Sufficient conditions for admissibility

Hans Zwart^a

^a*University of Twente
Department of Applied Mathematics
P.O. Box 217
NL-7500 AE Enschede
The Netherlands
h.j.zwart@math.utwente.nl*

Abstract

Sufficient conditions for the finite and infinite-time admissibility of an observation operator are given. It is shown that the estimates of Weiss are close to being sufficient. If the semigroup is surjective, then the estimate is sufficient.

Key words: Infinite-dimensional system, admissible observation operator, infinite-time admissible observation operator.

1 Introduction

Consider the system

$$\dot{z}(t) = Az(t), \quad y(t) = Cz(t), \quad z(0) = z_0, \quad (1)$$

where A is the infinitesimal generator of a C_0 -semigroup $T(t)$ on the Hilbert space Z , and C is a bounded operator from the domain of A , $D(A)$, to a second output space Y . An important question is whether the system as defined in (1) is well-posed. Of course since A generates a C_0 -semigroup on Z , we know that the state equation has the unique solution $z(t) = T(t)z_0$. However, since C is not a bounded operator on Z , it is not clear whether the output equation is well-posed. Pointwise in time the output equation only make sense if C is bounded operator on Z . However, one could relax this to the question if the output trajectory is locally square integrable. If this holds, then C is called an admissible observation operator, see Weiss [6].

In [7] it was conjectured¹ that C is an admissible output operator if and only if there exists an M such

$$\|C(sI - A)^{-1}\|^2 \leq M/\operatorname{Re}(s) \quad (2)$$

for s in some right-half plane. Already in [7] one can find that admissibility always implies (2). However, recently Zwart, Jacob and Staffans [9], Jacob and Zwart [4], and Jacob, Partington and Pott [3] showed that the converse does not hold. Hence the estimate (2) is not sufficient, and so it is natural to ask if there are sufficient conditions. Using the Hille-Yosida estimate, Grabowski and Callier [2] derived a necessary and sufficient condition. However, since this condition involves all the powers of $(sI - A)^{-1}$, it is not so simple to test. So there is a need for simple sufficient conditions. In this paper we present some of these conditions. For instance, we show that if there exists a $\gamma > 1$ such that $\operatorname{Re}(s)^\gamma \|C(sI - A)^{-1}\|^2$ is bounded on some right half plane, then C is an admissible observation operator.

Apart from the well-posedness of the equation (1), one would like to know when the output is square integrable over the whole time axis. This is known as infinite-time admissibility, see Grabowski [1]. In [8] it was conjectured that a scalar observation operator is infinite-time admissible if and only if

$$\|C(sI - A)^{-1}\|^2 \leq M/\operatorname{Re}(s) \quad (3)$$

for all s real part positive. For general output operators, Zwart, Jacob and Staffans [9] showed that condition (3) is not sufficient. For one-dimensional output operators one may find a similar result in Jacob and Zwart [4]. In Section 4 we present a simple sufficient condition when an observation operator is infinite-time admissible. Furthermore, we show that (3) the L^2 -norm of $CT(\cdot)$ over a time interval of length t grows at most as $\log(t)$.

In the proof of our theorems the following spaces and results are used frequently.

Definition 1.1 *Let Z and Y be Hilbert spaces, and let $\mathcal{L}(Z, Y)$ denote the bounded linear operators from Z to Y . Furthermore, we define*

$$L_2((0, \infty); Z) := \{h : [0, \infty) \rightarrow Z \mid \int_0^\infty \|h(t)\|^2 dt < \infty\};$$

¹ In [7] Weiss studies admissible input operators. Since they are dual notions, see [6], any result/conjecture for admissible input operator has its equivalent counterpart for admissible observation operators

$$H_2(Z) := \{f : \mathbb{C}_0^+ \rightarrow Z \mid f \text{ is analytic and } \sup_{r>0} \int_{-\infty}^{\infty} \|f(r + j\omega)\|^2 d\omega < \infty\};$$

$$H_\infty(\mathcal{L}(Z, Y)) := \{G : \mathbb{C}_0^+ \rightarrow \mathcal{L}(Y, Z) \mid G \text{ is analytic and } \sup_{\operatorname{Re}(s)>0} \|G(s)\| < \infty\}$$

Here $\mathbb{C}_0^+$ denotes the open right half plane of the complex plane.

It is well-known that the Laplace transform of a $f \in L_2(0, \infty); Z$ lies in $H_2(Z)$ and

$$\|f\|_{L_2(0, \infty); Z} = \frac{1}{\sqrt{2\pi}} \|\hat{f}\|_{H_2(Z)}, \quad (4)$$

where $\hat{f}$ denotes the Laplace transform of f . Furthermore, one has that for any $G \in H_\infty(\mathcal{L}(Z, Y))$ and any $f \in H_2(Z)$ the function $Gf \in H_2(Y)$ and

$$\|Gf\|_{H_2(Y)} \leq \|G\|_{H_\infty(\mathcal{L}(Z, Y))} \|f\|_{H_2(Z)}. \quad (5)$$

Hence G defines a bounded linear operator from $H_2(Z)$ to $H_2(Y)$.

2 General results

We begin with the formal definition of an admissible observation operator.

Definition 2.1 *Let $T(t)$ be a C_0 -semigroup on the Hilbert space Z , and let C be a bounded linear operator from the domain of A , $D(A)$ to a second Hilbert space Y . Then C is an admissible observation operator if there exist $t_1 > 0$ and $M > 0$ such that for all $z_0 \in D(A)$*

$$\int_0^{t_1} \|CT(t)z_0\|_Y^2 dt \leq M \|z_0\|^2.$$

Thus if C is an admissible observation operator, then the mapping $z_0 \mapsto CT(\cdot)z_0$ can be extended to a bounded mapping from Z to $L^2((0, t_1); Y)$.

It is easy to see that if C is an admissible observation operator for $T(t)$, then it is also an admissible observation operator for $e^{\omega t}T(t)$. Hence if we want to investigate admissible observation operators, then we may assume that the semigroup is exponentially stable.

Now we can formulate our first result. Note that this result was first obtained by Weiss [7]. However, our proof is much shorter.

Theorem 2.2 *Let $T(t)$ be an exponential stable, and surjective C_0 -semigroup with generator A on the Hilbert space Z , and let C be a bounded linear operator from $D(A)$ to Y . If $C(sI - A)^{-1}$ is bounded on the right half-plane, then C is an admissible observation operator.*

Proof Let x_0 be an element of Z . Since $T(t)$ is exponentially stable, we have that $((s+1)I - A)^{-1}x_0 \in H_2(Z)$. Using now that $C(sI - A)^{-1} \in H_\infty(\mathcal{L}(Z, Y))$, we obtain that

$$C(sI - A)^{-1}((s+1)I - A)^{-1}x_0 \in H_2(Z).$$

By the resolvent identity, we see that this expression equals

$$C(sI - A)^{-1}x_0 - C((s+1)I - A)^{-1}x_0.$$

Thus by taking the inverse Laplace transform, we have that

$$CT(t)x_0 - e^{-t}CT(t)x_0 = (1 - e^{-t})CT(t)x_0 \in L_2((0, \infty); Y). \quad (6)$$

Since $1 - e^{-t}$ is bounded away from zero on $[1, \infty)$, we get that

$$CT(t)T(1)x_0 = CT(t+1)x_0 \in L_2((0, \infty); Y).$$

Now using the fact that $T(1)$ is surjective we have that for every $z_0 \in Z$, ($z_0 = T(1)x_0$), $CT(t)z_0 \in L_2((0, \infty); Y)$. And hence we have admissibility. ■

3 Sufficient conditions for admissibility

In Weiss [7] it was conjectured that C is an admissible observation operator if and only if

$$\|C(sI - A)^{-1}\| \leq \frac{K}{\sqrt{\operatorname{Re}(s)}}, \quad (7)$$

for all complex s in some right half-plane. In Zwart, Jacob and Staffans [9] it is shown that this does not hold in general. Here we show that the estimate (7) almost implies admissibility. For the proof of this result we need the following simple observation.

Lemma 3.1 *Let A be a closed linear operator on Z , and let $C \in \mathcal{L}(D(A), Y)$. Then the following are equivalent.*

(1) *There exists an $M > 0$ such that for all $s \in \mathbb{C}_0^+$ we have*

$$\|C(sI - A)^{-1}\| \leq \frac{M}{\sqrt{\operatorname{Re}(s)}},$$

(2) *For all $\gamma > 0$ we have that $C((\cdot + \gamma)I - A)^{-1} \in H_\infty(\mathcal{L}(Z, Y))$ and*

$$\|C((\cdot + \gamma)I - A)^{-1}\|_{H_\infty(\mathcal{L}(Z, Y))} \leq \frac{M}{\sqrt{\gamma}}.$$

Furthermore, we need the following facts and notation.

- (1) There exist (unique) numbers $x_2 < 1 < x_1$ such that $x_i e^{-x_i} = 1/(2e)$, $i = 1, 2$.
- (2) For x_1 and x_2 as above, we have that

$$x e^{-x} > 1/(2e), \quad \text{for all } x \in (x_2, x_1).$$

The proof of this follows directly from the fact that $x e^{-x}$ is zero at 0 and at infinity, and that it is monotonically increasing and decreasing on $[0, 1)$ and $(1, \infty)$, respectively.

With these facts we prove the following result.

Theorem 3.2 *Let $T(t)$ be a strongly continuous semigroup on the Hilbert space Z , with generator A . Assume that $C \in \mathcal{L}(D(A), Y)$ satisfies the estimate*

$$\|C(sI - A)^{-1}\| \leq \frac{K}{\sqrt{\operatorname{Re}(s)}},$$

for all complex s in some right half-plane. Then the following holds

(1) *for every $r \in [1, 2)$ there exists a $M_r \geq 0$ such that for all $z_0 \in Z$*

$$\int_0^1 \|CT(t)z_0\|^r dt \leq M_r \|z_0\|^r. \tag{8}$$

Thus C is admissible observation operator for $Z, Y, T(\cdot)$ and r for any $r \in [1, 2)$, see section 6 of Weiss [6].

(2) *There exists a $\kappa > 0$ such that for all $z_0 \in Z$ and all $\varepsilon \in (0, x_2)$ we have that*

$$\int_{\varepsilon}^1 \|CT(t)z_0\|^2 dt \leq \kappa |\log(\varepsilon)| \|z_0\|^2, \tag{9}$$

Proof First we see that since the semigroup generated by $\omega I + A$ is given by $e^{\omega t}T(t)$, we can without loss of generality assume that $T(t)$ is exponentially stable, and

$$\|C(sI - A)^{-1}\| \leq \frac{K}{\sqrt{\operatorname{Re}(s)}},$$

for all complex $s \in \mathbb{C}_0^+$.

For $n \in \mathbb{N}$ define

$$\lambda_n = \left(\frac{x_1}{x_2}\right)^{n-1}, \quad x_n = \frac{x_1}{\lambda_n},$$

so that $\lambda_n \rightarrow \infty$, $x_n \rightarrow 0$ (monotone) and $\lambda_n x_n = x_1$, $\lambda_n x_{n+1} = x_2$. Hence

$$\lambda_n x_{n+1} e^{-\lambda_n x_{n+1}} = \frac{1}{2e} = \lambda_n x_n e^{-\lambda_n x_n}$$

and

$$\lambda_n t e^{-\lambda_n t} > \frac{1}{2e}, \quad \text{for all } t \in (x_{n+1}, x_n). \quad (10)$$

Since $T(t)$ is exponentially stable, we have that for all $\lambda > 0$ there holds

$$\|((\cdot + \lambda)I - A)^{-1} z_0\|_{H^2} \leq \frac{M}{\sqrt{2\lambda}} \cdot \|z_0\|.$$

Using (7) and Lemma 3.1, it follows that for all $\lambda > 0$ we have that

$$\begin{aligned} \|C((\cdot + \lambda)I - A)^{-2} z_0\|_{H^2(Y)} &\leq \|C((\cdot + \lambda)I - A)^{-1}\|_{H^\infty} \|((\cdot + \lambda)I - A)^{-1} z_0\|_{H^2(Z)} \\ &\leq \frac{K}{\sqrt{\lambda}} \cdot \frac{M}{\sqrt{2\lambda}} \|z_0\| \leq \frac{K_1}{\lambda} \|z_0\|, \end{aligned}$$

with $K_1 > 0$ independent of λ . In the time domain this means that, for all $\lambda > 0$ and all $z_0 \in D(A)$

$$\int_0^\infty \|\lambda t e^{-\lambda t} C T(t) z_0\|^2 dt \leq \frac{K_1^2}{2\pi} \|z_0\|^2. \quad (11)$$

Now we shall prove the first assertion.

Part 1. Let $r < 2$

$$\begin{aligned} \int_0^{x_1} \|CT(t)z_0\|^r dt &= \sum_{n=1}^{\infty} \int_{x_{n+1}}^{x_n} \|CT(t)z_0\|^r dt \\ &\leq (2e)^r \sum_{n=1}^{\infty} \int_{x_{n+1}}^{x_n} \|\lambda_n t e^{-\lambda_n t} CT(t)z_0\|^r dt, \end{aligned}$$

where we have used (10). Now we use the Hölder inequality with $q = 2/r$ (note that $q > 1$). With $1/p + 1/q = 1$ and using (11),

$$\begin{aligned} \int_0^{x_1} \|CT(t)z_0\|^r dt &\leq (2e)^r \sum_{n=1}^{\infty} \sqrt[p]{\int_{x_{n+1}}^{x_n} 1 dt} \cdot \sqrt[q]{\int_{x_{n+1}}^{x_n} \|\lambda_n t e^{-\lambda_n t} CT(t)z_0\|^{rq} dt} \\ &\leq (2e)^r \sum_{n=1}^{\infty} \sqrt[p]{(x_n - x_{n+1})} \cdot \frac{1}{\sqrt[q]{2\pi}} \sqrt[q]{K_1^2 \|z_0\|^2} \\ &= \sum_{n=1}^{\infty} \sqrt[p]{\left(\frac{x_2}{x_1}\right)^{n-1} (x_1 - x_2)} \cdot \frac{1}{\sqrt[q]{2\pi}} (2eK_1)^r \|z_0\|^r. \end{aligned}$$

Since $\sqrt[p]{\left(\frac{x_2}{x_1}\right)} < 1$, the sum converges and we obtain that for some $M_r^1 > 0$,

$$\int_0^{x_1} \|CT(t)z_0\|^r dt \leq M_r^1 \|z_0\|^r, \text{ for all } z_0 \in D(A). \quad (12)$$

Since $x_1 > 1$, we see that we have proved the first assertion.

Part 2. For $N \in \mathbb{N}$ we have that

$$\begin{aligned} \int_{x_N}^{x_1} \|CT(t)z_0\|^2 dt &= \sum_{n=1}^{N-1} \int_{x_{n+1}}^{x_n} \|CT(t)z_0\|^2 dt \\ &\leq (2e)^2 \sum_{n=1}^{N-1} \int_{x_{n+1}}^{x_n} \|\lambda_n t e^{-\lambda_n t} CT(t)z_0\|^2 dt, \end{aligned}$$

where we have used (10). With (11) we obtain

$$\int_{x_N}^{x_1} \|CT(t)z_0\|^2 dt \leq (2e)^2 \sum_{n=1}^{N-1} \int_0^{\infty} \|\lambda_n t e^{-\lambda_n t} CT(t)z_0\|^2 dt$$

$$\leq (2e)^2 \sum_{n=1}^{N-1} \frac{K_1^2}{2\pi} \|z_0\|^2 = \frac{2}{\pi} e^2 (N-1) K_1^2 \|z_0\|^2.$$

By definition of x_n we have that

$$x_N = x_1 \left(\frac{x_2}{x_1} \right)^{N-1}.$$

Thus

$$\log(x_N) = \log(x_1) - (N-1) \log\left(\frac{x_1}{x_2}\right) \quad (13)$$

Let ε be a number in the interval $(0, x_2)$. Since $x_n \rightarrow 0$, there exists a $N \in \mathbb{N}$ such that $x_N \leq \varepsilon < x_{N-1}$. For this N we have that

$$\begin{aligned} |\log(\varepsilon)|^{-1} \int_{\varepsilon}^1 \|CT(t)z_0\|^2 dt &\leq |\log(x_{N-1})|^{-1} \int_{x_N}^{x_1} \|CT(t)z_0\|^2 dt \\ &\leq |\log(x_{N-1})|^{-1} \frac{2}{\pi} e^2 (N-1) K_1^2 \|z_0\|^2. \end{aligned}$$

Now by (13) we see that the assertion of part 2. holds. ■

Hence we have seen that the estimate (7) almost implies the admissibility of C . In the following theorem we show that a slightly stronger estimate than (7) is sufficient for admissibility.

Theorem 3.3 *Let A be the infinitesimal generator of the C_0 -semigroup $T(t)$ on the Hilbert space Z and let $C \in \mathcal{L}(D(A), Y)$, where Y is a second Hilbert space. If there exists a $K > 0$ such that*

$$\|C(sI - A)^{-1}\| \leq \frac{K}{\log(\operatorname{Re}(s))\sqrt{\operatorname{Re}(s)}}, \quad (14)$$

for all complex s in some right half-plane, then C is an admissible observation operator.

Proof The proof goes along the same lines as the previous proof. Since admissibility of C for $T(t)$ is equivalent to admissibility of C for $e^{\omega t}T(t)$, we can without loss of generality assume that $T(t)$ is exponentially stable, and that for all $\gamma > 1$ there holds

$$\|C((\cdot + \gamma)I - A)^{-1}\|_{H_\infty(\mathcal{L}(Z, Y))} \leq \frac{K}{\log(\gamma)\sqrt{\gamma}}.$$

Similar as in the proof of Theorem 3.2, we obtain for $\lambda > 1$

$$\int_0^\infty \|\lambda e^{-\lambda t} CT(t) z_0\|^2 dt \leq K^2 \|z_0\|^2 \log(\lambda)^{-2}. \quad (15)$$

Now we proceed as in part 2. of the previous proof. For $N > 1$ we have

$$\begin{aligned} \int_{x_N}^{x_1} \|CT(t) z_0\|^2 dt &= (2e)^2 \sum_{n=1}^{N-1} \int_{x_{n+1}}^{x_n} \|\lambda_n t e^{-\lambda_n t} CT(t) z_0\|^2 dt \\ &\leq (2e)^2 \sum_{n=1}^{N-1} \int_0^\infty \|\lambda_n t e^{-\lambda_n t} CT(t) z_0\|^2 dt \\ &\leq (2e)^2 \sum_{n=1}^{N-1} K_1^2 \log(\lambda_n)^{-2} \|z_0\|^2. \end{aligned}$$

Since $\log(\lambda_n)$ behaves like n for n large, we see that the above inequality implies that

$$\int_{x_N}^{x_1} \|CT(t) z_0\|^2 dt \leq K_2 \|z_0\|^2$$

for all $N \in \mathbb{N}$ and all $z_0 \in D(A)$. Using now the fact that $x_N \rightarrow 0$, we conclude that C is admissible. ■

Looking carefully to the above proof, we see that we could conclude admissibility if

$$\|C(sI - A)^{-1}\| \leq \frac{K}{g(\operatorname{Re}(s))\sqrt{\operatorname{Re}(s)}},$$

for all s in some right-half plane with g a monotonically increasing function satisfying

$$\sum_{n=1}^{\infty} g(\alpha^n)^{-2} < \infty$$

for some $\alpha > 1$.

4 Sufficient conditions for infinite-time admissibility

In the previous section we studied the well-posedness of (1) on small time intervals. In this section, we study the well-posedness on the entire interval $[0, \infty)$. By well-posedness we again mean that the output is square integrable.

Definition 4.1 *Let $T(t)$ be a C_0 -semigroup on the Hilbert space Z , and let C be a bounded linear operator from the domain of A , $D(A)$ to a second Hilbert space Y . Then C is an infinite-time admissible observation operator if there exist $M > 0$ such that for all $z_0 \in D(A)$*

$$\int_0^\infty \|CT(t)z_0\|_Y^2 dt \leq M\|z_0\|^2.$$

Thus if C is an infinite-time admissible observation operator, then the mapping $z_0 \mapsto CT(\cdot)z_0$ can be extended to a bounded mapping from Z to $L^2((0, \infty); Y)$. It is easy to see that if the C_0 -semigroup is exponentially stable, then any admissible C is infinite-time admissible.

Let C be an infinite-time admissible observation operator. Since $C(sI - A)^{-1}$ is the Laplace transform of $CT(t)z_0$ there holds

$$\begin{aligned} \|C(sI - A)^{-1}z_0\|^2 &= \left\| \int_0^\infty e^{-st}CT(t)z_0 dt \right\|^2 \\ &\leq \|e^{-s\cdot}\|_{L_2(0,\infty)}^2 \|CT(t)z_0\|_{L_2(0,\infty),Y}^2 \\ &\leq \frac{1}{2\operatorname{Re}(s)} M\|z_0\|^2, \end{aligned} \tag{16}$$

where we have used the Cauchy-Schwarz inequality. Hence we see that if C is infinite-time admissible, then there exists an $K > 0$ such that

$$\|C(sI - A)^{-1}\| \leq \frac{K}{\sqrt{\operatorname{Re}(s)}}, \tag{17}$$

for all $s \in \mathbb{C}_0^+$.

From Zwart, Jacob and Staffans [9] we have that this condition is not sufficient. However, like estimate (7) it is very close to being sufficient as will become clear from Theorems 4.2 and 4.3. The proof of these theorems is very similar to the proof of Theorem 3.2 and 3.3, respectively. Because of this similarity, we present only the details of the first assertion.

Theorem 4.2 *Let $T(t)$ be a uniformly bounded C_0 -semigroup on the Hilbert space Z , with generator A . Assume that $C \in \mathcal{L}(D(A), Y)$ is an admissible observation operator which satisfies the estimate*

$$\|C(sI - A)^{-1}\| \leq \frac{K}{\sqrt{\operatorname{Re}(s)}},$$

for all complex $s \in \mathbb{C}_0^+$. Then there exists a $\kappa > 0$ such that for all $z_0 \in D(A)$ and all $\Omega \in (x_1, \infty)$ we have that

$$\int_0^\Omega \|CT(t)z_0\|^2 dt \leq \kappa \log(\Omega) \|z_0\|^2. \quad (18)$$

Here x_1 is the largest solution of $xe^{-x} = 1/(2e)$, see Section 3.

Proof Since $T(t)$ is uniformly bounded, we have that for all $\lambda > 0$

$$\begin{aligned} \|((\cdot + \lambda)I - A)^{-1}z_0\|_{H_2(Z)}^2 &= 2\pi \|e^{-\lambda\cdot}T(\cdot)z_0\|_{L_2((0,\infty);Z)}^2 \\ &\leq 2\pi \sup_{t \geq 0} \|T(t)\|^2 \|e^{-\lambda\cdot}\|_{L_2(0,\infty)}^2 \|z_0\|^2 \\ &\leq \frac{M_1}{\lambda} \|z_0\|^2. \end{aligned} \quad (19)$$

It is easy to see that (17) implies that

$$\|C((\cdot + \lambda)I - A)^{-1}\|_{H_\infty(\mathcal{L}(Z,Y))} \leq \frac{K}{\sqrt{\lambda}} \quad (20)$$

for all $\lambda > 0$. Combining (19) and (20) we obtain that for all $\lambda > 0$

$$\begin{aligned} \|C((\cdot + \lambda)I - A)^{-1}z_0\|_{H^2(Y)} &\leq \|C((\cdot + \lambda)I - A)^{-1}\|_{H_\infty} \|((\cdot + \lambda)I - A)^{-1}z_0\|_{H^2(Z)} \\ &\leq \frac{K}{\sqrt{\lambda}} \cdot \frac{M_1}{\sqrt{2\lambda}} \|z_0\| \leq \frac{K_1}{\lambda} \|z_0\|, \end{aligned}$$

with $K_1 > 0$ independent of λ . In the time domain this means that, for all $\lambda > 0$ and all $z_0 \in D(A)$

$$\int_0^\infty \|\lambda t e^{-\lambda t} CT(t)z_0\|^2 dt \leq \frac{K_1^2}{2\pi} \|z_0\|^2. \quad (21)$$

We define $y_1 = x_2$, and $y_2 = x_1$ and for $n \in \mathbb{N}$

$$\mu_n = \left(\frac{y_1}{y_2} \right)^{n-1}, \quad y_n = \frac{y_1}{\mu_n},$$

so that $\mu_n \rightarrow 0$, $y_n \rightarrow \infty$ and $\mu_n y_n = y_1$, $\mu_n y_{n+1} = y_2$. Hence

$$\mu_n t e^{-\mu_n t} > \frac{1}{2e}, \quad \text{for all } t \in (y_n, y_{n+1}). \quad (22)$$

Using this we see that

$$\begin{aligned} \int_{y_1}^{y_N} \|CT(t)z_0\|^2 dt &= \sum_{n=1}^{N-1} \int_{y_n}^{y_{n+1}} \|CT(t)z_0\|^2 dt \\ &\leq (2e)^2 \sum_{n=1}^{N-1} \int_{y_n}^{y_{n+1}} \|\mu_n t e^{-\mu_n t} CT(t)z_0\|^2 dt. \end{aligned}$$

With (21) we obtain that

$$\int_{y_1}^{y_N} \|CT(t)z_0\|^2 dt \leq \frac{2}{\pi} e^2 K_1^2 (N-1).$$

Since $N \sim \log(y_N)$ we have proved the assertion. ■

Theorem 4.3 *Let A be the infinitesimal generator of the uniformly bounded C_0 -semigroup $T(t)$ on the Hilbert space Z and let $C \in \mathcal{L}(D(A), Y)$, where Y is a second Hilbert space. If there exists a $K > 0$ such that for all $\gamma \in (0, 1)$*

$$\|C((\gamma + \cdot)I - A)^{-1}\|_{H_\infty(\mathcal{L}(Z, Y))} \leq \frac{K}{\log(\gamma)\sqrt{\gamma}}, \quad (23)$$

then C is an infinite-time admissible observation operator.

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