

Equivalence of switching linear systems by bisimulation

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A general notion of hybrid bisimulation is proposed for the class of switching linear systems. Connections between the notions of bisimulation-based equivalence, state-space equivalence, algebraic and input–output equivalence are investigated. An algebraic characterization of hybrid bisimulation and an algorithmic procedure converging in a finite number of steps to the maximal hybrid bisimulation are derived. Hybrid state space reduction is performed by hybrid bisimulation between the hybrid system and itself. By specializing the results obtained on bisimulation, also characterizations of simulation and abstraction are derived. Connections between observability, bisimulation-based reduction and simulation-based abstraction are studied.

1. Introduction

Hybrid systems have been the subject of intense research over the past few years because of their expressive power that is able to capture various non-smooth phenomena in diverse application areas. Furthermore, many interesting theoretical problems arise from the analysis and design of these systems. However in many situations the resulting hybrid systems are very complex, both in scale and in dynamical properties. Therefore the task of analysis and of synthesizing hybrid controllers to ensure some prescribed performances requirements is a complicated one, and formal methods for complexity reduction are essential for a feasible approach to the study of hybrid systems.

A powerful tool in this context, is the theory of bisimulation, introduced in the 1980s by Milner (1989) and Park (1981). This theory often provides an effective method for reducing the complexity of concurrent processes. The key idea in the notion of bisimulation is to find (and compute) an equivalence relation on the

class of hybrid systems under consideration that is preserving the properties of interest. Then reduction is performed by choosing the minimal (in the sense of size of state space) system in the sub-class of dynamical systems belonging to the same equivalence class of this bisimulation relation.

In the context of concurrent processes, the partition of the state space into equivalence classes induced by the bisimulation relation is in general finer than the partition induced by input–output equivalence or trace equivalence (Only for deterministic systems trace equivalence can be shown to imply bisimulation equivalence).

Also in system and control theory various equivalence notions have been formulated for continuous systems such as state-space and input–output equivalence. Developments in both areas have been rather independent; one of the reasons being that the employed mathematics are rather different. The rise of interest in hybrid systems has led to a reapproachment of the equivalence notions on concurrent processes developed in the computer science community with the equivalence notions for continuous systems in the control community.

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In particular, extensions of the notion of bisimulation to continuous systems have been explored before in Broucke (1998) and in a series of papers by Pappas and co-authors (e.g. Henzinger (1995), Lafferriere *et al.* (1998, 2000), Alur *et al.* (2000), Tabuada *et al.* (2002) and Pappas (2003, 2004)). The common denominator of those works is to associate a transition system with uncountable discrete state space to the continuous system under consideration, preserving reachability properties. Then, after reduction by bisimulation a simpler transition system is obtained, which hopefully has a finite discrete state space, thereby allowing the use of standard methods for analysis and verification of concurrent systems. This problem is not decidable in general as shown in Alur *et al.* (2000). However, by restricting the class of hybrid systems to timed automata, multirate automata, rectangular automata and O-minimal hybrid systems (Tabuada *et al.* 2002), the reduced transition system is finite. The continuous systems considered in Henzinger (1995), Lafferriere *et al.* (1998, 2000), Alur *et al.* (2000) and Tabuada *et al.* (2002) neither have continuous inputs or continuous outputs. Recently Pappas *et al.* proposed a new definition of bisimulation for linear (Pappas 2003, Tabuada *et al.* 2003,) and non-linear (Pappas 2004) systems that do include continuous control inputs. In particular Pappas (2003) derives for the case of linear control systems, interesting connections between the maximal bisimulation relation and the maximal controlled invariant subspace contained in the kernel of a given observation map, while Pappas (2004) shows how to reduce non-linear control systems to lower-dimensional systems by factoring out certain invariant distributions.

While in the previous approach the emphasis is on the preservation of the reachability properties, van der Schaft (2004a, b, c) focuses on external equivalence as the key property to be preserved in the definition of bisimulation. This latter approach is fundamental to a compositional modelling and control of hybrid systems as argued in van der Schaft and Schumacher (2001). In particular, it allows the design of a controller applied to the original continuous system on the basis of the reduced dynamical system. Furthermore in van der Schaft (2004a, c) it is shown that a linear deterministic dynamical system Σ is observable if and only if it equals the minimal system that is bisimilar to Σ , thus offering interesting links between the theory of bisimulation, the theory of realization, and the classical Kalman decomposition.

The aim of the present paper is to make another step in the rapprochement between the theory of concurrent processes and mathematical systems theory by defining and studying the notion of bisimulation for continuous-time switching linear systems. A switching linear system (SLS) (De Santis *et al.* 2003) can be

viewed as a combination of a discrete event (concurrent) system with linear continuous dynamical systems and includes as external variables both continuous inputs and outputs, discrete outputs, as well as hybrid (discrete and continuous) disturbance variables. These hybrid disturbances may be thought of as internal generators for non-determinism. The resulting class of systems are rather general since they may accept Zeno executions (Lygeros 1999), are non-deterministic, since a hybrid disturbance acts on the plant, and a reset map is defined over the set of continuous states.

Inspired by classical notions of bisimulation for concurrent processes (Clarke *et al.* 2002) and by the new notions introduced in (van der Schaft 2004a, c) for linear and non-linear dynamical systems, we propose a general notion of hybrid bisimulation for the class of switching linear systems. Given a pair of SLSs \mathbb{S}_1 and \mathbb{S}_2 , the proposed definition formalizes the intuitive idea of finding a relation $\mathcal{R} \subset \Xi_1 \times \Xi_2$ between the hybrid spaces Ξ_1 and Ξ_2 of \mathbb{S}_1 and \mathbb{S}_2 such that for any continuous input u and for any pair of initial conditions in \mathcal{R} , \mathcal{R} is invariant and the hybrid outputs coincide for any time and for a suitable choice of the hybrid disturbances. The proposed definition does not make use of the particular semantics of the class of SLSs involving only the notion of execution, and therefore may be also applied to more general hybrid systems where for example the discrete transitions depend on the continuous state x of the automata. A first step in this direction has been made in (van der Schaft 2004a, b). Moreover the definition applies to SLSs where multiple instantaneous jumps are allowed and where any execution χ_1 of an SLS \mathbb{S}_1 may be mimicked by an execution χ_2 of an SLS \mathbb{S}_2 and vice versa, while the jumps of χ_1 and χ_2 may be asynchronous (compare with the definition employed in (van der Schaft 2004a, b)). The proposed definition seems particularly appealing since it has clear links to well-known notions of algebraic, state-space and input–output equivalences (Callier and Desoer 1991) for dynamical systems. In fact bisimulation-based equivalence will be shown to be implied by algebraic equivalence, while implying state-space equivalence and input–output equivalence. Summarizing, we will prove the following sequence of implications:

$$\begin{aligned} \text{Algebraic E.} &\Rightarrow \text{Bisimilarity} \Rightarrow \text{State – Space E.} \\ &\Rightarrow \text{Input – Output E.} \end{aligned}$$

The proposed equivalence notions, except for algebraic equivalence, all coincide for the class of deterministic SLSs (no hybrid disturbances). By combining tools from concurrent systems analysis and from geometric control theory, a complete geometric characterization of hybrid bisimulations is derived. Moreover, inspired

by De Santis *et al.* (2004a), an algorithmic procedure converging in a finite number of steps to the maximal hybrid bisimulation is developed. This yields the construction of the minimal-dimensional SLS bisimilar to the original one. By specializing those results we propose a new definition of simulation that, roughly speaking, is a one-sided version of the notion of bisimulation, leading to a notion of abstraction for the class of SLSs. Inspired by the connections between bisimulation and observability for linear continuous systems (van der Schaft 2004a, b), we also investigate this issue in the context of SLSs. In view of the results developed in De Santis *et al.* (2003), that characterize observability of SLSs, we show that while an observable SLS cannot be reduced anymore by bisimulation, the converse implication is not true. More precisely, an example is provided where an SLS that cannot be reduced by bisimulation, is unobservable. Finally we use simulation-based abstraction, instead of bisimulation-based reduction for ‘extracting’ the observable sub-SLS of a given unobservable SLS.

Most of the results presented in this paper also hold for the class of discrete-time switching linear systems. Moreover even if most of the bisimulation analysis will be carried out in the context of switching linear systems, it may be naturally extended to the case of switching non-linear systems, in view of the results of van der Schaft (2004c), and to the case of switching systems with continuous dynamics given in pencil form in view of the results developed in van der Schaft (2004b), as will be briefly discussed in the end of the paper.

A preliminary and partial version of this paper has been published as (Pola *et al.* 2004).

The paper is organized as follows. In §2, a general notion of hybrid bisimulation is proposed and characterized, and relations to equivalence notions are given. In §3, an algorithmic procedure converging in a finite number of steps to the maximal hybrid bisimulation is proposed. In §4, reduction via bisimulation is studied. §5 is devoted to the definition and characterization of simulation and abstraction. Section 6 addresses the issue of connections between observability, reduction via bisimulation, and abstraction via simulation. §7 offers some guidelines for extending the results obtained for switching linear systems to the case of switching non-linear systems and switching systems in pencil form. The Appendix contains some technical proofs. Finally, §8 offers some concluding remarks.

2. Bisimilar switching linear systems

2.1. Preliminaries and basic definitions

Switching systems are an important subclass of hybrid systems that have been extensively studied in the past

few years (e.g. Wicks *et al.* (1998), Zefran and Burdick (1998), Dayawansa and Martin (1999), De Santis *et al.* (2003, 2004b, 2005) and Pola *et al.* (2004)). The motivation for considering this particular subclass of hybrid systems lies in their semantics that is able to capture different non-smooth phenomena, arising in the area of mechanical systems, power train control, aircraft and air traffic control, switching power converters and many other fields (e.g. Wicks *et al.* (1998), Zefran and Burdick (1998), Dayawansa and Martin (1999), de Santis *et al.* (2005) and the references therein). For example in (De Santis *et al.* 2005), an automotive engine in idle mode is modelled by means of a suitable switching system, where times of switchings between the different modes are lower bounded and upper bounded by two positive real parameters that depend themselves on the engine parameters. The hybrid state ξ of a switching linear system is composed of two components: the discrete state q belonging to a finite set Q and the continuous state x belonging to a linear space $X(q)$ depending on q . The evolution of the discrete state is governed by a discrete disturbance v that acts on the plant, while the evolution of x is given by a non-deterministic linear dynamical control system whose matrices depend on the current discrete state q . Whenever a discrete transition holds, the continuous state is instantly reset to a new value by means of a reset matrix depending on the discrete states before and after the transition. Moreover switching linear systems are characterized by a hybrid output y associated to the hybrid state ξ and allow multiple instantaneous transitions on the discrete states. The formal model of a switching linear system is given in the following definition that is based on (De Santis *et al.* 2003), following the general model of hybrid automata, see e.g. (Tomlin *et al.* 1998) and (Lygeros *et al.* 1999).

Definition 1: A Switching Linear System (SLS) \mathbb{S} is a tuple $(\Xi, U, D, Y, \Sigma, \gamma, E, M)$ where

- $\Xi = \cup_{q \in Q} \{q\} \times X(q)$ is the hybrid state space, where
 - $Q = \{q_1, q_2, \dots, q_{N_1}\}$ is the discrete state space, $N_1 \in \mathbb{N}$,
 - $\dim: Q \rightarrow \mathbb{N}$,
 - $\forall q \in Q, X(q) \subset \mathbb{R}^{\dim(q)}$ is the linear continuous state space.
- U is the linear input space.
- $D = V \times W$ is the hybrid disturbance space, where
 - $V = \{v_1, v_2, \dots, v_{N_2}\}$ is the set of the discrete disturbances, $N_2 \in \mathbb{N}$,
 - W is the linear continuous disturbance space.
- $Y = P \times H$ is the hybrid output space, where
 - $P = \{p_1, p_2, \dots, p_{N_3}\}$ is the discrete output space, $N_3 \in \mathbb{N}$,
 - H is the linear continuous output space.

- Σ is a function that associates to any discrete state $q \in Q$, the linear dynamical system

$$\Sigma(q): \begin{cases} \dot{x}(t) = A(q)x(t) + B(q)u(t) + G(q)w(t), \\ h(t) = C(q)x(t), \quad t \geq 0. \end{cases}$$

- $\gamma: Q \rightarrow P$ associates a discrete output to each discrete state.
- $E \subset Q \times V \times Q$ is a collection of discrete transitions.
- M is a function that associates to any $e = (q, v, q') \in E$, the reset matrix $M(e) \in \mathbb{R}^{\dim(q) \times \dim(q')}$.

Any element $\xi = (q, x) \in \Xi$ is called hybrid state, any element $d = (v, w) \in D$ is called hybrid disturbance and any element $y = (p, h) \in Y$ is called hybrid output of \mathbb{S} . Given an SLS \mathbb{S} , the tuple $\mathbb{D}_{\mathbb{S}} = (Q, P, V, E, \gamma)$ can be viewed as a Discrete Event System (DES) (Hopcroft and Ullman 1979), having state space Q , output set P , input set V , transition relation E and output function γ . The set $\text{succ}(q)$ is composed by the successors of the discrete state $q \in Q$, i.e. $\text{succ}(q) = \{q' \in Q \mid \exists v \in V: (q, v, q') \in E\}$. Given a set $Z \subset Z_1 \times Z_2$, the operator $\Pi|_{Z_i}(Z)$ is the projection of the set Z onto Z_i , $i = 1, 2$.

We now formally define the semantics of switching linear systems. First of all we assume throughout the paper that the hybrid disturbance is not available for measurements, thus yielding a non-deterministic system.

The discrete transitions in the class of SLSs are determined by the discrete disturbance v . We assume that multiple events (instantaneous transitions) are allowed. This can be formalized by using the notion of hybrid time basis proposed in (Lygeros *et al.* 1999). We recall that a hybrid time basis τ is an infinite or finite sequence of time intervals I_j , $j \in \{0, 1, \dots, J\}$, i.e. $\tau = \{I_j\}_{j=0}^J$, $J \in \mathbb{N} \cup \{\infty\}$, satisfying the following conditions:

- $I_j = \{t \in \mathbb{R}_0^+ : t_j \leq t \leq t'_j\}$, if $j < J$;
- I_J may be of the form $\{t \in \mathbb{R}_0^+ : t_J \leq t \leq t'_j\}$ or of the form $\{t \in \mathbb{R}_0^+ : t_J \leq t < \infty\}$ and $t'_j = \infty$;
- for all j , $t_j \leq t'_j$ and for $j > 0$, $t_j = t'_{j-1}$.

Denote by \mathcal{T} the set of all hybrid time bases. Since the SLSs under consideration are time-invariant continuous systems, there is no loss of generality in assuming $t_0 = 0$, for all $\tau \in \mathcal{T}$. Given a hybrid time basis $\tau \in \mathcal{T}$, denote by $[\tau] := \cup_{I_j \in \tau} I_j \times \{j\}$ the set of all hybrid times (t, j) , $t \in I_j$, $I_j \in \tau$, and define the ordering relation \leq on $[\tau]$ such that $(t, j) \leq (t', j')$ if $t \leq t'$ and $j \leq j'$. Given two hybrid time bases $\tau_1, \tau_2 \in \mathcal{T}$, such that $\sup\{t : (t, j) \in [\tau_1]\} = \sup\{t : (t, j) \in [\tau_2]\}$ denote by $[\tau_1, \tau_2] \subset [\tau_1] \times [\tau_2]$ any relation, satisfying the following conditions:

- $\Pi|_{[\tau_i]}([\tau_1, \tau_2]) = [\tau_i]$, $i = 1, 2$;
- $\forall ((t_a, j_a), (t_b, j_b)) \in [\tau_1, \tau_2]$, $t_a = t_b$;

- $\forall ((t_a, j_a), (t_b, j_b)), ((t'_a, j'_a), (t'_b, j'_b)) \in [\tau_1, \tau_2]$, if $(t_a, j_a) \leq (t'_a, j'_a)$ then $(t_b, j_b) \leq (t'_b, j'_b)$ and vice versa.

Remark 1: The last condition ensures that any relation $[\tau_1, \tau_2]$ preserves the ordering relation \leq in every hybrid time basis $[\tau_1]$ and $[\tau_2]$.

Given two sets Z_1, Z_2 , denote by $\mathcal{C}(Z_1, Z_2)$ and by $\mathcal{C}^0(Z_1, Z_2)$ the class of functions, respectively of piecewise continuous functions $z: Z_1 \rightarrow Z_2$. The switching system temporal evolution is then defined by means of the notion of execution.

Definition 2: An execution χ of an SLS \mathbb{S} is a collection $(\xi_0, \tau, u, d, \xi, y)$ with $\xi_0 = (q_0, x_0) \in \Xi$, $\tau \in \mathcal{T}$, $u \in \mathcal{C}^0(\mathbb{R}_0^+, U)$, $d = (v, w)$, where $v \in \mathcal{C}(\mathbb{N}, V)$, $w \in \mathcal{C}^0(\mathbb{R}_0^+, W)$, $\xi = (q, x)$, where $q \in \mathcal{C}(\mathbb{N}, Q)$, $x \in \mathcal{C}(\mathbb{R}_0^+ \times \mathbb{N}, X(\cdot))$, $y = (p, h)$, where $p \in \mathcal{C}(\mathbb{N}, P)$, $h \in \mathcal{C}(\mathbb{R}_0^+ \times \mathbb{N}, H)$, such that, by setting $\xi(t, j) = (q(j), x(t, j))$ and $y(t, j) = (p(j), h(t, j))$, for all $(t, j) \in [\tau]$, the following conditions are satisfied:

- Discrete evolution: $q(j) = q_0$; $q(j+1)$ is such that $e_j = (q(j), v(j+1), q(j+1)) \in E$; $p(j) = \gamma(q(j))$;
- Continuous evolution: $x(t_0, 0) = x_0$, $x(t_{j+1}, j+1) = M(e_{j+1})x(t'_j, j)$; moreover $x(t, j)$ and $h(t, j)$, for all $t \in I_j$ are respectively the unique solution and output at time t of the dynamical system $\Sigma(q(j))$, with initial state $x(t_j, j)$, initial time t_j , input function u and disturbance function w .

Switching linear systems are non-deterministic since a hybrid disturbance acts on the plant: note that even if the discrete disturbance would be measured then still the dynamics of the discrete variables could be non-deterministic.

We now introduce some equivalence notions, borrowed from the theory of concurrent processes, for the class of SLSs. In particular we consider the notions of bisimulation and simulation for SLSs. The proposed definitions are obtained by merging the classical notions for concurrent processes (e.g. Clarke *et al.* (2002)) with new definitions introduced in van der Schaft (2004a, c) for the classes of linear and non-linear continuous systems.

Definition 3: Consider two SLSs $\mathbb{S}_i = (\Xi_i, U_i, D_i, Y_i, \Sigma_i, \gamma_i, E_i, M_i)$, $i = 1, 2$ such that $U_1 = U_2$. A hybrid bisimulation between \mathbb{S}_1 and \mathbb{S}_2 is a subset $\mathcal{R} \subset \Xi_1 \times \Xi_2$ satisfying the following property. Take any $(\xi_{10}, \xi_{20}) \in \mathcal{R}$ and any input function $u_1 = u_2$. Then for any hybrid disturbance d_1 and for any execution $\chi_1 = (\xi_{10}, \tau_1, u_1, d_1, \xi_1, y_1)$ of \mathbb{S}_1 , there should exist a hybrid disturbance d_2 and an execution $\chi_2 = (\xi_{20}, \tau_2, u_2, d_2, \xi_2, y_2)$ of \mathbb{S}_2 satisfying the following conditions:

- $(\xi_1(t, j), \xi_2(t', j')) \in \mathcal{R}$,
- $y_1(t, j) = y_2(t', j')$,

$\forall((t,j),(t',j')) \in [\tau_1, \tau_2]$, for some $[\tau_1, \tau_2]$. Moreover the same holds with d_1 replaced by d_2 and vice versa.

Remark 2: The introduction of the relation $[\tau_1, \tau_2]$ allows the comparison of the hybrid state and output time-evolutions of the executions χ_1 and χ_2 with different hybrid time bases τ_1 and τ_2 , thus generalizing the definition of structural hybrid bisimulation introduced in (van der Schaft 2004a, b), where $\tau_1 = \tau_2$ was required, and where multiple instantaneous jumps were not considered.

Definition 4: Two SLSs \mathbb{S}_1 and \mathbb{S}_2 are bisimilar, and we write $\mathbb{S}_1 \sim \mathbb{S}_2$, if there exists a hybrid bisimulation $\mathcal{R} \subset \Xi_1 \times \Xi_2$ such that the projection of \mathcal{R} on each hybrid space equals this hybrid space, i.e.

$$\Pi|_{\Xi_i}(\mathcal{R}) = \Xi_i, \quad i = 1, 2. \quad (1)$$

Remark 3: The notion of bisimulation equivalence in the context of concurrent processes is usually given with respect to an initial state; on the contrary Definition 4 is given with respect to any hybrid initial state. The generalization to subsets of initial conditions $\Xi_i^0 \subset \Xi_i$, $i = 1, 2$ obviously can be done by relaxing (1) to $\Pi|_{\Xi_i}(\mathcal{R}) = \Xi_i^0$, $i = 1, 2$.

We recall that, given a set Z , a set $R \subset Z \times Z$ is an equivalence relation on Z if it is reflexive, i.e. $\forall z \in Z, (z, z) \in R$, symmetric, i.e. $\forall (z_1, z_2) \in R, (z_2, z_1) \in R$, and transitive, $\forall (z_1, z_2), (z_2, z_3) \in R, (z_1, z_3) \in R$. Bisimilarity between SLSs is an equivalence relation on the space of SLSs. A weaker notion than bisimulation is the so-called notion of simulation, as formalized hereafter.

Definition 5: Consider two SLSs $\mathbb{S}_i = (\Xi_i, U_i, D_i, Y_i, \Sigma_i, \gamma_i, E_i, M_i)$, $i = 1, 2$ such that $U_1 \subset U_2$. A hybrid simulation of \mathbb{S}_1 by \mathbb{S}_2 is a subset $\mathcal{R} \subset \Xi_1 \times \Xi_2$ satisfying the following property. Take any $(\xi_{10}, \xi_{20}) \in \mathcal{R}$ and any input function $u_1 = u_2$. Then for any hybrid disturbance d_1 and for any execution $\chi_1 = (\xi_{10}, \tau_1, u_1, d_1, \xi_1, y_1)$ of \mathbb{S}_1 , there should exist a hybrid disturbance d_2 and an execution $\chi_2 = (\xi_{20}, \tau_2, u_2, d_2, \xi_2, y_2)$ of \mathbb{S}_2 satisfying the following conditions:

- (i) $(\xi_1(t, j), \xi_2(t', j')) \in \mathcal{R}$;
- (ii) $y_1(t, j) = y_2(t', j')$;

$\forall((t,j),(t',j')) \in [\tau_1, \tau_2]$, for some $[\tau_1, \tau_2]$.

Definition 6: An SLS \mathbb{S}_1 is simulated by an SLS \mathbb{S}_2 (or equivalently \mathbb{S}_2 simulates \mathbb{S}_1), and we write $\mathbb{S}_1 \preceq \mathbb{S}_2$, if there exists a hybrid simulation $\mathcal{R} \subset \Xi_1 \times \Xi_2$ of \mathbb{S}_1 by \mathbb{S}_2 such that the projection of \mathcal{R} along the hybrid space Ξ_1 coincides with Ξ_1 , i.e. $\Pi|_{\Xi_1}(\mathcal{R}) = \Xi_1$.

Hybrid simulation is reflexive, transitive but not symmetric, and hence it is not an equivalence relation on the space of SLSs. An equivalence notion based on hybrid simulations can be stated as follows.

Definition 7: Two SLSs \mathbb{S}_1 and \mathbb{S}_2 are similar if $\mathbb{S}_1 \preceq \mathbb{S}_2$ and $\mathbb{S}_2 \preceq \mathbb{S}_1$.

2.2. Equivalent switching linear systems

Equivalence notions in the control theory usually deal with the characterization of whether two dynamical systems are state-space equivalent or input–output equivalent. Aim of this section is to define those notions for the class of SLSs and then to compare them with the notions of bisimilarity and similarity. The following definition extends to the class of SLSs well known concepts of algebraic, state-space and input–output equivalence for linear continuous systems (Callier and Desoer 1991) and DESs (Clarke *et al.* 2002).

Definition 8: Two SLSs \mathbb{S}_1 and \mathbb{S}_2 are algebraically equivalent if there exists an invertible mapping $T_{\mathbb{Q}}: Q_1 \rightarrow Q_2$ and for any $q_1 \in Q_1$, invertible linear mappings $T_{q_1}: X_1(q_1) \rightarrow X_2(T_{\mathbb{Q}}(q_1))$ such that,

- (i) $\gamma_1(q_1) = \gamma_2(T_{\mathbb{Q}}(q_1))$, for any $q_1 \in Q_1$;
- (ii) for any $e_1 = (q_1, v_1, q'_1) \in E_1$, there exists $v_2 \in V_2$ such that $e_2 = (T_{\mathbb{Q}}(q_1), v_2, T_{\mathbb{Q}}(q'_1)) \in E_2$ and vice versa;
- (iii) for any $q_1 \in Q_1$, the dynamical systems $\Sigma_1(q_1)$ and $\Sigma_2(T_{\mathbb{Q}}(q_1))$ are algebraically equivalent (Callier and Desoer 1991) with transformation matrix T_{q_1} , i.e.

$$A_1(q_1) = T_{q_1} A_2(T_{\mathbb{Q}}(q_1)) T_{q_1}^{-1},$$

$$B_1(q_1) = T_{q_1} B_2(T_{\mathbb{Q}}(q_1)),$$

$$G_1(q_1) = T_{q_1} G_2(T_{\mathbb{Q}}(q_1)),$$

$$C_1(q_1) = C_2(T_{\mathbb{Q}}(q_1)) T_{q_1}^{-1},$$

- (iv) $M_1(e_1) = T_{q'_1}^{-1} M_2(e_2) T_{q_1}$, for any $e_1 = (q_1, v_1, q'_1) \in E_1$, where $e_2 = (T_{\mathbb{Q}}(q_1), v_2, T_{\mathbb{Q}}(q'_1)) \in E_2$, for some $v_2 \in V_2$, and vice versa.

A notion of equivalence that is less conservative than algebraic equivalence is the notion of state-space equivalence as formalized below. This can be obtained by generalizing the corresponding notion given in (Callier and Desoer 1991) for linear continuous systems.

Definition 9: Let \mathbb{S}_1 and \mathbb{S}_2 be two (not necessarily distinct) SLSs with the same continuous control space, i.e. $U_1 = U_2$. Two hybrid states $\xi_{10} \in \Xi_1$ and $\xi_{20} \in \Xi_2$ are said to be state-equivalent if for any given input u , for any hybrid disturbance d_1 , for any execution $\chi_1 = (\xi_{10}, \tau_1, u, d_1, \xi_1, y_1)$ of \mathbb{S}_1 , there exists a hybrid

disturbance d_2 and an execution $\chi_2 = (\xi_{20}, \tau_2, u, d_2, \xi_2, y_2)$ of \mathbb{S}_2 such that $y_1(t, j) = y_2(t', j')$, $\forall ((t, j), (t', j')) \in [\tau_1, \tau_2]$, for some $[\tau_1, \tau_2]$. Moreover the same holds with d_1 replaced by d_2 and vice versa.

Definition 10: Two SLSs \mathbb{S}_1 and \mathbb{S}_2 are state-space equivalent if they have the same continuous input space, i.e. $U_1 = U_2$ and for any hybrid state $\xi_1 \in \Xi_1$ of \mathbb{S}_1 , there exists a hybrid state $\xi_2 \in \Xi_2$ of \mathbb{S}_2 that is equivalent to ξ_1 , and vice versa.

The following result highlights the connection between state-space equivalence and bisimilarity.

Lemma 1: Two SLSs \mathbb{S}_1 and \mathbb{S}_2 are state-space equivalent if and only if there exists a relation $\mathcal{R} \subset \Xi_1 \times \Xi_2$ such that $\Pi|_{\Xi_i}(\mathcal{R}) = \Xi_i$, $i = 1, 2$ and such that for any $(\xi_{10}, \xi_{20}) \in \mathcal{R}$, ξ_{10} and ξ_{20} are state-equivalent.

Another important equivalence notion in the context of control theory is the notion of input–output equivalence as given below.

Definition 11: Two SLSs \mathbb{S}_1 and \mathbb{S}_2 are input–output equivalent if $U_1 = U_2$ and for any $\xi_{10} \in \Xi_1$, for any given control law $u \in U_1$, for any hybrid disturbance d_1 , for any execution $\chi_1 = (\xi_{10}, \tau_1, u, d_1, \xi_1, y_1)$ of \mathbb{S}_1 , there exists $\xi_{20} \in \Xi_2$, a hybrid disturbance d_2 and an execution $\chi_2 = (\xi_{20}, \tau_2, u, d_2, \xi_2, y_2)$ of \mathbb{S}_2 such that $y_1(t, j) = y_2(t', j')$, $\forall ((t, j), (t', j')) \in [\tau_1, \tau_2]$, for some $[\tau_1, \tau_2]$. Moreover the same holds with ξ_{10} and d_1 replaced by ξ_{20} and d_2 and vice versa.

The above notions naturally specialize to the context of DESs (Clarke *et al.* 2002) and linear dynamical systems (Callier and Desoer 1991, van der Schaft 2004c). In particular in the context of DESs, algebraic equivalence is usually known as *graph isomorphism* and input–output equivalence as equivalence of the generated language. Moreover those notions easily extend to more general hybrid systems models.

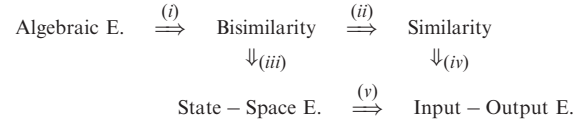
Algebraic, state-space and input–output equivalences are equivalence relations on the space of SLSs. Hereafter we provide some results that offer a ‘bridge’ between the control systems equivalence notions introduced above and the concurrent process equivalence notions introduced in the previous section.

Theorem 1: *The following statements are true:*

- (i) *Two algebraically equivalent SLSs are bisimilar;*
- (ii) *Two bisimilar SLSs are similar;*
- (iii) *Two bisimilar SLSs are state-space equivalent;*
- (iv) *Two similar SLSs are input–output equivalent;*
- (v) *Two state-space equivalent SLSs are input–output equivalent.*

Proof: See the Appendix. \square

The following picture highlights the relationships between the equivalence notions introduced so far:



Remark 4: Some remarks about the converse of the implications in the picture above are listed below.

- The converse implication of (i) is in general not true neither for the class of DESs nor for the class of linear continuous systems. Hence it is not true for the class of SLSs.
- The converse implication of (ii) is true for the class of linear dynamical systems (as shown in Proposition 5.3 of (van der Schaft 2004a) but in general not true for the class of DESs (as shown for example in (Clarke *et al.* 2002)), and hence it is not true for the class of SLSs.
- The converse implication of (iv) is not true for the class of linear dynamical systems (as shown in Example 1 of (van der Schaft 2004a)) and hence it is not true for the class of SLSs.
- The converse implication of (v) is in general not true neither for the class of DESs (as shown in (Clarke *et al.* 2002) nor for the class of linear dynamical systems (as shown in Example 1 of van der Schaft 2004a) and hence for the class of SLSs.

Remark 5: It is interesting to check those implications in the case of deterministic SLSs. An SLS \mathbb{S} is said to be deterministic if no discrete and continuous disturbances act on the plant, i.e. $V = \emptyset$, $W = \{0\}$. In this special case the converse of the implications (ii), (iii), (iv), (v) are true and hence bisimilarity, similarity, state space and input–output equivalences are equivalent notions. In the case of labelled transition systems the same reasoning applies: equivalence relations such as bisimulation, simulation and trace equivalences all coincide.

2.3. Characterizing hybrid bisimulations of SLSs

Definition 3 of hybrid bisimulation is general enough to be applied to more general hybrid systems than SLSs and in fact a first step in this direction has been done in (van der Schaft 2004a) and (van der Schaft 2004b). On the other hand for the class of SLSs a complete geometric characterization may be developed as this section shows. The semantics of tuples formally defining SLSs naturally induces a particular geometrical structure for a set to be a hybrid bisimulation, as the following result shows.

Proposition 1: *If \mathcal{R} is a hybrid bisimulation between two SLSs \mathbb{S}_1 and \mathbb{S}_2 , there exists $Q_{\mathcal{R}} \subset Q_1 \times Q_2$ and for any $(q_1, q_2) \in Q_{\mathcal{R}}$ suitable sets $\mathcal{R}(q_1, q_2) \subset X_1(q_1) \times X_2(q_2)$ such that*

$$\begin{aligned} ((q_1, x_1), (q_2, x_2)) \in \mathcal{R} &\iff (q_1, q_2) \in Q_{\mathcal{R}} \quad \text{and} \\ (x_1, x_2) &\in \mathcal{R}(q_1, q_2). \end{aligned}$$

Proof: The proof follows directly from the definition of \mathcal{R} being a subset of $\Xi_1 \times \Xi_2$, and by the definition of the hybrid spaces Ξ_1 and Ξ_2 . \square

By the result above, any hybrid bisimulation \mathcal{R} between two SLSs \mathbb{S}_1 and \mathbb{S}_2 can be represented as

$$\mathcal{R} = \left\{ ((q_1, x_1), (q_2, x_2)) \in \Xi_1 \times \Xi_2 \mid (q_1, q_2) \in Q_{\mathcal{R}}, \right. \\ \left. (x_1, x_2) \in \mathcal{R}(q_1, q_2) \right\}. \quad (2)$$

Moreover the linearity in the continuous dynamics and in the continuous variables spaces, leads to a particular structure for the set $\mathcal{R}(q_1, q_2)$ for any fixed pair $(q_1, q_2) \in Q_{\mathcal{R}}$. More precisely we now show that if \mathcal{R} is a hybrid bisimulation between two SLSs \mathbb{S}_1 and \mathbb{S}_2 , then the linear closure $\mathcal{L}(\mathcal{R})$ of \mathcal{R} is a hybrid bisimulation between \mathbb{S}_1 and \mathbb{S}_2 .

More formally, given a hybrid bisimulation \mathcal{R} as in (2), between two SLSs \mathbb{S}_1 and \mathbb{S}_2 , define the linear closure $\mathcal{L}(\mathcal{R})$ of \mathcal{R} as

$$\mathcal{L}(\mathcal{R}) = \left\{ ((q_1, x_1), (q_2, x_2)) \in \Xi_1 \times \Xi_2 \mid (q_1, q_2) \in Q_{\mathcal{R}}, \right. \\ \left. (x_1, x_2) \in \mathcal{L}(\mathcal{R}(q_1, q_2)) \right\},$$

where for any $(q_1, q_2) \in Q_{\mathcal{R}}$, $\mathcal{L}(\mathcal{R}(q_1, q_2))$ is the linear closure (Kelley and Namioka 1963) of $\mathcal{R}(q_1, q_2)$, i.e.

$$\mathcal{L}(\mathcal{R}(q_1, q_2)) = \left\{ \lambda^a \cdot (x_1^a, x_2^a) + \lambda^b \cdot (x_1^b, x_2^b), \quad \forall \lambda^a, \right. \\ \left. \lambda^b \in \mathbb{R}, \forall (x_1^a, x_2^a), (x_1^b, x_2^b) \in \mathcal{R}(q_1, q_2) \right\}.$$

By definition of $\mathcal{L}(\mathcal{R})$, $\mathcal{R} \subset \mathcal{L}(\mathcal{R})$; moreover the following result holds.

Proposition 2: *If \mathcal{R} is a hybrid bisimulation between two SLSs \mathbb{S}_1 and \mathbb{S}_2 , then $\mathcal{L}(\mathcal{R})$ is a hybrid bisimulation between \mathbb{S}_1 and \mathbb{S}_2 .*

Proof: See the Appendix. \square

By Propositions 1 and 2, from now on, any hybrid bisimulation \mathcal{R} between two SLSs \mathbb{S}_1 and \mathbb{S}_2 can be represented by (2) where $\mathcal{R}(q_1, q_2)$ is implicitly assumed to be a linear subspace of $X_1(q_1) \times X_2(q_2)$ for any $(q_1, q_2) \in Q_{\mathcal{R}}$.

The following result gives an algebraic characterization of hybrid bisimulations for SLSs.

Theorem 2: *Given two SLSs \mathbb{S}_1 and \mathbb{S}_2 , a set \mathcal{R} of the form (2) is a hybrid bisimulation between \mathbb{S}_1 and \mathbb{S}_2 if and only if for any $(q_1, q_2) \in Q_{\mathcal{R}}$ the following property holds:*

$\forall q'_1 \in \text{succ}(q_1), \exists q'_2 \in \text{succ}(q_2) \cup \{q_2\} : (q'_1, q'_2) \in Q_{\mathcal{R}}$ and

- (i) $\gamma_1(q_1) = \gamma_2(q_2)$ and $\mathcal{R}(q_1, q_2)$ is a bisimulation relation between $\Sigma_1(q_1)$ and $\Sigma_2(q_2)$;
- (ii) $\text{diag}(M_1(e_1), \bar{M}_2)\mathcal{R}(q_1, q_2) \subset \mathcal{R}(q'_1, q'_2)$, where $e_1 \in E_1$ takes q_1 into q'_1 and $e_2 \in E_2$ takes q_2 into q'_2 , and $\bar{M}_2 = M_2(e_2)$ if $q'_2 \neq q_2$, $\bar{M}_2 = I$ if $q'_2 = q_2$;

and vice versa, $\forall q'_2 \in \text{succ}(q_2), \exists q'_1 \in \text{succ}(q_1) \cup \{q_1\} : (q'_1, q'_2) \in Q_{\mathcal{R}}$ and conditions (i) and (ii) are satisfied.

Proof: (Sufficiency.) Theorem 2 (i) ensures that for any $(\xi_{10}, \xi_{20}) \in \mathcal{R}$ and any input function $u_1 = u_2$ for any hybrid disturbance d_1 and for any execution $\chi_1 = (\xi_{10}, \tau_1, u_1, d_1, \xi_1, \gamma_1)$ of \mathbb{S}_1 , there exists a hybrid disturbance d_2 and an execution $\chi_2 = (\xi_{20}, \tau_2, u_2, d_2, \xi_2, \gamma_2)$ of \mathbb{S}_2 , such that Definition 3 (i) and (ii) are satisfied for any $(t, 0) \in [\tau_1]$. Finally Theorem 2 (ii) ensures that once the switching has occurred in \mathbb{S}_1 from some (q_1, x_1) to some (q'_1, x'_1) and in \mathbb{S}_2 from some (q_2, x_2) to some (q'_2, x'_2) the pair of continuous states $(x'_1, x'_2) \in \mathcal{R}(q'_1, q'_2)$. Hence by induction the result follows.

(Necessity.) Suppose that \mathcal{R} is a hybrid bisimulation of the form (2) between \mathbb{S}_1 and \mathbb{S}_2 . Necessity of Theorem 2 (i) is obvious and can be proved by contradiction. As far for the necessity of Theorem 2 (ii), for any $((q_{10}, x_{10}), (q_{20}, x_{20})) \in \mathcal{R}$, for any $u_1 = u_2$ for any hybrid disturbance d_1 , consider $\tau_1 = \{I_0, I_1\}$, where $I_0 = \{t_0\} = \{t'_0\}$ and any execution $\chi_1 = (\xi_{10}, \tau_1, u_1, d_1, \xi_1, \gamma_1)$ of \mathbb{S}_1 : since \mathcal{R} is a hybrid bisimulation between \mathbb{S}_1 and \mathbb{S}_2 , there exists a hybrid disturbance d_2 and an execution $\chi_2 = (\xi_{20}, \tau_2, u_2, d_2, \xi_2, \gamma_2)$ of \mathbb{S}_2 such that Definition 3 (i) is satisfied $\forall ((t, j), (t', j')) \in [\tau_1, \tau_2]$, for some $[\tau_1, \tau_2]$. In particular by writing Definition 3 (i) in $t = t_1$, we have

$$(M_1(e_1)x_{10}, \bar{M}_2x_{20}) \in \mathcal{R}(q'_1, q'_2). \quad (3)$$

Since condition (3) is true for any $(x_{10}, x_{20}) \in \mathcal{R}(q_{10}, q_{20})$, then Theorem 2 (ii) is satisfied for any fixed $(q_{10}, q_{20}) \in Q_{\mathcal{R}}$. By repeating the same proof replacing q_{10} with q_{20} and vice versa, Theorem 2 (ii) is satisfied. \square

Remark 6: Note that in the result above we do not assume that $\mathcal{R} = \mathcal{L}(\mathcal{R})$ and hence Theorem 2 holds also for hybrid bisimulations \mathcal{R} for which $\mathcal{R} \neq \mathcal{L}(\mathcal{R})$.

Remark 7: Conditions outlined in Theorem 2 are sufficient for characterizing hybrid bisimulations for more general hybrid systems whose discrete transitions depend on the continuous state x .

By the result above, we may suppose w.l.o.g. that any hybrid bisimulation satisfies conditions of Theorem 2. Moreover a direct consequence of Theorem 2 is given in the following.

Corollary 1: *Two SLSs are bisimilar if and only if there exists a set $\mathcal{R} \subset \Xi_1 \times \Xi_2$ of the form (2) satisfying conditions of Theorem 2 and such that $\Pi|_{\Xi_i}(\mathcal{R}) = \Xi_i$, $i = 1, 2$.*

In the next section we will show how to check the conditions of the above result in a finite number of steps and hence how to check bisimilarity of a pair of SLSs.

If \mathbb{S}_1 and \mathbb{S}_2 are bisimilar with hybrid bisimulation \mathcal{R} , the dynamical systems $\Sigma(q_1)$ and $\Sigma(q_2)$ associated with any pair $(q_1, q_2) \in \mathcal{Q}_{\mathcal{R}}$ are not necessarily bisimilar as the following example shows.

Example 1: Let us consider a pair of SLS \mathbb{S}_1 and \mathbb{S}_2 whose DESs $\mathbb{D}_{\mathbb{S}_1} = \mathbb{D}_1$ and $\mathbb{D}_{\mathbb{S}_2} = \mathbb{D}_2$ are depicted in figure 1 and where

$$M_1(q, v, q') = \begin{cases} [I & 0] & \text{if } q = q_2, q' = q_3, \\ [I^T & 0^T]^T & \text{if } q = q_3, q' = q_2, \\ I & \text{otherwise,} \end{cases}$$

and $M_2(e) = I$ for any $e \in E_2$. Suppose that $\Sigma_1(q_1) = \Sigma_1(q_2) = \Sigma_2(q_4) = \Sigma_2(q_5)$ and $A(q_4) = \text{diag}\{A(q_3), A\}$, $B(q_4) = [B(q_3)^T \ B^T]^T$, $C(q_3) = [C(q_1) \ C]$, for some matrices A , B and C of appropriate dimensions. Consider a set \mathcal{R} of the form (2), where

$$\begin{aligned} \mathcal{Q}_{\mathcal{R}} &= \{(q_1, q_4), (q_2, q_5), (q_3, q_4)\}, \\ \mathcal{R}(q_1, q_4) &= \{(x_1, x_4) : x_1 = x_4\}, \\ \mathcal{R}(q_2, q_5) &= \{(x_2, x_5) : x_2 = x_5\}, \\ \mathcal{R}(q_3, q_4) &= \{(x_3, x_4) : (x_3, 0) = x_4\}. \end{aligned}$$

It is simple to check that \mathcal{R} is a hybrid bisimulation between \mathbb{S}_1 and \mathbb{S}_2 and $\Pi|_{\Xi_i}(\mathcal{R}) = \Xi_i$, $i = 1, 2$, thus \mathbb{S}_1 and \mathbb{S}_2 are bisimilar. However $\Sigma_2(q_4)$ and $\Sigma_1(q_3)$ are not bisimilar.

On the other hand the following result holds.

Proposition 3: *Consider two bisimilar SLSs \mathbb{S}_1 and \mathbb{S}_2 and a hybrid bisimulation \mathcal{R} between \mathbb{S}_1 and \mathbb{S}_2 such*

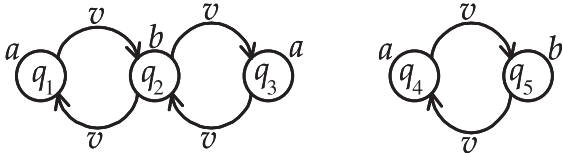


Figure 1. DES \mathbb{D}_1 in the left and DES \mathbb{D}_2 in the right.

that $\Pi|_{\Xi_i}(\mathcal{R}) = \Xi_i$, $i = 1, 2$. For any $q_1 \in \mathcal{Q}_1$, there exists $q_2 \in \mathcal{Q}_2$ such that $\Sigma_1(q_1) \sim \Sigma_2(q_2)$ and $(q_1, q_2) \in \mathcal{Q}_{\mathcal{R}}$ and vice versa.

Proof: The proof follows directly from Definitions 3 and 4. \square

When formalizing hybrid bisimulations in Definition 3, no restrictions were posed on hybrid time bases τ_1 and τ_2 of executions χ_1 and χ_2 while in the definition of structural hybrid bisimulation proposed in (van der Schaft 2004c), $\tau_1 = \tau_2$ was required. We now show that by associating to any SLS \mathbb{S} a suitable SLS \mathbb{S}° bisimilar to \mathbb{S} , there is no loss of generality into setting $\tau_1 = \tau_2$, in Definition 3.

More precisely given an SLS $\mathbb{S} = (\Xi, U, D, Y, \Sigma, \gamma, E, M)$, define the SLS

$$\mathbb{S}^\circ := (\Xi, U, D, Y, \Sigma, \gamma, E^\circ, M^\circ),$$

where $E^\circ := E \cup E'$ being $E' := \{(q, v, q), \forall q \in \mathcal{Q}$, for some $v \in V\}$ and $M^\circ(e) = M(e)$, if $e \in E$, $M^\circ(e) = I$, if $e \in E'$. Then the following holds.

Proposition 4: *Given two SLSs \mathbb{S}_1 and \mathbb{S}_2 ,*

- \mathbb{S}_1 and \mathbb{S}_1° are bisimilar;
- \mathcal{R} is a hybrid bisimulation between \mathbb{S}_1 and \mathbb{S}_2 if and only if \mathcal{R} is a hybrid bisimulation between \mathbb{S}_1° and \mathbb{S}_2 ;
- \mathcal{R} is a hybrid bisimulation between \mathbb{S}_1 and \mathbb{S}_2 if and only if \mathcal{R} is a hybrid bisimulation between \mathbb{S}_1° and \mathbb{S}_2° and in this last case \mathcal{R} satisfies Definition 3 with $\tau_1 = \tau_2$.

Remark 8: By considering \mathbb{S}_1° and \mathbb{S}_2° , instead of \mathbb{S}_1 and \mathbb{S}_2 , we implicitly force a time synchronization in the events driving the discrete transitions in the time evolution of each of the SLSs \mathbb{S}_1° and \mathbb{S}_2° .

We conclude this section by defining the sum of hybrid bisimulations between SLSs. Given two hybrid bisimulations \mathcal{R}^a and \mathcal{R}^b between two SLSs \mathbb{S}_1 and \mathbb{S}_2 , $\mathcal{R}^{ab} := \mathcal{R}^a + \mathcal{R}^b$ is called the *sum* of \mathcal{R}^a and \mathcal{R}^b if

$$\begin{aligned} \mathcal{R}^{ab} &:= \{((q_1, x_1), (q_2, x_2)) \in \Xi_1 \times \Xi_2 | (q_1, q_2) \\ &\in \mathcal{Q}_{\mathcal{R}^{ab}}, (x_1, x_2) \in \mathcal{R}^{ab}(q_1, q_2)\}, \end{aligned}$$

where $\mathcal{Q}_{\mathcal{R}^{ab}} := \mathcal{Q}_{\mathcal{R}^a} \cup \mathcal{Q}_{\mathcal{R}^b}$ and

$$\begin{aligned} \mathcal{R}^{ab}(q_1, q_2) &:= \begin{cases} \mathcal{R}^a(q_1, q_2) + \mathcal{R}^b(q_1, q_2), & \text{if } (q_1, q_2) \in \mathcal{Q}_{\mathcal{R}^a} \cap \mathcal{Q}_{\mathcal{R}^b}; \\ \mathcal{R}^a(q_1, q_2), & \text{if } (q_1, q_2) \in \mathcal{Q}_{\mathcal{R}^a} \setminus \mathcal{Q}_{\mathcal{R}^b}; \\ \mathcal{R}^b(q_1, q_2), & \text{if } (q_1, q_2) \in \mathcal{Q}_{\mathcal{R}^b} \setminus \mathcal{Q}_{\mathcal{R}^a}. \end{cases} \end{aligned}$$

Proposition 5: *Let \mathcal{R}^a and \mathcal{R}^b be hybrid bisimulations between bisimilar SLSs \mathbb{S}_1 and \mathbb{S}_2 . Then $\mathcal{R}^a + \mathcal{R}^b$ is a hybrid bisimulation between \mathbb{S}_1 and \mathbb{S}_2 .*

Proof: By applying Theorem 2 to $\mathcal{R}^a + \mathcal{R}^b$ the statement holds. \square

Remark 9: The results presented in this section are developed in the context of continuous-time switching linear systems. However the results of this section still hold when characterizing hybrid bisimulations for discrete-time switching linear systems.

3. Maximal hybrid bisimulation

In this section we characterize the properties and we propose a procedure for the computation of the maximal hybrid bisimulation.

The maximal hybrid bisimulation between (bisimilar) SLSs \mathbb{S}_1 and \mathbb{S}_2 is a hybrid bisimulation \mathcal{R}^* between \mathbb{S}_1 and \mathbb{S}_2 such that, for all hybrid bisimulations \mathcal{R} between \mathbb{S}_1 and \mathbb{S}_2 , $\mathcal{R} \subset \mathcal{R}^*$.

The following result summarizes some properties of the maximal hybrid bisimulation.

Theorem 3: *Given two SLSs \mathbb{S}_1 and \mathbb{S}_2 and the maximal hybrid bisimulation \mathcal{R}^* between \mathbb{S}_1 and \mathbb{S}_2 , the following property hold:*

- (i) $\Pi|_{\Xi_i}(\mathcal{R}^*) = \Xi_i$, $i=1,2$, if and only if \mathbb{S}_1 and \mathbb{S}_2 are bisimilar.

Moreover if \mathbb{S}_1 and \mathbb{S}_2 are bisimilar then:

- (ii) \mathcal{R}^* exists and is unique;
- (iii) $\mathcal{R}^* + \mathcal{R} = \mathcal{R}^*$, for any hybrid bisimulation \mathcal{R} between \mathbb{S}_1 and \mathbb{S}_2 ;
- (iv) $\mathcal{R}^* = \mathcal{L}(\mathcal{R}^*)$.

Proof: See the Appendix. \square

Hereafter we propose a procedure for the computation of \mathcal{R}^* . As in the case of linear dynamical systems, the proposed procedure is very close to procedures for the computation of the maximal safe set for SLSs: its main ingredient is the one developed in (De Santis *et al.* 2004a) which computes inner approximations of the maximal safe set for the class of discrete-time switching linear systems constrained to compact sets in the continuous state and input.

The key idea is to first compute the maximal bisimulation Q^* of the discrete layers associated with the pair of SLSs under consideration and then, to compute the maximal hybrid bisimulation \mathcal{R}^* on the basis of Q^* . The computation of Q^* may be performed by using standard algorithms as for example the one in (Clarke *et al.* 2002) which converges in a finite number

of steps to the maximal bisimulation relation for DESs. Note that $Q_{\mathcal{R}^*} \subset Q^*$, since in the computation of Q^* informations coming from the continuous dynamics have not been considered.

The computation of \mathcal{R}^* may be done by combining Algorithm 2 of (van der Schaft 2004a), which computes the maximal bisimulation relation for linear dynamical systems, with Procedure Switching of De Santis *et al.* (2004) for the computation of maximal safe sets for switching systems. Computing \mathcal{R}^* requires the analysis of the topological properties of DESs $\mathbb{D}_{\mathbb{S}_1}$ and $\mathbb{D}_{\mathbb{S}_2}$ associated to SLSs \mathbb{S}_1 and \mathbb{S}_2 . For this purpose, it is useful to define the DES \mathbb{D}^* , naturally induced by the bisimulation relation Q^* . More formally, let $\mathbb{D}^* = (Q^*, P^*, V^*, E^*, \gamma^*)$ be a DES where:

- $P^* = P_1 \cup P_2$,
- $V^* = V_1 \times V_2$,
- $E^* = \{((q_1, q_2), (v_1, v_2), (q'_1, q'_2)) : (q_1, v_1, q'_1) \in E_1, (q_2, v_2, q'_2) \in E_2, \gamma_1(q_1) = \gamma_2(q_2), \gamma_1(q'_1) = \gamma_2(q'_2)\}$,
- $\gamma^*(q_1, q_2) = \gamma_1(q_1) = \gamma_2(q_2), \forall (q_1, q_2) \in Q^*$.

Proposition 6: *Given two bisimilar SLSs \mathbb{S}_1 and \mathbb{S}_2 , \mathbb{D}^* is well defined.*

The computation of the maximal hybrid bisimulation \mathcal{R}^* exploits the topological properties of \mathbb{D}^* .

Before explaining the basic steps of the proposed procedure for the computation of \mathcal{R}^* , we need to recall well known facts about DESs (Hopcroft and Ullman 1979).

A Strongly Connected Component (SCC) of \mathbb{D}^* is the maximal set of mutually reachable states. We denote by \mathbf{F} , the set of all SCCs associated to \mathbb{D}^* . SCCs determine a Directed Acyclic Graph (DAG). Moreover $\mathbf{F}^0 \subset \mathbf{F}$ denotes the set of all SCCs not reached by any SCCs, $\mathbf{F}^i \subset \mathbf{F}$ denotes the set of all SCCs, reached in one step by a SCC in \mathbf{F}^{i-1} and so on. Let $n_{\mathbf{F}}$ be the maximal integer i for which \mathbf{F}^i is nonempty. Notice that the intersection $\mathbf{F}^i \cap \mathbf{F}^j$ with $i \neq j$ may be non-empty. Any SCC in \mathbf{F}^0 (resp. in $\mathbf{F}^{n_{\mathbf{F}}}$) is called a root (resp. a leaf) of the DAG associated to \mathbb{D}^* . For any $F \in \mathbf{F}$, we denote by $Q_F \subset Q_1 \times Q_2$, the set of *extended discrete states* belonging to F . Moreover for any extended discrete state $(q_1, q_2) \in Q_F$ for some F , $\text{succ}(q_1, q_2)$ denotes the set of all extended discrete states that are successor of (q_1, q_2) in F . Sets $Q_F, \forall F \in \mathbf{F}$ are a partition of Q^* . For any F of \mathbb{D}^* , the set $\text{succ}(F)$ is composed by those SCCs, reached by F in one step and $\mathbb{S}_i(F)$, $i=1,2$ denote the pair of bisimilar SLSs naturally induced by the DES \mathbb{D}^* and by continuous dynamics associated to its extended discrete states. $\mathcal{G}(F^+, F^-) \subset Q_{F^+}$ denotes the set of extended discrete states of F^+ , reachable in one step by an

extended discrete state in F^- . Moreover M^\dagger denotes the Moore–Penrose pseudo-inverse of a given matrix M and $\text{Bisim}(\Sigma_1(q_1), \Sigma_2(q_2), \text{Init})$ denotes the maximal linear bisimulation between Σ_1 and Σ_2 constrained to a subspace Init (see (van der Schaft 2004c) for an algorithmic procedure computing $\text{Bisim}(\Sigma_1(q_1), \Sigma_2(q_2), \text{Init})$ in a finite number of steps).

The computation of \mathcal{R}^* is based on Theorem 2 and is carried out using a two step procedure:

- At the lower-level, we give a procedure for the computation of the maximal hybrid bisimulation between bisimilar SLSs $\mathbb{S}_1(F)$ and $\mathbb{S}_2(F)$ naturally induced by a SCC F , constrained to a given subspace $\text{Init}(F)$.
- At the higher-level, the computation of \mathcal{R}^* is proposed and based on the lower-level procedure.

We start by describing how to compute the maximal hybrid bisimulation $\text{BisimSCC}(\text{Init}(F), F)$ between SLSs $\mathbb{S}_1(F)$ and $\mathbb{S}_2(F)$ induced by the SCC F of \mathbb{D}^* and constrained in the hybrid subspace:

$$\text{Init}(F) = \bigcup_{(q_1, q_2) \in Q_F} \{(q_1, q_2)\} \times \text{Init}^F(q_1, q_2).$$

We will define an appropriate recursion that, by exploiting topological structure of F , computes a sequence of sets $K(i)$, $i = 0, 1, \dots$, converging to the maximal hybrid bisimulation between $\mathbb{S}_1(F)$ and $\mathbb{S}_2(F)$.

At first we set $i = 0$ and the initial maximal hybrid bisimulation $K(0)$ between $\mathbb{S}_1(F)$ and $\mathbb{S}_2(F)$ as

$$K(0) := \bigcup_{(q_1, q_2) \in Q_F} \{(q_1, q_2)\} \times Z((q_1, q_2), 0),$$

where $Z((q_1, q_2), 0) := \text{Init}^F(q_1, q_2)$ for any $(q_1, q_2) \in Q_F$. For any $(q_1, q_2) \in Q_F$, we first update the constraining subspace Z_0 where $\text{Bisim}(\Sigma_1(q_1), \Sigma_2(q_2), Z_0)$ lies: for any $(q'_1, q'_2) \in \text{succ}(q_1, q_2)$, by Theorem 2 (ii), $\text{Bisim}(\Sigma_1(q_1), \Sigma_2(q_2), Z_0)$ has to belong to $M^\dagger(e_1, e_2)Z((q'_1, q'_2), 0)$, where e_1 and e_2 connect discrete states (q_1, q_2) to (q'_1, q'_2) . Compute

$$Z_0 := \bigcap_{(q'_1, q'_2) \in \text{succ}(q_1, q_2)} M^\dagger(e_1, e_2)Z((q'_1, q'_2), 0). \quad (4)$$

Then it is possible to compute $Z((q_1, q_2), 0) := \text{Bisim}(\Sigma_1(q_1), \Sigma_2(q_2), Z_0)$ between $\Sigma_1(q_1)$ and $\Sigma_2(q_2)$. Finally we update $K(0) := \bigcup_{(q_1, q_2) \in Q_F} \{(q_1, q_2)\} \times Z((q_1, q_2), 0)$ and $i := i + 1$. By iterating this step again the maximal hybrid bisimulation between $\mathbb{S}_1(F)$ and $\mathbb{S}_2(F)$ corresponds to a fixed point $K(i) = K(i - 1)$, for some $i \in \mathbb{N}$, of the recursion above. The proposed procedure is formalized in the following Function.

Function 1: $R(F) := \text{BisimSCC}(\text{Init}(F), F)$

```

set i := 0
forall (q1, q2) in QF
  set Z((q1, q2), i) := Init^F(q1, q2)
set K(i) := union_{(q1, q2) in QF} {(q1, q2)} x Z((q1, q2), i)
while K(i) != K(i-1) repeat
  for any (q1, q2) in QF do
    compute Z0 := intersection_{(q'1, q'2) in succ(q1, q2)} M^dagger(e1, e2)Z((q'1, q'2), i)
    where ek = (qk, vk, q'k), vk in V_k, k = 1, 2
    compute Z((q1, q2), i) = Bisim(Sigma1(q1), Sigma2(q2), Z0)
  end do
  set K(i) := union_{(q1, q2) in QF} {(q1, q2)} x Z((q1, q2), i)
  set i := i + 1
end while
return R(F) := K(i)
end Function

```

We now provide the high-level algorithm. The computation of the maximal hybrid bisimulation starts from the leaves $F \in \mathbf{F}^{n_F}$ and going backwards, ends to the roots $F \in \mathbf{F}^0$ of the DAG associated to \mathbb{D}^* . For any $F \in \mathbf{F}$, $\text{Init}(F)$ represents the constraining subspace where the maximal hybrid bisimulation has to be computed. Firstly we set

$$\text{Init}(F) := \bigcup_{(q_1, q_2) \in Q_F} \{(q_1, q_2)\} \times \text{Init}^F(q_1, q_2), \quad \forall F \in \mathbf{F}.$$

Any $F \in \mathbf{F}^{n_F}$ has no successors. The first step consists of computing for any $F \in \mathbf{F}^{n_F}$, the maximal hybrid bisimulation $\text{BisimSCC}(\text{Init}(F), F)$ associated to SLSs $\mathbb{S}_1(F)$ and $\mathbb{S}_2(F)$, induced by F . Then we need to update the constraining subspace $\text{Init}(F^-)$ of SCCs $F^- \in \text{succ}^{-1}(\mathbf{F}^{n_F})$: consider all SCCs $F^+ \in \text{succ}(F^-)$ and all extended discrete states $(q_1, q_2) \in Q_{F^-}$ that reach in one step the extended discrete states $(q'_1, q'_2) \in \mathcal{G}(F^+, F^-) \subset Q_{F^+}$ of F^+ . By Theorem 2 (ii), for any fixed $F^+ \in \text{succ}(F^-)$ and for any fixed $(q'_1, q'_2) \in \mathcal{G}(F^+, F^-)$, the set $\text{Bisim}((q_1, q_2), Z_0)$ has to belong to

$$\alpha(q'_1, q'_2) := M^\dagger(e_1, e_2)\text{Bisim}((q'_1, q'_2), \text{Init}(F^+)), \quad (5)$$

where e_1 and e_2 connect $(q_1, q_2) \in Q_{F^-}$ to $(q'_1, q'_2) \in Q_{F^+}$. By considering all extended discrete states $(q'_1, q'_2) \in \mathcal{G}(F^+, F^-)$ and all SCCs $F^+ \in \text{succ}(F^-)$, the maximal hybrid bisimulation between $\mathbb{S}_1(F^-)$ and $\mathbb{S}_2(F^-)$, has to belong to

$$\text{Init}(F^-) := \text{Init}(F^-) \cap I^0(F^-)$$

where

$$I^0(F^-) := \bigcap_{F^+ \in \text{succ}(F^-)} \left(\bigcap_{(q'_1, q'_2) \in \mathcal{G}(F^+, F^-)} \alpha(q'_1, q'_2) \right).$$

Finally it is possible to go backwards and consider the SCCs of \mathbf{F}^{n_F-1} and so on. This procedure ends when all SCCs are visited. The resulting set \hat{R} can be thought of as being of the form (2); moreover \hat{R} as returned by the procedure may contain some empty sets. For this reason define

$$\text{Clean}(\hat{R}) = \left\{ ((q_1, x_1), (q_2, x_2)) \in \hat{R} : \hat{R}(q_1, q_2) \neq \emptyset \right\},$$

and finally set $\hat{R} := \text{Clean}(\hat{R})$. The proposed procedure is formalized in the following algorithm.

Algorithm 1: MAXIMAL HYBRID BISIMULATION

```

set  $i := n_F, \hat{R} := \emptyset$ 
 $\forall F \in \mathbf{F}$ 
  set  $\text{Init}(F) := \bigcup_{(q_1, q_2) \in Q_F} \{(q_1, q_2)\} \times \text{Init}^F(q_1, q_2)$ 
while  $i > 0$  repeat
   $\forall F \in \mathbf{F}^i$ 
    compute  $R(F) := \text{BisimSCC}(\text{Init}(F), F)$ 
  set  $\hat{R} := \hat{R} \cup R(F)$ 
   $\forall F^- \in \text{succ}^{-1}(F)$ 
    compute  $\text{Init}(F^-) := \text{Init}(F^-) \cap I^0(F^-)$ ,  $i := i - 1$ 
end while
 $\hat{R} := \text{Clean}(\hat{R})$ 
end Algorithm

```

Remark 10: Algorithm 1 makes use of geometric linear control theory and therefore there are various efficient tools in the literature for the effective computation of the required sets.

Convergence properties of the procedure above is now characterized.

Theorem 4: Algorithm 1 converges in a finite number of steps to the maximal hybrid bisimulation $\hat{R} = \mathcal{R}^*$ or to the empty set.

The result above is also important because it gives a reformulation of Theorem 2 that is checkable in a finite number of steps, as the following result shows.

Corollary 2: Two SLSs \mathbb{S}_1 and \mathbb{S}_2 are bisimilar if and only if the returned set \hat{R} of Algorithm 1 is such that $\Pi_{|\Xi_i}(\hat{R}) = \Xi_i$, $i = 1, 2$.

We conclude this section by giving some results highlighting the computational complexity of the proposed approach. The following results give an upper bound on the number of steps for which the convergence of Function BisimSCC and Algorithm 1 is ensured.

Proposition 7: Given a SCC F and an initial subspace $\text{Init}(F)$, Function BisimSCC($\text{Init}(F), F$) converges in at most $\aleph(F)$ steps, where

$$\aleph(F) = \max \{ \dim X_1(q_1) + \dim X_2(q_2), (q_1, q_2) \in Q_F \}.$$

Proposition 8: Algorithm 1 converges in at most \aleph steps, where $\aleph = \sum_{F \in \mathbf{F}} \aleph(F)$.

The proof of the results above is based on the structure of the sets involved being linear subspaces.

4. Reduction via hybrid bisimulation

Reduction via bisimulation is a well-known technique to reduce the topological complexity of concurrent processes (see for example (Hermanns 2002)). The basic idea is to find a bisimulation between the process and itself and then to factorize the state space of the process under the equivalence relation induced by the bisimulation. In this section, we extend results in (Hermanns 2002) and (van der Schaft 2004a) to SLSs.

Therefore in the following we will consider an SLS \mathbb{S} and a copy of itself and we show how to perform a hybrid state space reduction of \mathbb{S} . The following obvious facts hold.

Lemma 2: Given an SLS \mathbb{S} , the identity relation $\mathcal{R}_{id} := \{(\xi_1, \xi_2) \mid \xi_1 = \xi_2\}$ is a hybrid bisimulation between \mathbb{S} and itself.

Lemma 3: Given an SLS \mathbb{S} , for any hybrid bisimulation \mathcal{R} between \mathbb{S} and itself, $\mathcal{R}^{-1} := \{(\xi_2, \xi_1) \mid (\xi_1, \xi_2) \in \mathcal{R}\}$ is a hybrid bisimulation between \mathbb{S} and itself.

Every $\mathcal{R} \subset \Xi \times \Xi$ naturally induces a relation on Ξ by saying that $\xi_1, \xi_2 \in \Xi$ are related by \mathcal{R} if and only if $(\xi_1, \xi_2) \in \mathcal{R}$. For performing the hybrid state space reduction of a given SLS, we employ an equivalence relation on the hybrid state space Ξ in such a way that all hybrid states belonging to the same equivalence class of the equivalence relation are reduced to the same hybrid state. The following result shows that any hybrid bisimulation naturally induces an equivalence relation on the hybrid state space.

Proposition 9: For any hybrid bisimulation \mathcal{R} between an SLS \mathbb{S} and itself there exists a hybrid bisimulation \mathcal{R}' between \mathbb{S} and itself that is also an equivalence relation on the hybrid state space Ξ of \mathbb{S} .

Proof: Denote by Q' the set of all $(q_1, q_3) \in (Q_1 \times Q_2) \setminus Q_{\mathcal{R}}$ such that $(q_1, q_2), (q_2, q_3) \in Q_{\mathcal{R}}$ for some $q_2 \in Q_{\mathcal{R}}$ and set

$$\mathcal{R}'' = \mathcal{R} \cup \left(\bigcup_{(q_1, q_3) \in Q'} \{(q_1, q_3)\} \times \mathcal{R}''(q_1, q_3) \right),$$

where $\mathcal{R}''(q_1, q_3) = \{(x_1, x_3) \in X(q_1) \times X(q_3) \mid \exists x_2 \in X(q_2): (x_1, x_2) \in \mathcal{R}(q_1, q_2), (x_2, x_3) \in \mathcal{R}(q_2, q_3)\}$ is a linear bisimulation between $\Sigma(q_1)$ and $\Sigma(q_3)$. Set

$$\mathcal{R}' = \mathcal{R}'' + (\mathcal{R}'')^{-1} + \mathcal{R}_{id}.$$

By construction \mathcal{R}' is an equivalence relation on the hybrid state space Ξ of \mathbb{S} . We now show that \mathcal{R}'' is a hybrid bisimulation between \mathbb{S} and itself, and then by Proposition 5, that \mathcal{R}' is a hybrid bisimulation between \mathbb{S} and itself. For any $(q_1, q_2) \in Q_{\mathcal{R}}$ Theorem 2 (i) and (ii) are satisfied. For any $(q_1, q_2) \in Q'$, there exists $q_3 \in Q$ such that $(q_1, q_2), (q_2, q_3) \in Q_{\mathcal{R}}$. Then by definition, for any $q'_1 \in succ(q_1)$, there exists $q'_2 \in succ(q_2)$, and for any $q'_2 \in succ(q_2)$, there exists $q'_3 \in succ(q_3)$ such that $(q'_1, q'_2), (q'_2, q'_3) \in Q_{\mathcal{R}}$ and hence $(q'_1, q'_3) \in Q_{\mathcal{R}}$. Moreover $\gamma(q_1) = \gamma(q_2) = \gamma(q_3)$ and $\mathcal{R}''(q_1, q_3)$ is a bisimulation between $\Sigma(q_1)$ and $\Sigma(q_3)$ and therefore Theorem 2 (i) is satisfied. For any $(x_1, x_3) \in \mathcal{R}(q_1, q_3)$ there exists $x_2 \in X(q_2)$ such that $(x_1, x_2) \in \mathcal{R}(q_1, q_2)$, $(x_2, x_3) \in \mathcal{R}(q_2, q_3)$ and $(M_1(e_1)x_1, M_2(e_2)x_2) \in \mathcal{R}(q'_1, q'_2)$, $(M_2(e_2)x_2, M_3(e_3)x_3) \in \mathcal{R}(q'_2, q'_3)$, with appropriate discrete transitions e_i , $i = 1, 2, 3$, and therefore $(M_1(e_1)x_1, M_3(e_3)x_3) \in \mathcal{R}(q'_1, q'_3)$, i.e. Theorem 2 (ii) is satisfied. By repeating the same proof replacing the role of q_1 by q_2 , the statement follows. \square

Remark 11: \mathcal{R}' as constructed in the proof above can be seen as the closure of \mathcal{R} with respect to the properties of reflexivity, symmetry and transitivity.

Remark 12: Given a hybrid bisimulation \mathcal{R} , the hybrid bisimulation \mathcal{R}' , constructed in the proof of Proposition 9, is such that $\mathcal{R} \subset \mathcal{R}'$, and therefore it is easy to see that the maximal hybrid bisimulation \mathcal{R}^* between \mathbb{S} and itself is also an equivalence relation.

By the result above we may assume w.l.o.g. that the hybrid bisimulations under consideration are equivalence relations on the hybrid state space Ξ of \mathbb{S} .

Given a hybrid bisimulation and equivalence relation \mathcal{R} , we now show how to perform a hybrid state space reduction and how to define the reduced SLS bisimilar to \mathbb{S} .

Denote by Ω_i , the equivalence class induced by $Q_{\mathcal{R}}$ such that $q_j, q_k \in \Omega_i$ if and only if $(q_j, q_k) \in Q_{\mathcal{R}}$. For any Ω_i choose the set of representatives Ω_i^* such that

- for any $q \in \Omega_i$ there exists $q_i^* \in \Omega_i^*$ such that $\Pi|_{X(q)}(\mathcal{R}(q, q_i^*)) = X(q)$,
- for any $q, q' \in \Omega_i^*$, $q \neq q'$, $\Pi|_{X(q')}(\mathcal{R}(q, q')) \neq X(q')$.

The existence of Ω_i^* is guaranteed by Proposition 3. Denote by $Q^{\mathcal{R}}$ the set of all canonical representatives of the sequence Ω_i , $i = 1, 2, \dots$, i.e. $Q^{\mathcal{R}} = \bigcup_i \Omega_i^*$. For any $q \in Q^{\mathcal{R}}$ define $\mathcal{E}(q) := \{q' \in Q: (q, q') \in Q_{\mathcal{R}}\}$, $\bar{\mathcal{R}}(q) := \{x_1 - x_2 \mid (x_1, x_2) \in \mathcal{R}(q, q)\}$ and finally

$$\bar{\mathcal{R}} := \bigcup_{q \in Q^{\mathcal{R}}} \mathcal{E}(q) \times \bar{\mathcal{R}}(q).$$

The hybrid state space Ξ of the SLS \mathbb{S} under consideration may be now factorized by $\bar{\mathcal{R}}$. We write $\Xi/\bar{\mathcal{R}}$ to denote the reduced hybrid state space of \mathbb{S} , naturally induced by $\bar{\mathcal{R}}$, i.e.

$$\Xi/\bar{\mathcal{R}} = \bigcup_{q \in Q^{\mathcal{R}}} \{q\} \times X(q)/\bar{\mathcal{R}}(q).$$

Let $\Pi_{\mathbb{Q}}^{\mathcal{R}}: Q \rightarrow Q^{\mathcal{R}}$ be the canonical projection map associating to each element of Q its unique canonical representative in $Q^{\mathcal{R}}$, and for any $q \in Q^{\mathcal{R}}$ let $\Pi_q^{\mathcal{R}}: X(q) \rightarrow X(q)/\bar{\mathcal{R}}(q)$ be the canonical projection. Define the reduced SLS:

$$\mathbb{S}^{\mathcal{R}} = (\Xi^{\mathcal{R}}, U, D, Y, \Sigma^{\mathcal{R}}, \gamma^{\mathcal{R}}, E^{\mathcal{R}}, M^{\mathcal{R}}),$$

where

- $X^{\mathcal{R}}(q) = X(q)/\bar{\mathcal{R}}(q)$ and $\dim^{\mathcal{R}}(q) = \dim(X^{\mathcal{R}}(q))$, $\forall q \in Q^{\mathcal{R}}$;
- For any $q \in Q^{\mathcal{R}}$, $\Sigma^{\mathcal{R}}(q)$ is given by equations

$$\Sigma^{\mathcal{R}}(q) : \begin{cases} \dot{x}(t) = A^{\mathcal{R}}(q)x(t) + B^{\mathcal{R}}(q)u(t) + G^{\mathcal{R}}(q)w(t), \\ y(t) = C^{\mathcal{R}}(q)x(t), \end{cases}$$

where the dynamical systems above are defined as in van der Schaft (2004a); for the sake of completeness, there exists a 'feedback' map $K(q)$ such that

$$(A(q) + G(q)K(q))\bar{\mathcal{R}}(q) \subset \bar{\mathcal{R}}(q),$$

and thus $A(q) + G(q)K(q)$ projects to a linear map $A^{\mathcal{R}}(q): X^{\mathcal{R}}(q) \rightarrow X^{\mathcal{R}}(q)$ satisfying $A^{\mathcal{R}}(q)\Pi_q^{\mathcal{R}} = \Pi_q^{\mathcal{R}}(A(q) + G(q)K(q))$; $B^{\mathcal{R}}(q) = \Pi_q^{\mathcal{R}}B(q)$; $G^{\mathcal{R}}(q) = \Pi_q^{\mathcal{R}}G(q)$; $C^{\mathcal{R}}(q)$ is such that $C^{\mathcal{R}}(q)\Pi_q^{\mathcal{R}} = C(q)$;

- $\gamma^{\mathcal{R}}: Q^{\mathcal{R}} \rightarrow P$ such that $\gamma^{\mathcal{R}}(q) = \gamma(q)$, $\forall q \in Q^{\mathcal{R}}$;
- $e = (q'_1, v, q'_2) \in E^{\mathcal{R}}$ if and only if there exist $q_1 \in \mathcal{E}^{-1}(q'_1)$ and $q_2 \in \mathcal{E}^{-1}(q'_2)$ such that $(q_1, v, q_2) \in E$;
- $\forall e = (q, v, q') \in E^{\mathcal{R}}$, $M^{\mathcal{R}}(e)\Pi_q^{\mathcal{R}} = \Pi_{q'}^{\mathcal{R}}M(e)$.

The reduced SLS $\mathbb{S}^{\mathcal{R}}$ depends on the choice of the set $Q^{\mathcal{R}}$ of canonical representatives of equivalence classes induced by $Q_{\mathcal{R}}$ on Q . The following holds.

Proposition 10: *Let \mathcal{R} be a hybrid bisimulation and equivalence relation between \mathbb{S} and itself. Then for any canonical representative $Q^{\mathcal{R}}, \mathbb{S}$ and $\mathbb{S}^{\mathcal{R}}$ are bisimilar and*

$$\text{card}(Q^{\mathcal{R}}) \leq \text{card}(Q), \dim(X(\Pi_{\mathbb{Q}}^{\mathcal{R}}(q))) \leq \dim(X(q)), \forall q \in Q.$$

Proof: For any $Q^{\mathcal{R}}$ define $\mathcal{R}' \subset \Xi \times \Xi/\bar{\mathcal{R}}$ such that $((q, x), (q', x')) \in \mathcal{R}'$ if and only if $\Pi_{\mathbb{Q}}^{\mathcal{R}}(q) = q'$ and $\Pi_q^{\mathcal{R}}(x) = x'$. It is easily seen that \mathcal{R}' is a hybrid bisimulation such that $\Pi|_{\Xi}(\mathcal{R}') = \Xi$ and $\Pi|_{\Xi/\bar{\mathcal{R}}}(\mathcal{R}') = \Xi/\bar{\mathcal{R}}$. The second part of the statement is obvious by construction. \square

Remark 13: The second part of the statement of Proposition 10 formalizes the intuitive idea that $\mathbb{S}^{\mathcal{R}}$ is in some way ‘smaller’ than \mathbb{S} .

The following result is a direct consequence of Proposition 10 and of the definition of reduced SLSs.

Corollary 3: *Let \mathcal{R}_1 and \mathcal{R}_2 be hybrid bisimulations and equivalence relations between \mathbb{S} and itself such that $\mathcal{R}_2 \subset \mathcal{R}_1$. Then,*

$$\begin{aligned} \text{card}(Q^{\mathcal{R}_1}) &\leq \text{card}(Q^{\mathcal{R}_2}), \\ \dim(X(\Pi_{\mathbb{Q}}^{\mathcal{R}_1}(q))) &\leq \dim(X(\Pi_{\mathbb{Q}}^{\mathcal{R}_2}(q))), \forall q \in Q. \end{aligned}$$

These results reveal a sort of monotony in the functions $\text{card}(Q^{\mathcal{R}})$ and $\dim(X(\Pi_{\mathbb{Q}}^{\mathcal{R}}(q)))$, $\forall q \in Q$ depending on \mathcal{R} . Moreover, in view of the result above one may think to find the minimal bisimilar SLS of an SLS \mathbb{S} , by reducing \mathbb{S} by means of \mathcal{R}^* . We now show that this intuitive idea is true.

We formally refer to a minimal bisimilar SLS of a given SLS \mathbb{S} as an SLS \mathbb{S}' that is bisimilar to \mathbb{S} , and such that the cardinality of its discrete state space Q' and the dimensions of its continuous state space $X'(q)$, $q \in Q'$ are minimal among all other SLSs that are bisimilar to \mathbb{S} . Denote by $\min(\mathbb{S})$ the class of minimal bisimilar SLSs of \mathbb{S} . From Corollary 3 and the definition of maximal hybrid bisimulation we derive the following result.

Corollary 4: *Let \mathcal{R}^* be the maximal hybrid bisimulation between \mathbb{S} and itself, then $\mathbb{S}^{\mathcal{R}^*} \in \min(\mathbb{S})$.*

Remark 14: It is worth to point out that, by the procedures illustrated in §3, the computation of $\mathbb{S}^{\mathcal{R}^*}$ can be done in a finite number of steps. This result is also important because shows that the minimal bisimilar SLS of a given SLS can be always computed, whereas the same reasoning does not apply to the general case of hybrid systems, as shown in (Alur *et al.* 2000).

We recall the following result that will be instrumental in the subsequent developments.

Lemma 4 (van der Schaft 2004c): *If Σ_1 and Σ_2 are bisimilar linear dynamical systems, then any $\Sigma'_1 \in \min(\Sigma_1)$ and any $\Sigma'_2 \in \min(\Sigma_2)$ are algebraically equivalent.*

Since $\mathbb{S}^{\mathcal{R}^*}$ depends on the set $Q^{\mathcal{R}^*}$ of canonical representatives of Q , it is not unique. However, the following result holds.

Proposition 11: *The family of $\mathbb{S}^{\mathcal{R}^*}$ parametrized by $Q^{\mathcal{R}^*}$, is composed of SLSs that are algebraically equivalent.*

Proof: See the Appendix. \square

Finally, using the same arguments as in the proof above, it is simple to derive a generalization of Lemma 4 to SLSs:

Corollary 5: *If \mathbb{S}_1 and \mathbb{S}_2 are bisimilar, then any $\mathbb{S}'_1 \in \min(\mathbb{S}_1)$ and $\mathbb{S}'_2 \in \min(\mathbb{S}_2)$ are algebraically equivalent.*

5. Simulation and abstraction

Aim of this section is to characterize the notion of simulation as introduced in Definition 5 and to introduce the notion of abstraction for the class of SLSs.

By specializing Theorem 2, the following result is obtained.

Theorem 5: *Given two SLSs \mathbb{S}_1 and \mathbb{S}_2 , a set \mathcal{R} of the form (2) is a simulation of \mathbb{S}_1 by \mathbb{S}_2 if and only if for any $(q_1, q_2) \in Q_{\mathcal{R}}$ the following property holds:*

$\forall q'_1 \in \text{succ}(q_1), \exists q'_2 \in \text{succ}(q_2) \cup \{q_2\}: (q'_1, q'_2) \in Q_{\mathcal{R}}$ and

- (i) $\gamma_1(q_1) = \gamma_2(q_2)$ and $\mathcal{R}(q_1, q_2)$ is a simulation relation of $\Sigma_1(q_1)$ by $\Sigma_2(q_2)$;
- (ii) $\text{diag}(M_1(e_1), \bar{M}_2)\mathcal{R}(q_1, q_2) \subset \mathcal{R}(q'_1, q'_2)$, where $e_1 \in E_1$ takes q_1 into q'_1 and $e_2 \in E_2$ takes q_2 into q'_2 , and $\bar{M}_2 = M_2(e_2)$ if $q'_2 \neq q_2$, $\bar{M}_2 = I$ if $q'_2 = q_2$;

Remark 15: On the basis of the above result and by specializing the proposed procedure for the computation of the maximal hybrid bisimulation given in §3, it is possible to give a procedure for the computation of the maximal hybrid simulation of an SLS \mathbb{S}_1 by an SLS \mathbb{S}_2 . This is done by replacing the algorithms for the computation of the maximal bisimulation of DESs and of linear dynamical systems (see §3) with the ones in (Clarke *et al.* 2002) and (van der Schaft 2004a), computing respectively the maximal simulation of a DES \mathbb{D}_1 by a DES \mathbb{D}_2 , and the maximal simulation of a dynamical system Σ_1 by a dynamical system Σ_2 . The proposed procedure converges to the maximal simulation of \mathbb{S}_1 by \mathbb{S}_2 in a finite number of steps.

By combining the results on reduction of SLSs given in §4 with the above results on hybrid simulations:

Proposition 12: *Let \mathbb{S}_1 and \mathbb{S}_2 be two SLSs, such that \mathbb{S}_2 is simulated by \mathbb{S}_1 . Let \mathcal{R}_1 and \mathcal{R}_2 be respectively hybrid bisimulations and equivalence relations, between \mathbb{S}_1 and itself and \mathbb{S}_2 and itself. Then $\mathbb{S}_2^{\mathcal{R}_2} \preceq \mathbb{S}_1^{\mathcal{R}_1}$.*

Proof: $\mathbb{S}_2^{\mathcal{R}_2} \preceq \mathbb{S}_2 \preceq \mathbb{S}_1 \preceq \mathbb{S}_1^{\mathcal{R}_1}$. \square

The notion of simulation is very close to the notion of abstraction. *Abstraction* of concurrent processes and dynamical systems has been studied in (Henzinger 1995), (Alur *et al.* 2000), (Tabuada *et al.* 2002) and (Pappas 2003). The main idea is to ‘simplify’ the given process under consideration in such a way that the resulting process simulates the original one. In other words, we may say that abstraction is related to simulation in the same way as reduction is related to bisimulation. Following (Pappas 2003) and (van der Schaft 2004c) abstractions of SLSs are defined as follows. Let $\mathbb{S} = (\Xi, U, D, Y, \Sigma, \gamma, E, M)$ be an SLS where $\Xi = \bigcup_{q \in Q} \{q\} \times X(q)$. Define a suitable set

$$\Xi^A = \bigcup_{q \in Q^A} \{q\} \times X^A(q),$$

such that $Q^A \subset Q$ and $\gamma(Q^A) = \gamma(Q)$ and for any $q \in Q^A$, let $X^A(q)$ be a linear subspace of $X(q)$. Define a surjective map $\mathcal{M}: \Xi \rightarrow \Xi^A$, such that

$$\mathcal{M}(q, x) = (\mathcal{M}_Q(q), \mathcal{M}_q x), \quad \forall (q, x) \in \Xi,$$

where $\mathcal{M}_Q: Q \rightarrow Q^A$ and for any $q \in Q$, $\mathcal{M}_q x$ is linear in x . Suppose that for any $q \in Q$, $\ker \mathcal{M}_q \subset \ker C(q)$. The map \mathcal{M} naturally induces the SLS

$$\mathbb{S}^A := (\Xi^A, U, D, Y, \Sigma^A, \gamma^A, E^A, M^A),$$

where

- For any $q \in Q^A$, $\Sigma^A(q)$ is given by equations

$$\begin{aligned} \dot{x}(t) &= A^A(q)x(t) + B^A(q)u(t) + G^A(q)w(t), \\ y(t) &= C^A(q)x(t), \end{aligned}$$

where $A^A(q) := \mathcal{M}_q A(q) \mathcal{M}_q^\dagger$, $B^A(q) := \mathcal{M}_q B(q)$, $C^A(q) := C(q) \mathcal{M}_q^\dagger$, $G^A(q) := \mathcal{M}_q [G(q): A(q)z_1(q) \dots : A(q)z_r(q)]$, where $z_1(q), \dots, z_r(q)$ span $\ker \mathcal{M}_q$;

- $\gamma^A(q') = \gamma(\mathcal{M}_Q^{-1}(q))$ for some $q \in Q$;
- $e = (q'_1, w, q'_2) \in E^A$ if and only if there exist $q_1 \in \mathcal{M}_Q^{-1}(q'_1)$ and $q_2 \in \mathcal{M}_Q^{-1}(q'_2)$ such that $(q_1, w, q_2) \in E$;
- $M^A(e) = \mathcal{M}_{q'} M(e) \mathcal{M}_{q'}^\dagger$, for any $(q, w, q') \in E^A$.

We think of \mathbb{S}^A as an abstraction of \mathbb{S} .

Remark 6: It is easily seen that one can associate to any hybrid system \mathbb{H} whose discrete transitions depend on the continuous state x , a suitable switching system \mathbb{S} , whose discrete transitions are caused by external discrete disturbances, and which is an abstraction of \mathbb{H} . The following result holds.

Proposition 13: $\mathbb{S} \preceq \mathbb{S}^A$.

Proof: Define the following set $\mathcal{R}' \subset \Xi \times \Xi^A$ such that $((q, x), (q', x')) \in \mathcal{R}'$ if and only if $\mathcal{M}_Q(q) = q'$ and $\mathcal{M}_q(x) = x'$. It is easily seen that \mathcal{R}' is a hybrid simulation of \mathbb{S} by \mathbb{S}^A such that $\Pi|_{\Xi}(\mathcal{R}') = \Xi$ and $\Pi|_{\Xi^A}(\mathcal{R}') = \Xi^A$. \square

We conclude this section by establishing connections between bisimulation-based reduction and simulation-based abstraction.

Proposition 14: *Given an SLS \mathbb{S} and a hybrid bisimulation \mathcal{R} between \mathbb{S} and itself, then $\mathbb{S}^{\mathcal{R}}$ is an abstraction of \mathbb{S} .*

Proof: Set $\Xi^A := \Xi/\mathcal{R}$. By definition $\gamma(Q^A) = \gamma(Q)$ and $X^A(q)$ is a linear subspace of $X(q)$ for any $q \in Q^A$. Finally define $\mathcal{M}_Q := \Pi_Q^{\mathcal{R}}$ and $\mathcal{M}_q := \Pi_q^{\mathcal{R}}$, for any $q \in Q$. Since by definition for any $q \in Q$, $C^{\mathcal{R}}(q) \Pi_q^{\mathcal{R}} = C(q)$, then for any $x \in \ker \Pi_q^{\mathcal{R}}$, $x \in \ker C(q)$: thus, $\ker \Pi_q^{\mathcal{R}} \subset \ker C(q)$ for any $q \in Q$, and the statement holds. \square

The converse of the previous statement obviously is not true in general.

6. Connections with observability of SLSs

We outlined in §2.2 that bisimilarity between SLSs implies their input–output equivalence. This means that any SLS \mathbb{S} has the nice property that any reduced SLS $\mathbb{S}^{\mathcal{R}}$ of \mathbb{S} is input–output equivalent to \mathbb{S} . In this section, we will analyze the preservation of the structural property of observability under bisimulation-based reduction and simulation-based abstraction.

The class of SLSs that we consider in this context are characterized by deterministic continuous dynamics, i.e. $W = \{0\}$, unconstrained input and output functions, i.e. $U = \mathbb{R}^m$ and $H = \mathbb{R}^s$, $m, s \in \mathbb{N}$, and a minimum dwell time $\theta_m > 0$, (Morse 1996) such that for any hybrid time basis $\tau, \forall I_j \in \tau, t'_j - t_j \geq \theta_m$.

We recall here the definition of observability proposed in (De Santis *et al.* 2003)

Definition 12: An SLS $\mathbb{S} = (\Xi, \mathbb{R}^m, V \times \{0\}, P \times \mathbb{R}^s, \Sigma, \gamma, E, M)$ is observable if there exist a function

$$\varphi: \mathcal{C}(\mathbb{R}_0^+ \times \mathbb{N}, \mathbb{R}^s) \times \mathcal{C}(\mathbb{N}, P) \times \mathcal{C}^0(\mathbb{R}_0^+, U) \rightarrow \Xi,$$

an integer $\mathbf{j} \geq 0$ and a real number $\theta \in (0, \theta_m)$ such that $\forall \xi_0 \in \Xi, \forall \tau \in \mathcal{T}, \forall v \in \mathcal{C}(\mathbb{N}, V)$ there exists an execution $\chi = (\xi_0, \tau, u, d, \xi, \gamma)$ such that $\varphi(y|_{[(t_0, 0), (t, \mathbf{j})]}, u|_{[t_0, t]}) = \xi(t, \mathbf{j}), \forall t \in (t_j + \theta, t'_j], \forall \mathbf{j} = \mathbf{j}, \mathbf{j} + 1, \dots, \text{card}(\tau) - 1$, where $y|_{[(t_0, 0), (t, \mathbf{j})]}$ is the restriction of the output $y(t, \mathbf{j})$ to $[t_j, t'_j], 0 \leq j \leq \mathbf{j}$. \mathbb{S} is said to be *unobservable* if it is not observable. A necessary and sufficient condition for testing observability of SLSs is given in the following.

Theorem 6 (De Santis et al. 2003): *An SLS $\mathbb{S} = (\Xi, \mathbb{R}^m, V \times \{0\}, P \times \mathbb{R}^s, \Sigma, \gamma, E, M)$ is observable if and only if*

- (i) $\Sigma(q_i)$ is observable, $\forall q_i \in Q$;
- (ii) $\forall q_i, q_j \in Q, q_i \neq q_j$, one of the following conditions hold:
 - $\gamma(q_i) \neq \gamma(q_j)$,
 - $\exists k \in \mathbb{N} \cup \{0\} : C(q_i)A(q_i)^k B(q_i) \neq C(q_j)A(q_j)^k B(q_j)$.

The problem that we address now is the preservation of observability under bisimulation reduction. The following result holds.

Proposition 15: *If an SLS \mathbb{S} is observable then $\mathbb{S} \in \min(\mathbb{S})$.*

Proof: Let \mathcal{R} be any hybrid bisimulation and equivalence relation between \mathbb{S} and itself. Since Theorem 6 (ii) holds, by Theorem 2, $Q_{\mathcal{R}} = \{(q, q) : q \in Q\}$. Moreover for any $q \in Q$, since $\Sigma(q)$ is observable, by Corollary 6.4 in (van der Schaft 2004c), $\mathcal{R}(q, q) = \{(x, x) : x \in X(q)\}$. Therefore any hybrid bisimulation and equivalence relation between \mathbb{S} and itself coincides with \mathcal{R}_{id} and then $\mathbb{S} \in \min(\mathbb{S})$. \square

The converse of the result above is proved in (van der Schaft 2004c) for the class of linear dynamical systems. However the following counterexample shows that it is not true for the class of SLSs.

Example 2: Let us consider an SLS \mathbb{S}_3 , whose DES $\mathbb{D}_{\mathbb{S}_3} = \mathbb{D}_3$ is depicted in figure 2 and whose continuous dynamics are such that $\Sigma_3(q_6) = \Sigma_3(q_7) = \Sigma_3(q_8) = \Sigma_3(q_9)$ and where $M_3(e) = I$, for any $e \in E_3$. Suppose that $\Sigma_3(q_i)$ are observable for any $q_i, i = 6, \dots, 9$. $\mathbb{S}_3 \in \min(\mathbb{S}_3)$ and is unobservable since Theorem 6 (ii) is not satisfied for q_6, q_7 .

It is important to emphasize that unobservable SLSs may give rise to reduced observable or unobservable SLSs. This motivates the introduction of the following unobservable SLSs classification.

Definition 13: An SLS \mathbb{S} is said to be

- Non-properly unobservable if it is unobservable and there exists a hybrid bisimulation \mathcal{R} between \mathbb{S} and itself, such that the reduced SLS $\mathbb{S}^{\mathcal{R}}$ is observable,
- Properly unobservable if it is unobservable and for any hybrid bisimulation \mathcal{R} between \mathbb{S} and itself the reduced SLS $\mathbb{S}^{\mathcal{R}}$ is unobservable.

Example 3: Let us consider an SLS \mathbb{S}_2 , whose DES $\mathbb{D}_{\mathbb{S}_2} = \mathbb{D}_2$ is depicted in figure 2 and whose continuous dynamics are such that $\Sigma_2(q_3) = \Sigma_2(q_4) = \Sigma_2(q_5)$ and observable and where $M_2(e) = I$, for any $e \in E_2$. \mathbb{S}_2 is unobservable since Theorem 6 (ii) is not satisfied for q_3, q_5 . It is simple to check that any minimal bisimilar SLS $\mathbb{S}'_2 \in \min(\mathbb{S}_2)$ of \mathbb{S}_2 , whose DES is algebraically equivalent to \mathbb{D}_1 , with $\Pi_{Q^{\mathcal{R}^*}}(q_3) = \Pi_{Q^{\mathcal{R}^*}}(q_5) = q_2$, $\Pi_{Q^{\mathcal{R}^*}}(q_4) = q_1$, is observable. Then \mathbb{S}_2 is nonproperly unobservable.

Example 4: Let us consider an SLS \mathbb{S}_3 , whose DES $\mathbb{D}_{\mathbb{S}_3} = \mathbb{D}_3$ is depicted in figure 2 and whose continuous dynamics are such that $\Sigma_3(q_6) = \Sigma_3(q_7) = \Sigma_3(q_8) = \Sigma_3(q_9)$ and where $M_3(e) = I$, for any $e \in E_3$. Suppose that $\Sigma_3(q_i)$ are observable for any $q_i, i = 6, \dots, 9$. \mathbb{S}_3 is unobservable since Theorem 6 (ii) is not satisfied for q_6, q_7 . Moreover it is simple to check that $\mathbb{S}_3 \in \min(\mathbb{S}_3)$. Since the only hybrid bisimulation between \mathbb{S}_3 and itself is the identity relation \mathcal{R}_{id} , $\mathbb{S}_3^{\mathcal{R}_{id}}$ is unobservable and hence \mathbb{S}_3 is properly unobservable.

Non-proper and proper unobservability can be characterized as follows.

Proposition 16: *An SLS \mathbb{S} is non-properly unobservable if and only if \mathbb{S} is unobservable and any $\mathbb{S}' \in \min(\mathbb{S})$ is observable.*

Proof: Sufficiency holds by definition. For the necessity part, suppose by contradiction that \mathbb{S} is observable or $\exists \mathbb{S}' \in \min(\mathbb{S})$ unobservable. If \mathbb{S} is

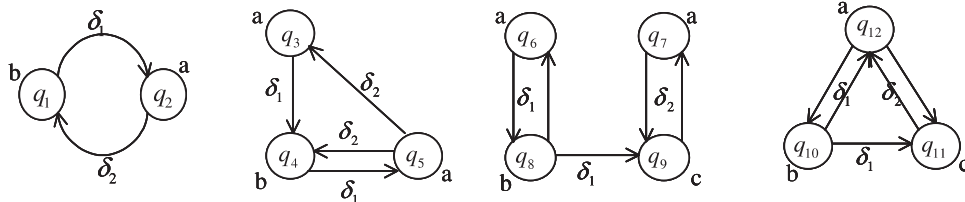


Figure 2. DESs $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3$ and \mathbb{D}_4 .

observable a contradiction holds. Now suppose $\exists \mathbb{S}' \in \min(\mathbb{S})$ unobservable. If \mathbb{S}' is unobservable, by Theorem 6 and Corollary 6.4 in (van der Schaft 2004c), there exist $q_1, q_2 \in Q^{\mathcal{R}^*}$, such that $\gamma^{\mathcal{R}^*}(q_1) = \gamma^{\mathcal{R}^*}(q_2)$ and $\Sigma^{\mathcal{R}^*}(q_1) \sim \Sigma^{\mathcal{R}^*}(q_2)$. It is simple to check by definition of \mathbb{S}' , that any pair $(q'_1, q'_2) \in \Pi_Q^{-1}(q_1) \times \Pi_Q^{-1}(q_2) \subset Q \times Q$ is such that for any hybrid bisimulation and equivalence relation \mathcal{R} between \mathbb{S} and itself, $(q'_1, q'_2) \notin Q_{\mathcal{R}}$, and then the reduced SLS $\mathbb{S}^{\mathcal{R}}$ will have two discrete states $\Pi_Q(q'_1), \Pi_Q(q'_2) \in Q^{\mathcal{R}}$ such that $\gamma^{\mathcal{R}}(\Pi_Q(q'_1)) = \gamma^{\mathcal{R}}(\Pi_Q(q'_2))$ and $\Sigma^{\mathcal{R}}(\Pi_Q(q'_1)) \sim \Sigma^{\mathcal{R}}(\Pi_Q(q'_2))$; then $\mathbb{S}^{\mathcal{R}}$ is unobservable for any \mathcal{R} , and hence \mathbb{S} is not nonproperly unobservable. \square

Proposition 17: *An SLS \mathbb{S} is properly unobservable if and only if \mathbb{S} is unobservable and any $\mathbb{S}' \in \min(\mathbb{S})$ is unobservable.*

The proof of the result above is an easy consequence of Definition 13 and Proposition 16, and is therefore omitted.

Remark 16: Conditions of Propositions 16 and 17 may be easily checked by constructing the minimal bisimilar SLS of the given SLS \mathbb{S} . As pointed out in Remark 14, this construction reduces to the computation of the maximal hybrid bisimulation between the SLS \mathbb{S} and itself, as formalized in Corollary 4.

In view of the discussion above, reduction via hybrid bisimulation is a good tool to extract the observable dynamics of any non-properly unobservable SLS. This result clearly links to the well known Kalman decomposition of linear dynamical systems, that has been recently extended to the class of SLSs (see (De Santis *et al.* 2003, 2004b)). However if one wants to extract the observable dynamics of a properly unobservable SLS, reduction is not the right tool as shown by Example 4. Therefore the key idea is to use abstraction instead of reduction. The following result formalizes that idea.

Proposition 18: *For any properly unobservable SLS \mathbb{S} , there exists an abstraction \mathbb{S}^A of \mathbb{S} which is observable and which simulates \mathbb{S} .*

Proof: Define Q^A such that $\exists q_i, q_j \in Q^A, q_i \neq q_j$ such that $\gamma(q_i) = \gamma(q_j)$ and for any $q \in Q$ set $\mathcal{M}_q := \Pi_q^{\mathcal{R}^*}$ and $\Pi_q^{\mathcal{R}^*}: X(q) \rightarrow X(q)/_{\mathcal{R}^*(q)}$, $\mathcal{R}^*(q)$ being the maximal linear bisimulation between $\Sigma(q)$ and itself, for any $q \in Q$. Since, as shown in the proof of Proposition 14, $\ker \mathcal{M}_q \subset \ker C(q)$, the SLS \mathbb{S}^A is an abstraction of \mathbb{S} . By construction $\Sigma^A(q)$ is observable for any $q \in Q$ (see van der Schaft (2004c)) and then \mathbb{S}^A satisfies conditions of Theorem 6: therefore \mathbb{S}^A is observable and finally by Proposition 13 the statement holds. \square

The following example shows an application of Proposition 18.

Example 5: Consider SLS \mathbb{S}_3 of Example 4. Define an SLS \mathbb{S}_4 , whose DES $\mathbb{D}_{\mathbb{S}_4} = \mathbb{D}_4$ is depicted in figure 2 and where $\Sigma_4(q_{10}) = \Sigma_4(q_{11}) = \Sigma_4(q_{12}) := \Sigma_3(q_6)$ and $M_4(e) = I$, for any $e \in E_4$. \mathbb{S}_4 is observable, is an abstraction of \mathbb{S}_3 and simulates \mathbb{S}_3 .

7. Some possible extensions

Aim of this section is to briefly discuss some possible extensions of the proposed methodology for reducing the complexity of the class of switching linear systems. In fact the proposed results are easily extendable to switching systems where the continuous dynamics are governed by non-linear control systems or by linear systems in pencil forms.

The key results to be generalized in these new frameworks are:

- Theorem 2, characterizing the algebraic properties for a set to be a hybrid bisimulation for the class of switching systems under consideration,
- The procedures described in §3 for computing the maximal hybrid bisimulation,
- The definition of reduced switching systems as formalized in §4.

In the following we offer some remarks for generalizing these key results in the case of switching non-linear systems and of switching systems in pencil form.

We refer to a switching non-linear system, respectively to a switching linear system in pencil form, as a tuple $(\Xi, U, D, Y, \Sigma, \gamma, E, M)$ as in Definition 1 where Σ is replaced by the following function that associates to any discrete state $q \in Q$, the non-linear dynamical system

$$\Sigma(q) : \begin{cases} \dot{x}(t) = a(q, x(t)) + b(q, x(t))u(t) + g(q, x(t))w(t), \\ h(t) = c(q, x(t)), \quad t \geq 0; \end{cases}$$

$a(\cdot, \cdot)$, $b(\cdot, \cdot)$, $g(\cdot, \cdot)$ and $c(\cdot, \cdot)$ being sufficiently smooth vector fields, respectively by the following function associating to any discrete state $q \in Q$, the linear dynamical system in pencil form

$$\Sigma(q) : \begin{cases} L(q)\dot{x}(t) = A(q)x(t), \\ w(t) = H(q)x(t), \end{cases}$$

with $L(\cdot)$, $A(\cdot)$ and $H(\cdot)$ being matrices of appropriate dimensions. The notion of executions of switching non-linear systems and switching linear systems in

pencil form can be easily generalized from Definition 2 and is therefore omitted.

It is worth to point out that Theorem 2 is still true both for the class of switching non-linear systems and for the class of switching linear systems in pencil form. In particular note that Theorem 2 (i) implicitly requires that $\mathcal{R}(q_1, q_2)$ is a bisimulation relation between the pair of dynamical systems $\Sigma_1(q_1)$ and $\Sigma_2(q_2)$. An algebraic characterization of bisimulations for non-linear systems can be found in Proposition 7.9 of (van der Schaft 2004c) and an algebraic characterization of bisimulations for linear dynamical systems in pencil form can be found in Theorem 3.2. of (van der Schaft 2004b).

As far as for the computation of the maximal hybrid bisimulation, it can be done by generalizing procedures described in §3, where the operator $\text{Bisim}(\cdot, \cdot)$ should be replaced by Algorithm 7.5 of (van der Schaft 2004c) in the case of switching non-linear systems, and by Algorithm 3.3 of (van der Schaft 2004b), in the case of switching linear systems in pencil forms.

Finally in view of the remarks above showing how to compute the maximal hybrid bisimulation, the reduced switching non-linear system and the reduced switching linear system in pencil form may be defined following the guidelines of §4; we leave the details to the reader.

8. Conclusions and outlook

In this paper we studied bisimulation for the class of switching linear systems. The proposed definition is mainly inspired by the classical notions given for concurrent processes (Park (1981), Milner (1989), Clarke *et al.* (2002), Hermanns (2002)) and by the definitions introduced in (van der Schaft 2004a, b, c). The definition considers the general case where the switching mechanisms in the switching linear systems may be asynchronous. Moreover the definition includes more general frameworks of hybrid systems as for example the case of discrete transitions depending on the continuous state x .

A complete algebraic characterization of bisimulation has been developed by combining tools from the theory of concurrent processes (Clarke *et al.* (2002), Hermanns (2002)) with those from control theory (De Santis *et al.* (2004a), van der Schaft 2004a)). Moreover an algorithmic procedure converging to the maximal hybrid bisimulation relation in a finite number of steps has been developed: this procedure allows the computation of the minimal bisimilar SLS of a given SLS and offers some interesting links to the analysis of observability of SLSs as given in (De Santis *et al.* 2003).

The proposed approach easily extends to the case of discrete-time switching linear systems. Moreover, §7 gives some guidelines for generalizing the results of this paper to the case of switching non-linear systems and switching systems in pencil form. The presented results give also sufficient conditions for characterizing hybrid bisimulations for more general hybrid systems models with x -dependent discrete transitions. For instance an SLS \mathbb{S} can be easily constructed to mimic the behaviour of a hybrid system \mathbb{H} whose discrete transitions are x -dependent in such a way that \mathbb{S} is an abstraction of \mathbb{H} ; therefore a preliminary sub-optimal state space reduction of \mathbb{H} can be pursued by reducing \mathbb{S} : this last approach is important also in view of the results shown in (Alur *et al.* 2000) that proves the non-decidability of bisimulation-based reduction for general hybrid systems.

Following the guidelines of the proposed approach an extension of the present results would be the study of bisimulation theory for hybrid systems with x -dependent discrete transitions.

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Appendix: Technical proofs

Proof of Theorem 1: (*Proof of (i)*). With reference to Definition 8, given two SLSs \mathbb{S}_1 and \mathbb{S}_2 let $\mathcal{R} \subset \Xi_1 \times \Xi_2$ be such that $((q_1, x_1), (q_2, x_2)) \in \mathcal{R}$ if and only if $q_2 = T_{\mathbb{Q}}(q_1)$ and $x_2 = T_{q_1}(x_1)$. By construction \mathcal{R} is a hybrid bisimulation between \mathbb{S}_1 and \mathbb{S}_2 , such that $\Pi|_{\Xi_i}(\mathcal{R}) = \Xi_i$, $i = 1, 2$.

(*Proof of (ii)*). Obvious by definition.

(*Proof of (iii)*). Since \mathbb{S}_1 and \mathbb{S}_2 are bisimilar there exists a hybrid bisimulation $\mathcal{R} \subset \Xi_1 \times \Xi_2$ such that $\Pi|_{\Xi_i}(\mathcal{R}) = \Xi_i$, $i = 1, 2$ and that therefore satisfies conditions of Lemma 1.

(*Proof of (iv)*). For any given control law $u \in U_1$, for any $\xi_{10} \in \Xi_1$, consider a hybrid state $\xi_{20} \in \Xi_2$ such that $(\xi_{10}, \xi_{20}) \in \mathcal{R}$ for some hybrid simulation \mathcal{R} of \mathbb{S}_1 by \mathbb{S}_2 . By definition for any hybrid disturbance d_1 , for any execution $\chi_1 = (\xi_{10}, \tau_1, u, d_1, \xi_1, y_1)$ of \mathbb{S}_1 , there exists a hybrid disturbance d_2 and an execution $\chi_2 = (\xi_{20}, \tau_2, u, d_2, \xi_2, y_2)$ of \mathbb{S}_2 such that $y_1(t, j) = y_2(t', j')$, $\forall ((t, j), (t', j')) \in [\tau_1, \tau_2]$, for some $[\tau_1, \tau_2]$. Moreover the

same holds with ξ_{10} and d_1 replaced by ξ_{20} and d_2 and vice versa and hence the statement holds.

(*Proof of (v)*). For any given control law $u \in U_1$, for any $\xi_{10} \in \Xi_1$, consider a state-equivalent hybrid state $\xi_{20} \in \Xi_2$. By definition for any hybrid disturbance d_1 , for any execution $\chi_1 = (\xi_{10}, \tau_1, u, d_1, \xi_1, y_1)$ of \mathbb{S}_1 , there exists a hybrid disturbance d_2 and an execution $\chi_2 = (\xi_{20}, \tau_2, u, d_2, \xi_2, y_2)$ of \mathbb{S}_2 such that $y_1(t, j) = y_2(t', j')$, $\forall((t, j), (t', j')) \in [\tau_1, \tau_2]$, for some $[\tau_1, \tau_2]$. Moreover the same holds with ξ_{10} and d_1 replaced by ξ_{20} and d_2 and vice versa and hence the statement holds. \square

Proof of Proposition 2: By construction, for any $(\xi_{10}, \xi_{20}) \in \mathcal{L}(\mathcal{R})$, where $\xi_{10} = (q_{10}, x_{10})$ and $\xi_{20} = (q_{20}, x_{20})$, there exists $\lambda^a, \lambda^b \in \mathbb{R}$ and $(x_{10}^a, x_{20}^a), (x_{10}^b, x_{20}^b) \in \mathcal{R}(q_{10}, q_{20})$ such that $(x_{10}, x_{20}) = \lambda^a(x_{10}^a, x_{20}^a) + \lambda^b(x_{10}^b, x_{20}^b)$. Take any input function $u_1 = u_2 = u$ and any hybrid disturbance $d_1 = (v_1, w_1)$. Choose some $u_1^a, u_1^b \in \mathcal{C}^0(\mathbb{R}_0^+, U_1)$ and some $w_1^a, w_1^b \in \mathcal{C}^0(\mathbb{R}_0^+, W_1)$ such that

$$u_1 = \lambda^a u_1^a + \lambda^b u_1^b, \quad w_1 = \lambda^a w_1^a + \lambda^b w_1^b.$$

For any execution $\chi_1 = ((q_{10}, x_{10}), \tau_1, u_1, (v_1, w_1), \xi_1, y_1)$ of \mathbb{S}_1 , there exists executions $\chi_1^a = ((q_{10}, x_{10}^a), \tau_1, u_1^a, (v_1, w_1^a), \xi_1^a, y_1^a)$ and $\chi_1^b = ((q_{10}, x_{10}^b), \tau_1, u_1^b, (v_1, w_1^b), \xi_1^b, y_1^b)$ of \mathbb{S}_1 , such that

$$\begin{aligned} x_1(t, j) &= \lambda^a x_1^a(t, j) + \lambda^b x_1^b(t, j), \\ h_1(t, j) &= \lambda^a h_1^a(t, j) + \lambda^b h_1^b(t, j), \quad \forall(t, j) \in [\tau_1], \end{aligned} \quad (6)$$

where $\xi_1 = (q_1, x_1), \xi_1^a = (q_1^a, x_1^a), \xi_1^b = (q_1^b, x_1^b), y_1 = (p_1, h_1), y_1^a = (p_1^a, h_1^a), y_1^b = (p_1^b, h_1^b)$. Moreover, since $(x_{10}^a, x_{20}^a), (x_{10}^b, x_{20}^b) \in \mathcal{R}(q_{10}, q_{20})$, there exist hybrid disturbances $d_2^a = (v_2, w_2^a) \in D_2$ and $d_2^b = (v_2, w_2^b) \in D_2$ and executions $\chi_2^a = ((q_{20}, x_{20}^a), \tau_2, u_2^a, (v_2, w_2^a), \xi_2^a, y_2^a)$ and $\chi_2^b = ((q_{20}, x_{20}^b), \tau_2, u_2^b, (v_2, w_2^b), \xi_2^b, y_2^b)$ of \mathbb{S}_2 , such that

$$\begin{aligned} (\xi_1^a(t, j), \xi_2^a(t', j')) &\in \mathcal{R}, \quad y_1^a(t, j) = y_2^a(t', j'), \\ (\xi_1^b(t, j), \xi_2^b(t', j')) &\in \mathcal{R}, \quad y_1^b(t, j) = y_2^b(t', j'), \end{aligned} \quad (7)$$

$\forall((t, j), (t', j')) \in [\tau_1, \tau_2]$, for some $[\tau_1, \tau_2]$. By choosing a hybrid disturbance $d_2 = (v_2, w_2)$ where $w_2 = \lambda^a w_2^a + \lambda^b w_2^b$, there exists an execution $\chi_2 = ((q_{20}, x_{20}), \tau_2, u_2, (v_2, w_2), \xi_2, y_2)$ of \mathbb{S}_2 such that

$$\begin{aligned} x_2(t, j) &= \lambda^a x_2^a(t, j) + \lambda^b x_2^b(t, j), \\ h_2(t, j) &= \lambda^a h_2^a(t, j) + \lambda^b h_2^b(t, j), \quad \forall(t, j) \in [\tau_2], \end{aligned} \quad (8)$$

and hence by combining (6), (8) and (7),

$$\begin{aligned} (x_1(t, j), x_2(t', j')) &= \lambda^a \cdot (x_1^a(t, j), x_2^a(t', j')) \\ &\quad + \lambda^b \cdot (x_1^b(t, j), x_2^b(t', j')) \\ &\in \mathcal{L}(\mathcal{R}(q_1(j), q_2(j))), \end{aligned}$$

$h_1(t, j) = h_2(t', j'), \forall((t, j), (t', j')) \in [\tau_1, \tau_2]$, for some $[\tau_1, \tau_2]$.

Therefore $(\xi_1(t, j), \xi_2(t', j')) \in \mathcal{L}(\mathcal{R})$ and $y_1(t, j) = y_2(t', j'), \forall((t, j), (t', j')) \in [\tau_1, \tau_2]$, for some $[\tau_1, \tau_2]$. By repeating the same proof with d_1 replaced by d_2 and vice versa the statement holds. \square

Proof of Theorem 3: We only have to prove Theorem 3 (ii) since the other properties follow directly by the definition of \mathcal{R}^* . Let \mathcal{R}^* be a hybrid bisimulation such that $Q_{\mathcal{R}^*}$ is of maximal cardinality and $\mathcal{R}^*(q_1, q_2)$ is of maximal dimension for any $(q_1, q_2) \in Q_{\mathcal{R}^*}$ and satisfying hypotheses of Theorem 2 (note that \mathcal{R}^* exists since the cardinality of the state spaces Q_1 and Q_2 are finite and the dimensions of the linear spaces $X_1(\cdot)$ and $X_2(\cdot)$ are finite). We now show that any hybrid bisimulation \mathcal{R} between \mathbb{S}_1 and \mathbb{S}_2 is such that $\mathcal{R} \subset \mathcal{R}^*$. Suppose by contradiction that $\mathcal{R} \not\subset \mathcal{R}^*$. Then $Q_{\mathcal{R}} \not\subset Q_{\mathcal{R}^*}$ or there exists $(q_1, q_2) \in Q_{\mathcal{R}} \subset Q_{\mathcal{R}^*}$ such that $\mathcal{R}(q_1, q_2) \not\subset \mathcal{R}^*(q_1, q_2)$. If $Q_{\mathcal{R}} \not\subset Q_{\mathcal{R}^*}$ then $\mathcal{R} + \mathcal{R}^* \not\subset \mathcal{R}^*$ and by Proposition 5, $\mathcal{R} + \mathcal{R}^*$ is a hybrid bisimulation: therefore a contradiction holds in the maximal cardinality of $Q_{\mathcal{R}^*}$. Suppose now that $Q_{\mathcal{R}} \subset Q_{\mathcal{R}^*}$ and that there exists $(q_1, q_2) \in Q_{\mathcal{R}} \subset Q_{\mathcal{R}^*}$ such that $\mathcal{R}(q_1, q_2) \not\subset \mathcal{R}^*(q_1, q_2)$. If $\mathcal{R}(q_1, q_2) \not\subset \mathcal{R}^*(q_1, q_2)$ then $\mathcal{R} + \mathcal{R}^* \not\subset \mathcal{R}^*$ and by Proposition 5, $\mathcal{R} + \mathcal{R}^*$ is a hybrid bisimulation: therefore a contradiction holds in the maximal dimension of $\mathcal{R}^*(q_1, q_2)$. The uniqueness of \mathcal{R}^* can be easily proved by contradiction. \square

Proof of Proposition 11: Consider two sets $Q_1^{\mathcal{R}^*}$ and $Q_2^{\mathcal{R}^*}$ of canonical representatives of equivalence classes induced by $Q_{\mathcal{R}^*}$ on Q and the corresponding reduced SLSs $\mathbb{S}_1^{\mathcal{R}^*}$ and $\mathbb{S}_2^{\mathcal{R}^*}$. By construction $\mathbb{S}_1^{\mathcal{R}^*}$ and $\mathbb{S}_2^{\mathcal{R}^*}$ are bisimilar and therefore there exists a hybrid bisimulation \mathcal{R}' between $\mathbb{S}_1^{\mathcal{R}^*}$ and $\mathbb{S}_2^{\mathcal{R}^*}$ such that $\Pi|_{\Xi}(\mathcal{R}') = \Xi_i, i = 1, 2$. By definition of $Q_1^{\mathcal{R}^*}$ and $Q_2^{\mathcal{R}^*}$, the relation $Q_{\mathcal{R}'} \subset Q_1^{\mathcal{R}^*} \times Q_2^{\mathcal{R}^*}$ is one-to-one, i.e. there exists an invertible mapping $T_{\mathbb{Q}} : Q_1^{\mathcal{R}^*} \rightarrow Q_2^{\mathcal{R}^*}$ such that $(q_1, q_2) \in Q_{\mathcal{R}'}$ if and only if $T_{\mathbb{Q}}(q_1) = q_2$. It is easily seen that $T_{\mathbb{Q}}$ satisfies Definition 8 (i) and (ii). Moreover for any $(q_1, q_2) \in Q_{\mathcal{R}'}$, the dynamical systems $\Sigma_1^{\mathcal{R}^*}(q_1)$ and $\Sigma_2^{\mathcal{R}^*}(q_2)$ are bisimilar and since $\Sigma_1^{\mathcal{R}^*}(q_1) \in \min(\Sigma_1^{\mathcal{R}^*}(q_1)), \Sigma_2^{\mathcal{R}^*}(q_2) \in \min(\Sigma_2^{\mathcal{R}^*}(q_2))$, by Lemma 4, they are algebraically equivalent, i.e. there exists an invertible matrix T_{q_1} such that Definition 8 (iii) is satisfied and $(x_1, x_2) \in \mathcal{R}'(q_1, q_2)$ if and only if

$$x_1 = T_{q_1} x_2. \quad (9)$$

Finally by rewriting Theorem 2 (ii) for \mathcal{R}' , one obtains

$$\forall (x_1, x_2) \in \mathcal{R}(q_1, q_2), (M_1(e_1)x_1, M_2(e_2)x_2) \in \mathcal{R}(q'_1, q'_2), \quad (10)$$

where $T_{\mathbb{Q}}(q_1) = q_2, T_{\mathbb{Q}}(q'_1) = q'_2$ and $e_i, i = 1, 2$ are appropriate discrete transitions. By condition (9), condition (10) can be rewritten as $M_1(e_1)T_{q_1}x_2 = T_{q'_1}M_2(e_2)x_2, \forall x_2 \in X(q_2)$ and then

$$M_1(e_1)T_{q_1} = T'_{q'_1}M_2(e_2). \quad (11)$$

By combining condition (11) with Definition 8 (ii), Definition 8 (iv) holds. \square

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