \xi + \begin{pmatrix} 0 \\ I \end{pmatrix} \tilde{\zeta}, \end{cases} \tag{12}$$

with constraint $\tilde{z}(t) \in \mathscr{S}_z$ for all $t \geqslant 0$, where $A_\kappa = A + \kappa I$ for $\kappa \geqslant 0$. It is more difficult to keep the state inside a convex set $\mathscr{V}$ (containing 0) for this system due to the fact that the extra term $\kappa\xi$ always points outside the set $\mathscr{V}$.

All the technical difficulties mentioned before are resolved in this way. If we choose a direction for the derivative to point tangent or inside the set for this auxiliary system then by reducing κ we can guarantee that, for a slightly smaller κ, we can make the set positive invariant. Moreover, by reducing κ we obtain some flexibility which enables us to make the feedback continuous and even Lipschitz continuous. Finally, if the state stays in the set for some positive κ then for the original system the state converges to zero exponentially.

Note that the recoverable set of this auxiliary system is close to the recoverable set of the original system by Lemma 20. The technical details of the above are in Appendix F and yield the proof of Theorem 18 for the special case of system Σ_0 given in (11). However, in order to expand a controller of the subsystem $\bar{\Sigma}_1$ to a controller for the system Σ, we need a strengthened version of Theorem 18 which can handle small exponentially decaying disturbances. The details of this expansion from $\bar{\Sigma}_1$ to Σ are in the next subsection.

Theorem 22 (Special case with disturbance). *Given a linear time-invariant system*

$$\Sigma_0^d: \begin{cases} \dot{\xi} = A_0\xi + B_0\zeta + d, \\ \eta = \begin{pmatrix} C_0 \\ 0 \end{pmatrix}\xi + \begin{pmatrix} 0 \\ I \end{pmatrix}\zeta, \end{cases} \tag{13}$$

with the unobservable modes of (C_0, A_0) *antistable and* (A_0, B_0) *stabilizable. Given* $M>0$ *and a compact subset* $\mathscr{K} \subset \operatorname{int}\mathscr{R}_C(\Sigma_0^d, \mathscr{S}_z)$, *there exists* $\delta > 0$ *and a Lipschitz-continuous feedback* $\zeta = f(\xi)$ *such that the equilibrium point* 0 *is asymptotically stable for all initial conditions in* $\mathscr{K}$ *and for any disturbance* d *satisfying*

$$\|d(t)\| \leqslant M\mathrm{e}^{-\delta t}, \tag{14}$$

the closed-loop system satisfies $\xi(t) \to 0$ *as* $t \to \infty$ *and* $\eta(t) \in \mathscr{S}_z$ *for all* $t \geqslant 0$.

4.2. Proof of Theorem 18

In the above, we have decomposed the original system Σ into two subsystems Σ_1 and Σ_2 and then we established that the computational effort for determining the recoverable set is concentrated in system Σ_1. If we look more closely at the system Σ_1, we can extract another subsystem $\bar{\Sigma}_1$ and the recoverable set of this last subsystem is the core of the computational effort needed in determining the recoverable set.

This time, we want to establish a suitable controller with a domain of attraction containing an arbitrarily chosen compact set $\mathscr{K}$ which is itself contained in the interior of $\mathscr{R}_C(\Sigma, \mathscr{S})$. First note that the recoverable set of the full system satisfies the structure established in Theorem 13. Therefore, we can find a compact set $\mathscr{K}_1$ such that $\mathscr{K}_1$ is contained in the interior of $\mathscr{R}_C(\Sigma_1, \mathscr{S}_z)$ and

$$\mathscr{K} \subset T_x\left\{ \begin{pmatrix} x_{ab} \\ x_{cd} \end{pmatrix} \middle|\, x_1 \in \mathscr{K}_1 \right\} \cap \mathscr{A}(\Sigma, \mathscr{S}),$$

where x_{ab} and x_{cd} denote the initial conditions of Σ_1 and Σ_2, respectively. As noted in the beginning of this section we assume without loss of generality that we have no zeros on the imaginary axis which implies in the scb structure that the dynamics of x_a^{-0} is asymptotically stable and hence can be exempted from stabilization. Consider the system

$$\bar{\Sigma}_{1\kappa}: \begin{cases} \dot{x}_a^+ = A_{a\kappa}^+ x_a^+ + K_{ab}^+ + C_b x_b + K_{a2}^+\zeta, \\ \dot{x}_b = (A_{b\kappa} + K_{bb}C_b)x_b + K_{b2}\zeta, \\ \bar{z} = \begin{pmatrix} C_b \\ 0 \end{pmatrix} x_b + \begin{pmatrix} 0 \\ I \end{pmatrix}\zeta, \end{cases} \tag{15}$$

with $A_{a\kappa}^+ = A_a^+ + \kappa I$ and $A_{b\kappa} = A_b + \kappa I$. The associated recoverable set $\mathscr{R}_C(\bar{\Sigma}_{1\kappa}, \mathscr{S})$ has the following property:

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle|\, x_1 \in \mathscr{R}_C(\bar{\Sigma}_{1\kappa}, \mathscr{S}_z) \right\} \subset \mathscr{R}_C(\Sigma_1, \mathscr{S}_z),$$

where x_1 denotes the initial condition for x_a^+ and x_b while x_2 denotes the initial condition for x_a^{-0}. Moreover, similar to Lemma 20 it is easy to verify that for κ small enough:

$$\mathscr{K}_1 \subset T_x\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle|\, x_1 \in \mathscr{R}_C(\bar{\Sigma}_{1\kappa}, \mathscr{S}_z) \right\}.$$

Choose κ small enough such that this latter inclusion is satisfied. Then we can design a controller f for $\bar{\Sigma}_{1\kappa}$ according to Theorem 22 and it is easily verified that this controller when applied to Σ_1 creates an exponentially stable system with $\mathscr{K}_1$ contained in its domain of attraction which can handle exponentially decaying disturbances satisfying (14).

Next, we consider the second subsystem Σ_2 given by (5). This system has the nice structure that the mapping from $(\tilde{u}_d, u_0)$ to ζ is strongly controllable. Assume the initial state of Σ is in the interior of the set

$$T_x\left\{ \begin{pmatrix} x_{ab} \\ x_{cd} \end{pmatrix} \middle|\, x_1 \in \mathscr{K}_1 \right\} \cap \mathscr{A}(\Sigma, \mathscr{S}). \tag{16}$$

Following the design methodology in Saberi et al. (2002), we can then design a feedback for inputs $(\tilde{u}_d, u_0)$ which stabilizes Σ_2 and such that $\zeta = f(x_1) + d$ with d satisfying (14) while satisfying the constraints. This controller is then easily seen to satisfy the conditions of Theorem 18.

5. Example

We consider the following system:

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u,$$

$$z = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x,$$

with $\mathscr{S}$ given by

$$\mathscr{S} = \{z \in \mathbb{R}^2 \mid -1 \leqslant z_1 \leqslant 4,\ -1 \leqslant z_2 \leqslant 1\}.$$

In this case, we note that in the scb context x_a^0 corresponds to x_3 while x_d corresponds to x_4. Moreover, x_c is not present since the system is left-invertible. Note that the system is not right-invertible and hence we cannot rely on the relatively easy structure we obtain for right-invertible systems as outlined in our previous paper (Saberi et al., 2002).

In order to obtain the recoverable set we first compute the system Σ_1 which is given by

$$\Sigma_1: \begin{cases} \dot{\tilde{x}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & -2 & 0 \end{pmatrix} \tilde{x} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \zeta, \\ z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tilde{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \zeta \end{cases}$$

and then the system $\bar{\Sigma}_1$ given by

$$\bar{\Sigma}_1: \begin{cases} \dot{\bar{x}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{x} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \zeta, \\ z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \bar{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \zeta. \end{cases}$$

The recoverable region for this system can be computed using the techniques available from the work of Ryan, see (Ryan, 1982). We obtain the recoverable set $\mathscr{R}(\bar{\Sigma}_1, \mathscr{S})$ given in Fig. 1. Next, consider the boundary. The dashed line does not belong to the recoverable set while the solid line is part of the recoverable set. The theory developed in this paper then tells us that

$$\overline{\mathscr{R}(\Sigma, \mathscr{S})} = \left\{ x \in \mathbb{R}^4 \mid \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \overline{\mathscr{R}(\bar{\Sigma}_1, \mathscr{S})}, -1 \leqslant x_4 \leqslant 1 \right\}.$$

Assume we have a compact set $\mathscr{K}$ contained in the interior of $\mathscr{R}(\Sigma, \mathscr{S})$ and we want to obtain a controller

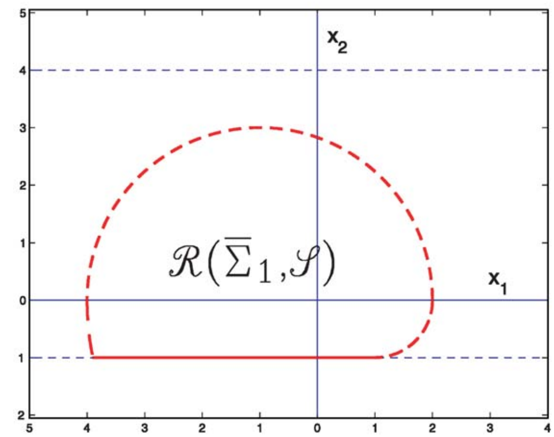

Fig. 1. Recoverable region $\mathscr{R}(\bar{\Sigma}_1, \mathscr{S})$.

which stabilizes the system and contains $\mathscr{K}$ in its domain of attraction while avoiding constraint violation when starting in the set $\mathscr{K}$.

The theory developed in this paper tells us that we need to look at a modification of the system Σ_1

$$\Sigma_{1,\varepsilon}: \begin{cases} \dot{\tilde{x}} = \begin{pmatrix} \varepsilon & 1 & 0 \\ -1 & \varepsilon & 0 \\ 1 & -2 & \varepsilon \end{pmatrix} \tilde{x} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \zeta, \\ z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tilde{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \zeta, \end{cases}$$

for some $\varepsilon > 0$ small enough. We first need to design a controller which stabilizes this system and contains $\mathscr{K}_1$ in its domain of attraction while avoiding constraint violation when the initial condition is in the set $\mathscr{K}_1$ where,

$$\mathscr{K}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mathscr{K}.$$

Efficient design methods for this are not known. However, gridding can be one option, and working with a lower dimensional subsystem Σ_1 will definitely make the gridding method more attractive.

Acknowledgements

The research leading to the results presented in this paper was initiated by the first author in collaboration with his student Camiel van Moll. The results were described in the thesis (van Moll, 1999) and was for the case of input constraints only but addressed several issues also described in this paper. The work of Camiel van Moll is gratefully acknowledged.

Appendix A. A special coordinate basis

In this section we recall from Saberi and Sannuti (1990), Sannuti and Saberi (1987) special coordinate basis (scb) for system Σ in (1). A system in scb reveals the inherent finite and infinite zero structures, which are crucial components in classifying the constraints and in facilitating the design.

For a general linear system Σ in (1), one can choose appropriate coordinates in the state, input, and output spaces

$$x = T_x\,\bar{x}, \quad u = T_u\,\bar{u} + \bar{F}x, \quad z = T_z\,\bar{z},$$

where T_x, T_u, and T_z are transformation matrices and $\bar{F}$ is a preliminary feedback which will make the structure of the system more visible. With this objective, we also use the following decomposition for the state, output and input of the system.

$$\bar{x} = \begin{pmatrix} x_a \\ x_b \\ x_c \\ x_d \end{pmatrix}, \quad \bar{z} = \begin{pmatrix} z_b \\ \zeta \end{pmatrix}, \quad \zeta = \begin{pmatrix} z_0 \\ z_d \end{pmatrix} \quad \text{and } \bar{u} = \begin{pmatrix} u_0 \\ u_c \\ u_d \end{pmatrix},$$

after which the system (1) takes the following form:

$$\bar{\Sigma}: \begin{cases} \dot{x}_a = A_a x_a + K_{ab} z_b + K_{a2}\zeta, \\ \dot{x}_b = A_b x_b + K_{bb} z_b + K_{b2}\zeta, \\ \dot{x}_c = A_c x_c + B_c u_c + K_{cb} z_b + K_{c2}\zeta, \\ \dot{x}_d = A_d x_d + B_d u_d + K_{db} z_b + K_{d2}\zeta, \\ \bar{z} = \begin{pmatrix} z_b \\ \zeta \end{pmatrix} = \begin{pmatrix} C_b x_b \\ u_0 \\ C_d x_d \end{pmatrix}. \end{cases} \tag{A.1}$$

Furthermore, the x_a equation can be decomposed as

$$\dot{x}_a^{-0} = A_a^{-0} x_a^{-0} + K_{ab}^{-0} C_b x_b + K_{a2}^{-0}\zeta,$$
$$\dot{x}_a^{+} = A_a^{+} x_a^{+} + K_{ab}^{+} C_b x_b + K_{a2}^{+}\zeta,$$

where

$$x_a = \begin{pmatrix} x_a^{-0} \\ x_a^{+} \end{pmatrix}, \quad A_a = \begin{pmatrix} A_a^{-0} & 0 \\ 0 & A_a^{+} \end{pmatrix},$$

$$K_{ab} = \begin{pmatrix} K_{ab}^{-0} \\ K_{ab}^{+} \end{pmatrix}, \quad K_{a2} = \begin{pmatrix} K_{a2}^{-0} \\ K_{a2}^{+} \end{pmatrix},$$

with all eigenvalues of A_a^{-0} in the closed left-half plane, and all eigenvalues of A_a^{+} in the open right-half plane.

The scb components have the following dimensions. For the state, $x_a \in \mathbb{R}^{n_a}$, $x_b \in \mathbb{R}^{n_b}$, $x_c \in \mathbb{R}^{n_c}$, and $x_d \in \mathbb{R}^{n_d}$, with $n_a + n_b + n_c + n_d = n$. $x_a^{-0} \in \mathbb{R}^{n_a^{-0}}$, $x_a^{+} \in \mathbb{R}^{n_a^{+}}$, and $n_a^{-0} + n_a^{+} = n_a$. For the input, $u_0 \in \mathbb{R}^{m_0}$, $u_c \in \mathbb{R}^{m_c}$, and $u_d \in \mathbb{R}^{\ell}$, with $m_0 + m_c + \ell = m$. And for the output, $z_0 \in \mathbb{R}^{m_0}$, $z_b \in \mathbb{R}^{m_b}$, and $z_d \in \mathbb{R}^{\ell}$, with $m_0 + m_b + \ell = p$. Finally $\zeta \in \mathbb{R}^{n_2}$ with $n_2 = m_0 + \ell$.

The components involved in scb have lots of nice properties. Among others we mention the following that are relevant to this work:

(i) The eigenvalues of A_a are the invariant zeros of the system Σ.
(ii) The infinite zeros are associated with the dynamics of x_d.
(iii) The matrix pair (A_c, B_c) is controllable.
(iv) The matrix pair (C_b, A_b) is observable.
(v) If system Σ is right invertible, then the dimension of x_b is zero. In this case, the components x_b and z_b disappear.
(vi) If system Σ is left invertible, then the dimension of x_c is zero. In this case, the components x_c and u_c disappear.

Appendix B. Proof of Lemma 9

We start with showing property (i). Since $0 \in \operatorname{int} \mathscr{S}$ and the system is linear and controllable, there exists a ball $\mathscr{B}(0, \varepsilon)$ around the origin with radius ε and time $T > 0$ such that for any $x(0) \in \mathscr{B}(0, \varepsilon)$ there exists a control u which steers the state to the origin in time T without violating the constraint. By definition, for any $x(0) \in \mathscr{R}_C(\Sigma, \mathscr{S})$, there exists an input u such that $x(t) \to 0$ as $t \to \infty$ while satisfying the constraints. Hence, there exists a time $T_1 > 0$ so that $x(t) \in \mathscr{B}(0, \varepsilon)$ for $t \geqslant T_1$. Therefore, it is possible to drive any initial state in $\mathscr{R}_C(\Sigma, \mathscr{S})$ to the origin in time $T + T_1$.

Property (ii) follows from the assumption that $\mathscr{S}$ is convex and $0 \in \mathscr{S}$.

To show property (iii), we note that already in LeMay (1964) it was established that in the case of only input constraints the recoverable set is open. In the case of general state and input constraints the set $\mathscr{R}_C(\Sigma, \mathscr{S})$ need not be open. This is seen from the simple example $\dot{x} = u$ with $z = x$ and constraint set $\mathscr{S} = \{z \mid z \in [-1, 1]\}$ which yields $\mathscr{R}_C(\Sigma, \mathscr{S}) = \mathscr{S}$ which is obviously closed.

Finally, we consider property (iv). Under the condition that system Σ has relative degree at most one, is left invertible, and with all invariant zeros antistable the system Σ in scb takes the following form (see Appendix A):

$$\begin{cases} \dot{x}_a = A_a x_a + K_{ab} z_b + K_{a2}\zeta, \\ \dot{x}_b = A_b x_b + K_{bb} z_b + K_{b2}\zeta, \\ \dot{x}_d = A_d x_d + B_d u_d + K_{dd} z_b + K_{d2}\zeta, \\ \bar{z} = \begin{pmatrix} z_b \\ \zeta \end{pmatrix} = \begin{pmatrix} C_b x_b \\ u_0 \\ x_d \end{pmatrix}. \end{cases} \tag{B.1}$$

Firstly, since $\mathscr{S}$ is bounded, we find that $x_d(t)$ must be bounded. Secondly, the x_a^+ dynamics is antistable and controlled by the virtual input z_b and ζ which are bounded. It is a classical result that the recoverable region for this subsystem must be bounded. It remains to show that the recoverable region for $x_b(t)$ is also bounded. Consider the following subsystem:

$$\dot{x}_b = A_b x_b + K_{bb} z_b + K_{b2}\zeta,$$

$$z_b = C_b x_b.$$

It is known that (C_b, A_b) is observable. Moreover, the inputs z_b and ζ and the output z_b of this system are both bounded. It is then not very hard to verify that this implies that the state $x_b(t)$ must be bounded.

Appendix C. Proof of Theorem 16

We note that Theorem 13 helps us to relate the recoverable regions of $\bar{\Sigma}_1$ and Σ and we obtain (10). In order to obtain (9), we need to do a bit more work. Our proof is strongly motivated by results from LeMay (1964). One inclusion is basically obvious:

$$\mathscr{R}_C(\Sigma_1, \mathscr{S}_z) \subset \mathbb{R}^{n_a^{-0}} \times \mathscr{R}_C(\bar{\Sigma}_1, \mathscr{S}_z). \tag{C.1}$$

For notational ease we will denote an initial condition of Σ_1 by (x_1, x_2) with x_1 equal to the initial condition for x_a^{-0} and x_2 a vector consisting of initial conditions for x_a^+ and x_b.

We first note that for any $x_2 \in \mathscr{R}_C(\bar{\Sigma}_1, \mathscr{S}_z)$ we can find x_1 such that $(x_1, x_2) \in \mathscr{R}_C(\Sigma_1, \mathscr{S}_z)$. After all if we choose an input u for $\bar{\Sigma}_1$ which steers x_2 to 0 at time T without violating constraints, then for this same input u we can always choose x_1 such that for the initial condition (x_1, x_2) the system Σ_1 reaches the origin at time T. Moreover, since x_1 does not affect the constraints, the initial condition (x_1, x_2) is steered to zero without constraint violations.

Next, we note that for any $x_1 \in \mathbb{R}^{n_a^{-0}}$ we have that $(x_1, 0) \in \mathscr{R}_C(\Sigma_1, \mathscr{S}_z)$. It is well known that we can locally stabilize a system using a linear feedback $u = -B'Px$ with P a solution of an algebraic Riccati equation and such that an ellipsoid of the form $x'Px \leqslant c$ is invariant for the closed loop system while constraint violations are avoided. If we apply this to the system Σ_1, we find that P restricted to the part of the system composed of x_1 can be made arbitrarily small and this yields that we can guarantee that for any x_1 there exists a solution of the Riccati equation P such that $(x_1, 0)$ is contained in this invariant ellipsoidal set and for which we can hence avoid constraint violation. This clearly implies that $(x_1, 0) \in \mathscr{R}_C(\Sigma_1, \mathscr{S}_z)$. For further details regarding this type of arguments we refer to Hu et al. (2001a,b).

We claim that for all (x_1, x_2) with $x_2 \in \mathscr{R}_C(\bar{\Sigma}_1, \mathscr{S}_z)$ we have that $(x_1, x_2) \in \mathscr{R}_C(\Sigma_1, \mathscr{S}_z)$. In other words

$$\mathscr{R}_C(\Sigma_1, \mathscr{S}_z) \supset \mathbb{R}^{n_a^{-0}} \times \mathscr{R}_C(\bar{\Sigma}_1, \mathscr{S}_z).$$

and combined with (C.1) the proof of (9) would be complete.

Let $\varepsilon > 0$ be given. Choose any (x_1, x_2) with $x_2 \in \mathscr{R}_C(\bar{\Sigma}_1, \mathscr{S}_z)$. We know there exists $\tilde{x}_1$ such that $(\tilde{x}_1, x_2)$ is in $\mathscr{R}_C(\Sigma_1, \mathscr{S}_z)$.

Choose $\lambda \in (0, 1)$. We have

$$(x_1 - \lambda\tilde{x}_1/1 - \lambda, 0) \in \mathscr{R}_C(\Sigma_1, \mathscr{S}_z).$$

This implies

$$(x_1, x_2) = \lambda(\tilde{x}_1, x_2) + (1 - \lambda)(x_1 - \lambda\tilde{x}_1/1 - \lambda, 0),$$

is an element of the set $\mathscr{R}_C(\Sigma_1, \mathscr{S}_z)$ due to convexity.

Appendix D. Proof of Lemma 20

We need the following lemma in order to prove Lemma 20. which can be easily proven.

Lemma 23. *Consider system Σ in (1) and a convex, compact constraint set $\mathscr{S}$ containing 0 in the interior. Assume that (A, B) is stabilizable. For any compact subset $\mathscr{K} \subset \operatorname{int}\mathscr{R}_C(\Sigma, \mathscr{S})$ and any ball $\mathscr{B}(0, \varepsilon) \subset \mathscr{K}$ with $\varepsilon > 0$ sufficiently small, there exists a time $T > 0$ such that for any initial condition in $\mathscr{K}$ there exists an input u for which $x(T) \in \mathscr{B}(0, \varepsilon)$ and $z(t) \in \mathscr{S}$, $\forall\, t \in [0, T]$.*

Proof of Lemma 20. We first note that for any $\kappa > 0$ we have

$$\mathscr{R}_C(\Sigma^\kappa, \mathscr{S}) \subset \mathscr{R}_C(\Sigma, \mathscr{S}). \tag{D.1}$$

This follows from a simple observation that if $x(t)$ and $u(t)$ satisfy system Σ^κ with initial condition $x(0)$ and the constraint, then $e^{-\kappa t}x(t)$ and $e^{-\kappa t}u(t)$ satisfy system Σ with the same initial condition and the constraint. Also, it is clear from the same observation that

$$\mathscr{R}_C(\Sigma^\ell, \mathscr{S}) \subset \mathscr{R}_C(\Sigma^\kappa, \mathscr{S}), \tag{D.2}$$

for $\ell > \kappa \geqslant 0$.

Next, we show that for any compact set $\mathscr{K}$ satisfying

$$\mathscr{K} \subset \alpha\mathscr{R}_C(\Sigma, \mathscr{S}), \quad \alpha \in (0, 1),$$

there exists a $\kappa > 0$ such that

$$\mathscr{K} \subset \mathscr{R}_C(\Sigma^\kappa, \mathscr{S}). \tag{D.3}$$

Note that if (A, B) is stabilizable, then there exists a sufficiently small $\kappa_0 > 0$ such that for all $0 \leqslant \kappa \leqslant \kappa_0$ the pair (A_κ, B) is also stabilizable. Also note that by stabilizability, there exists $\varepsilon > 0$ sufficiently small such that one can find for any point in $\mathscr{B}(0, \varepsilon)$, a control u such that the resulting trajectory goes to zero asymptotically without violating the constraints. We choose such a small $\varepsilon > 0$ for which this property holds for system Σ^{κ_0}.

Next, we consider system Σ. By Lemma 23 there exists a uniform $T > 0$ such that any initial state in $\mathscr{K}$ can be driven to the ball $\mathscr{B}(0, \alpha\varepsilon)$ in time T by a suitable u while respecting the constraints. Let $\tilde{x}_0 \in \mathscr{K} \subset \alpha\mathscr{R}_C(\Sigma, \mathscr{S})$.

Then there exist $\hat{x}(t)$ and $\hat{u}(t)$ satisfying

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B\hat{u}(t), \quad \hat{x}(0) = \tilde{x}_0,$$

$$\hat{z}(t) = C_z\hat{x}(t) + D_z\hat{u}(t) \in \alpha\mathscr{S},$$

for $t \in [0,T]$ and $\hat{x}(T) \in \mathscr{B}(0,\alpha\varepsilon)$, where T does not depend on $\tilde{x}_0$. By choosing $\kappa > 0$ small enough so that $\alpha e^{\kappa T} < 1$, it is straightforward that $\tilde{x}(t) = e^{\kappa t}\hat{x}(t)$ and $\tilde{u}(t) = e^{\kappa t}\hat{u}(t)$ satisfy for $t \in [0,T]$

$$\dot{\tilde{x}}(t) = (A + \kappa I)\tilde{x}(t) + B\tilde{u}(t), \quad \tilde{x}(0) = \tilde{x}_0,$$

$$\tilde{z}(t) = C_z\tilde{x}(t) + D_z\tilde{u}(t) \in \mathscr{S},$$

while $\tilde{x}(T) = x(T) \in \mathscr{B}(0,\varepsilon)$. Hence, $\tilde{x}_0 \in \mathscr{R}_C(\Sigma^\kappa, \mathscr{S})$. □

Appendix E. Proof of Lemma 21

In general the recoverable region is not closed and its closure is not easily characterized. However, for system (11) the closure of the recoverable region, $\overline{\mathscr{R}_C(\Sigma_0,\mathscr{S})}$, can easily be characterized by the set of initial conditions of system (11) for which there exists an input that keeps the state bounded while avoiding constraint violation. This result is stated in the following lemma, which leads to the proof of Lemma 21. Similar results have for instance been obtained in LeMay (1964). Due to space limitations the proof is omitted.

Lemma 24. *Consider system Σ_0 in (11) with compact, convex constraint set $\mathscr{S}$ containing 0 in the interior. Assume that the unobservable modes of (C_0, A_0) are antistable and (A_0, B_0) stabilizable. Then we have*

$$\overline{\mathscr{R}_C(\Sigma_0,\mathscr{S})} = \Omega(\Sigma_0,\mathscr{S}),$$

where

$$\Omega(\Sigma_0,\mathscr{S}) := \{\xi_0 \in \mathbb{R}^n \mid \exists\, \zeta \text{ such that the solution of } \Sigma_0 \text{ with } \xi(0) = \xi_0 \text{ satisfies } \xi \in L_\infty[0,\infty) \text{ and } \bar{z}(t) \in \mathscr{S},\ \forall t \geqslant 0\}.$$

Appendix F. Proof of Theorem 22

Some preparation is needed before we proceed to the proof of this theorem. Define

$$\mathscr{C}_0(\kappa) := \mathscr{R}_C(\Sigma_0^\kappa, \mathscr{S}).$$

From Lemmas 9 and 21 we know that $\mathscr{C}_0(\kappa)$ is convex and bounded. Also, by Lemma 20, for any compact subset $\mathscr{K} \subset \operatorname{int} \mathscr{R}_C(\Sigma_0,\mathscr{S})$ there exist $\ell > \kappa > 0$ such that

$$\mathscr{K} \subset \mathscr{C}_0(\ell) \subset \mathscr{C}_0(\kappa) \subset \mathscr{R}_C(\Sigma_0,\mathscr{S}). \tag{F.1}$$

Next we show the following:

(i) There exists a continuous feedback for system (12), so that the closure of $\mathscr{C}_0(\ell)$ is an invariant set.

(ii) This feedback can be slightly modified to be Lipschitz, and when applied to the original system (11), $\overline{\mathscr{C}}_0(\ell)$ is again an invariant set while the state of the system with any initial condition in $\overline{\mathscr{C}}_0(\ell)$ will converge to the origin.

From these we conclude that $\overline{\mathscr{C}}_0(\ell)$, and hence $\mathscr{K}$, is contained in the domain of attraction.

As stated before, we will try to achieve this by trying to guarantee that the trajectory points inwards or tangent to this set in every boundary point by an appropriate choice of the input. In order to formalize this we need the following set:

$$N_{\mathscr{V}}(\xi) := \{\eta \in \mathbb{R}^n \mid \|\eta\| = 1 \text{ and } \langle \xi' - \xi, \eta \rangle \leqslant 0,\ \forall\, \xi' \in \mathscr{V}\}.$$

Note that $N_{\mathscr{V}}(\xi)$ is the set of normals in the point ξ to the set $\mathscr{V}$ (as studied in for instance Rockafellar, 1970). It is also shown in Rockafellar (1970) that for a convex set $\mathscr{V}$ the set of normals is nonempty whenever ξ is a boundary point of $\mathscr{V}$. Its importance is due to the fact that if we start in ξ in the direction v then this direction is tangent to or pointing inside $\mathscr{V}$ if and only if $\langle v, \eta \rangle \leqslant 0$ for all $\eta \in N_{\mathscr{V}}(\xi)$.

Let the relation (F.1) hold for $\ell > \kappa > 0$. Define $T_\kappa : \partial\mathscr{C}_0(\ell) \to \mathscr{P}(\mathbb{R}^n)$, where $\mathscr{P}(\mathbb{R}^n)$ denotes the collection of all subsets of $\mathbb{R}^n$, by

$$T_\kappa(\xi) := \left\{ A_\kappa\xi + B_0\zeta \,\middle|\, \begin{pmatrix} C_0\xi \\ \zeta \end{pmatrix} \in \mathscr{S} \text{ and } \langle A_\kappa\xi + B_0\zeta, \eta \rangle \leqslant 0,\ \forall\, \eta \in N_{\mathscr{C}_0(\ell)}(\xi) \right\}.$$

The next lemma states some properties of $T_\kappa(\xi)$. The proof is omitted due to page limitations.

Lemma 25. *Assume $\kappa < \ell$. Then we have:*

(i) *$T_\kappa(\xi)$ is convex and closed for every $\xi \in \partial\mathscr{C}_0(\ell)$.*

(ii) *For any point $\xi \in \partial\mathscr{C}_0(\ell)$ the sets $T_\kappa(\xi)$ and $T_\ell(\xi)$ are nonempty.*

Next we ask: will the state trajectory stay in $\overline{\mathscr{C}}(\kappa)$ for all $t \geqslant 0$ if we choose a feedback such that $\dot{\xi}(0) \in T_\kappa(x_0)$ for all initial conditions $\xi(0) \in \partial\mathscr{C}(\kappa)$? This can be addressed using Nagumo's theorem (see Aubin, 1991; Nagumo, 1942).

Theorem 26 (Nagumo). *Consider system (12). Let the relation (F.1) hold for $\ell > \kappa > 0$. Assume that there is a Lipschitz continuous feedback $\zeta = f(\xi)$ such that $A_\kappa\xi + B_0 f(\xi) \in T_\kappa(\xi)$ for all $\xi \in \partial\mathscr{C}_0(\ell)$. Then for any initial condition inside $\mathscr{C}_0(\ell)$ the solution of the differential equation remains in $\overline{\mathscr{C}}_0(\ell)$.*

Since $T_\kappa(\xi(t))$ is nonempty for $\xi(t) \in \partial\mathscr{C}_0(\ell)$, there exists $\zeta(t)$ such that $\dot{\xi}(t) \in T_\kappa(x(t))$. In order to apply Nagumo's Theorem, we need a continuous $\zeta(t)$ for feedback. The existence of a continuous feedback is assured by Michael's Theorem. We first recall the formal definition of upper and lower semicontinuity of set valued functions, see for instance Aubin (1991, Sections 2.1.2 and 6.5.3).

Definition 27. Let X and Y be normed spaces, $D \subset X$ and $F(\cdot)$ a set-valued function from D to subsets of Y such that $F(x)$ is nonempty for all $x \in D$.

F is called upper semicontinuous at $x_0 \in D$ if for any neighborhood U of $F(x_0)$, there exists $\varepsilon > 0$ such that for all $x' \in D$ with $\|x' - x\| < \varepsilon$ we have $F(x') \subset U$. F is called upper semicontinuous if F is upper semicontinuous at every point of D.

F is called lower semicontinuous at $x_0 \in D$ if for any $y \in F(x_0)$ and for any sequence $\{x_n\} \in D$ that converges to x_0 there exists a sequence $\{y_n\}$ with $y_n \in F(x_n)$ that converges to y. F is lower semicontinuous if F is lower semicontinuous at every point of D.

Using the above, we can formulate Michael's theorem:

Theorem 28 (Michael, 1956; Aubin and Frankowska, 1990). *Let D be a compact metric space and Y a Banach space. Every lower semicontinuous function $F(\cdot)$ from D to the non-empty, closed, and convex subsets of Y admits a continuous selection.*

In our case $D = \partial\mathscr{C}_0(\ell)$ which is clearly a compact metric space and $Y = \mathbb{R}^n$ is clearly a Banach space. Lemma 25 assures that $T_\kappa(\xi)$ is not empty for all $\xi \in \partial\mathscr{C}_0(\ell)$; that is, $F = T_\kappa$ maps into nonempty subsets of Y. A continuous selection means that we can find a continuous function $h : \partial\mathscr{C}_0(\ell) \to \mathbb{R}^n$ such that $h(\xi) \in T_\kappa(\xi)$ for all $\xi \in \partial\mathscr{C}_0(\ell)$, which is the result we need. But, to apply Michael's Theorem we need to establish that T_κ is lower semicontinuous and $T_\kappa(\xi)$ is closed and convex for all $\xi \in \partial\mathscr{C}_0(\ell)$. The set is closed and convex by lemma 25 and lower semicontinuity is the content of next lemma.

Lemma 29. *Let the relation* (F.1) *hold for $\ell > \kappa > 0$. Then T_κ is lower semicontinuous on $\partial\mathscr{C}_0(\ell)$.*

Michael's Theorem leads to the existence of a continuous function h such that $h(\xi) \in T_\kappa(\xi)$ for all $\xi \in \partial\mathscr{C}_0(\ell)$. Let

$$\zeta = f(\xi) = B_0^\dagger[h(\xi) - A_\kappa \xi], \tag{F.2}$$

where $B_0^\dagger$ is the Moore–Penrose generalized inverse of B_0. Clearly, this $\zeta(\xi)$ is a continuous feedback on $\partial\mathscr{C}_0(\ell)$.

The control law in this proposition does not guarantee asymptotic stability. After a slight modification, we obtain a stabilizing continuous control law that achieves our goal.

Proof of Theorem 22. Given that the system has a bounded input there exists t_1 such that at time t_1 for all initial conditions in $\mathscr{K}_1$ and any input satisfying the constraint, we are guaranteed to be inside the set $\mathscr{R}_C(\Sigma_0, \mathscr{S})$. We consider the system from time t_1 onward.

Let f be the continuous controller given by (F.2) whose existence followed from Michael's theorem. Let $\rho > 0$ be such that $\mathscr{B}(0, \rho) \subset \mathscr{C}_0(\ell)$. Then it is readily verified that

$$\langle \xi, \eta \rangle \geqslant \rho, \quad \forall\, \xi \in \partial\mathscr{C}_0(\ell),\ \forall\, \eta \in N_{\mathscr{C}_0(\ell)}(\xi).$$

For any $M > 0$, choose δ such that $d(t)$ satisfies $\|\mathrm{d}(t)\| \leqslant \rho\kappa/4$ for all $t > t_1$. It also follows that for all $\xi \in \partial\mathscr{C}_0(\ell)$ and $\eta \in N_{\mathscr{C}_0(\ell)}(\xi)$:

$$\langle A_{\kappa/2}\xi + B_0 f(\xi), \eta \rangle < -\frac{\rho\kappa}{2}. \tag{F.3}$$

Since f is a continuous function defined on the compact set $\partial\mathscr{C}_0(\ell)$, there exists a differentiable function f_0 on $\partial\mathscr{C}_0(\ell)$ such that $\|B[f(x) - f_0(x)]\| < \rho\kappa/2$. Thus, by (4) f_0 satisfies for all $\xi \in \partial\mathscr{C}_0(\ell)$ and for all $\eta \in N_{\mathscr{C}_0(\ell)}(\xi)$:

$$\langle A_{\kappa/2}\xi + B_0 f_0(\xi), \eta \rangle \leqslant 0. \tag{F.4}$$

Next we extend the differentiable feedback f_0 defined on $\partial\mathscr{C}_0(\kappa)$ to a globally Lipschitz feedback f_1 defined on $\mathscr{C}_0(\ell)$. Define $\beta : \mathbb{R}^d \to \mathbb{R}_0^+$, where $d = \dim \xi$, as

$$\beta(\xi) := \inf\{\beta \geqslant 0 \mid \xi \in \beta\mathscr{C}_0(\ell)\}.$$

Clearly, $\xi \in \beta(\xi)\partial\mathscr{C}_0(\ell)$ for all $\xi \neq 0$. It is easily seen that the function β is Lipschitz and there exists $M > 0$ such that:

$$|\beta(\xi) - \beta(\xi')| \leqslant \beta(\xi - \xi') \leqslant M\|\xi - \xi'\|.$$

Define $f_1 : \mathbb{R}^d \to \mathbb{R}^m$ by

$$f_1(\xi) := \begin{cases} \beta(\xi) f_0(\xi/\beta(\xi)), & \xi \neq 0, \\ 0, & \xi = 0. \end{cases}$$

Since β is globally Lipschitz and f_0 differentiable, it is easily verified that f_1 is globally Lipschitz. Moreover, f_1 is positively homogeneous, i.e. $f_1(\gamma\xi) = \gamma f_1(\xi)$ for all $\xi \in \mathbb{R}^n$ and $\gamma > 0$, because $\beta(\gamma\xi) = \gamma(\beta\xi)$.

Noting that $f_1(\xi) = f_0(\xi)$ for $\xi \in \partial\mathscr{C}_0(\ell)$, utilizing (F.4), we find for all $\xi \in \partial\mathscr{C}_0(\ell)$ and $\eta \in N_{\mathscr{C}_0(l)}(\xi)$:

$$\langle A_{\kappa/4}\xi + B_0 f(\xi) + d, \eta \rangle \leqslant 0$$

for all $t \geqslant t_1$. Then from Nagumo's theorem we conclude that for all $\xi \in \mathscr{C}_0(\ell)$ the state $\xi(t)$ remains in $\overline{\mathscr{C}}_0(\ell)$ for all $t \geqslant t_1$. But if we apply the feedback $u = f_1(\xi)$ to system (11) with the same initial condition $\xi(0) \in \mathscr{C}_0(\ell)$ and let $\tilde{\xi}(t)$ be the solution of system (11), it is easy to see that

$$\tilde{\xi}(t) = \mathrm{e}^{-\kappa t/4}\xi(t),$$

where we used the property that f_1 is positive homogeneous. Since $\xi(t)$ remains in $\overline{\mathscr{C}}_0(\ell)$, a bounded set, we conclude that $\tilde{\xi}(t)$ converges to zero exponentially, which shows that $\mathscr{K}$ is a subset of the domain of attraction of system Σ_0. □

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