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Controlled Invariance for Hamiltonian Systems

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Abstract. A notion of controlled invariance is developed which is suited to Hamiltonian control systems. This is done by replacing the controlled invariant *distribution*, as used for general nonlinear control systems, by the controlled invariant *function group*. It is shown how Lagrangian or coisotropic controlled invariant function groups can be made invariant by static, respectively dynamic, Hamiltonian feedback. This constitutes a first step in the development of a geometric control theory for Hamiltonian systems that explicitly uses the given structure.

1. Introduction

In the last fifteen years the so-called *geometric theory* of linear systems has proved to be a powerful tool in the solution of various control and synthesis problems (see the trendsetting book of Wonham (1979)). The basic concept in this theory is the notion of *controlled invariance* or (A, B)-invariance of a linear subspace of the state space. Recently, due to the work of Isidori-Krener-Gori-Giorgi-Monaco (1981a) and Hirschorn (1981) this basic notion has been successfully generalized to *nonlinear* systems (Firstly to nonlinear systems which are affine in the inputs and in Nijmeijer-van der Schaft (1982b) also to general nonlinear systems). In this nonlinear generalization the linear subspaces are replaced by (involutive) *distributions* on the state space, or their corresponding *foliations*. Roughly speaking, a distribution on the state space of a system is controlled invariant if it can be made invariant (in a precise geometric sense) by applying (nonlinear) *feedback* to the system. This notion of a controlled invariant distribution has already been used in problems like nonlinear disturbance decoupling, non-interacting control and invertibility.

Although the development of this theory has been very successful, it is clear that for many control and synthesis purposes it will not be possible to develop an adequate theory covering *all* nonlinear systems. For instance the treatment of *stability*, which is missing up till now in the nonlinear geometric theory, seems very hard in the general case. Therefore it will also be necessary to focus on special types of nonlinear systems. In our opinion, a natural candidate for such a subclass of nonlinear systems is formed by the *Hamiltonian* systems, as originally proposed by Brockett (1977), and developed in a series of papers by the author and others, see e.g. van der Schaft (1981, 1982, 1983b,c). A prototype of a Hamiltonian system are the classical Euler-Lagrange equations with external forces. Although in many applications the Hamiltonian description constitutes an idealization (neglection of friction, dissipation etc.) it has proved to be at least a very natural starting point. Outstanding examples are robot manipulators, large space structures and in general (conservative) mechanical systems.

The basic philosophy of this paper is that in dealing with these Hamiltonian systems it is worthwhile to explicitly *use* the Hamiltonian structure in the solution of control and synthesis problems, and to look for solutions which "remain within the Hamiltonian framework". Most importantly, the *feedback* which is applied, can and/or should be of a Hamiltonian (and therefore physically interpretable!) form. Of course one *could* apply the nonlinear geometric theory immediately to Hamiltonian systems. However the feedback which is needed to make a controlled invariant distribution invariant will in general *affect* the Hamiltonian form of the equations. Since we want to take advantage of the Hamiltonian structure and not to reduce the system to an "ordinary" nonlinear system, this is clearly not satisfying. Therefore as a basic step in building a geometric theory for Hamiltonian systems we have to develop a notion of controlled invariance which is particularly suited to Hamiltonian systems. Preliminary work on such a concept of *Hamiltonian controlled invariance* has already been done in van der Schaft (1983a) for the case of linear Hamiltonian systems. In this paper this will be extended to the nonlinear case. The basic contribution will be the introduction of the *controlled invariant function group*, which will replace the controlled invariant distribution. We will prove, under certain conditions, that a *Lagrangian* controlled invariant function group can be made invariant by *Hamiltonian feedback*, i.e. feedback which leaves the Hamiltonian form invariant, while *coisotropic* controlled invariant function groups can be made invariant by dynamic Hamiltonian feedback, i.e. the addition of a Hamiltonian compensator. From a mathematical point of view the notion of a Lagrangian controlled invariant function group is related to the classical concept of *complete integrability* of Hamiltonian vectorfields.

Certainly, the theory in this paper should be seen as only a first step in the development of a geometric control theory of Hamiltonian systems, and so we will only use Lagrangian and coisotropic controlled invariant function groups in the solution of the somewhat ubiquitous *disturbance decoupling problem* for Hamiltonian systems. Apart from Lagrangian or coisotropic function groups also *symplectic* controlled invariant function groups are of much interest. They seem to be the natural tool in the Hamiltonian non-interacting control problem. This will be dealt with in a future paper (Nijmeijer & van der Schaft (1984c), see also (1984b)).

Hamiltonian systems. We will briefly review the definition of a Hamiltonian system, see e.g. van der Schaft (1982, 1983b,c). Let M be a $2n$-dimensional connected manifold with symplectic form ω. By Darboux's theorem there exist local coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ such that $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$. Such coordinates are called *canonical*. Given a function $F: M \to \mathbb{R}$ we define the Hamiltonian vectorfield X_F on M by $\omega(X_F, -) = -dF$. In canonical coordinates

$$X_F = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial}{\partial p_i} \right) \tag{1.1}$$

Given another function $G: M \to \mathbb{R}$ we define the *Poisson bracket* $\{F, G\} = X_F(G) = \omega(X_F, X_G)$. In canonical coordinates

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right) \tag{1.2}$$

A (coordinate) transformation $\varphi: M \to M$ is *canonical* if φ preserves the Poisson bracket, i.e.

$$\{F \circ \varphi, G \circ \varphi\} = \{F, G\} \circ \varphi \qquad \forall F, G \tag{1.3}$$

An (affine) *Hamiltonian system* on M with internal energy H is now defined as

$$\dot{x} = X_H(x) - \sum_{j=1}^{m} u_j X_{C_j}(x) \qquad y_j = C_j(x) \tag{1.4}$$

with $x = (q, p)$ canonical coordinates, $u = (u_1, \ldots, u_m)$ the inputs, $y = (y_1, \ldots, y_m)$ the outputs, and $C = (C_1, \ldots, C_m)$ the output mapping. This constitutes a direct generalization of the classical Euler-Lagrange equations with external forces u_i

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} &= \begin{cases} u_i & i = 1, \ldots, m \\ 0 & i = m+1, \ldots, n \end{cases} \\ y_j = q_j & \qquad\qquad j = 1, \ldots, m \end{aligned} \tag{1.5}$$

or in Hamiltonian form (with $p_i = \frac{\partial L}{\partial \dot{q}_i}$ the momenta and $H(q, p) = \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L(q, \dot{q})$ the internal energy)

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} & i &= 1, \ldots, n \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} + u_i & i &= 1, \ldots, m \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} & i &= m+1, \ldots, n \\ y_j &= q_j & j &= 1, \ldots, m \end{aligned} \tag{1.6}$$

In fact, if we take $C_j(q, p) = q_j$ in (1.4) we obtain (1.6), and conversely if we allow for canonical coordinate transformations on (q,p) in (1.6), then (1.6) becomes of the form (1.4).

If we interpret $y = (y_1, \dots, y_m)$ as coordinates for an m-dimensional output manifold Y, then $(y_1, \dots, y_m, u_1, \dots, u_m)$ can be most naturally interpreted as *natural* coordinates for the cotangent bundle T^*Y. (If $(y_1, \dots, y_m)$ are arbitrary coordinates for Y, then we define natural coordinates $(y_1, \dots, y_m, u_1, \dots, u_m)$ for T^*Y by letting a point $(\bar{y}_1, \dots, \bar{y}_m, \bar{u}_1, \dots, \bar{u}_m)$ in these coordinates correspond to the one-form $\sum_{j=1}^{m} \bar{u}_j \, dy_j$ on Y in the point $(\bar{y}_1, \dots, \bar{y}_m)$.) Being a cotangent bundle T^*Y has a natural symplectic form ω^e. In fact, if $(y_1, \dots, y_m, u_1, \dots, u_m)$ are natural coordinates, then ω^e equals $\sum_{j=1}^{m} du_j \wedge dy_j$. So natural coordinates are always canonical. Notice that if we choose another set of coordinates $(y_1', \dots, y_m')$ for Y, then the coordinates $(u_1, \dots, u_m)$ have to change to $(u_1', \dots, u_m')$ in such a way that $\omega^e = \sum_{j=1}^{m} du_j' \wedge dy_j'$. For instance, if we transform the outputs from Cartesian to angular coordinates, then the inputs change from translational forces to torques.

Function groups. We briefly collect some facts about function groups and Poisson structures, which date back to Lie (1890) and were recently rediscovered by various authors (cf. Weinstein (1983), Hermann (1977)). Let M be a connected symplectic manifold with Poisson bracket $\{F, G\} = \omega(X_F, X_G)$. We call a collection $\mathscr{F}$ of smooth functions from M to $\mathbb{R}$ a *function space*, if

1) $\mathscr{F}$ is a linear subspace (over $\mathbb{R}$) of $C^\infty(M)$, the smooth functions on M.

2) If $F_1, \dots, F_s \in \mathscr{F}$ and $G: \mathbb{R}^s \to \mathbb{R}$ is a smooth function, then $G(F_1, \dots, F_s) \in \mathscr{F}$. Furthermore, we call $\mathscr{F}$ a *function group* if also

3) $\mathscr{F}$ is closed under Poisson bracket, i.e. if $F_1, F_2 \in \mathscr{F}$, then $\{F_1, F_2\} \in \mathscr{F}$.

Notice that by 2) a non-empty function space always contains $\mathbb{R}$, the constant functions on M (actually by this fact condition 2) implies condition 1)!). Given some functions $F_1, \dots, F_k$ on M we denote by span $\{F_1, \dots, F_k\}$ the smallest function space containing these functions. Furthermore the *sum* $\mathscr{F}^1 + \mathscr{F}^2$ of two function spaces $\mathscr{F}^1, \mathscr{F}^2$ will be the smallest function space containing $\mathscr{F}^1$ as well as $\mathscr{F}^2$. Given a function space $\mathscr{F}$, we denote by $\bar{\mathscr{F}}$ the *closure* of $\mathscr{F}$ under Poisson bracket, i.e. the smallest function *group* containing $\mathscr{F}$. Furthermore we define

$$\mathscr{F}^\perp = \{G \in C^\infty(M) \mid \{G, F\} = 0, \forall F \in \mathscr{F}\} \tag{1.7}$$

Let now $G_1, G_2 \in \mathscr{F}^\perp$ and $K: \mathbb{R}^2 \to \mathbb{R}$, then

$$\{K(G_1, G_2), F\} = \frac{\partial K}{\partial x_1}(G_1, G_2)\{G_1, F\} + \frac{\partial K}{\partial x_2}(G_1, G_2)\{G_2, F\} = 0 \tag{1.8}$$

and hence $\mathscr{F}^{\perp}$ is a function space. Furthermore by the Jacobi-identity

$$\{\{G_1, G_2\}, F\} + \{\{G_2, F\}, G_1\} + \{\{F, G_1\}, G_2\} = 0 \tag{1.9}$$

for any $G_1, G_2 \in \mathscr{F}^{\perp}$ and $F \in \mathscr{F}$. Hence $\{G_1, G_2\} \in \mathscr{F}^{\perp}$ and $\mathscr{F}^{\perp}$ is actually a function group, called the *polar group*. Finally $\mathscr{F} \cap \mathscr{F}^{\perp}$ is a function space (resp. group) if $\mathscr{F}$ is a function space (resp. group), and the elements of $\mathscr{F} \cap \mathscr{F}^{\perp}$ are called the distinguished or *Casimir* functions. It is clear that for any function space $\mathscr{F}$

$$\mathscr{F} \cap \mathscr{F}^{\perp} \subset \mathscr{F} \subset \bar{\mathscr{F}} \subset (\mathscr{F}^{\perp})^{\perp} \tag{1.10}$$

and for any two function spaces F_1, F_2

$$\begin{aligned} &\mathscr{F}_1 + \mathscr{F}_2 \subset \overline{\mathscr{F}_1 + \mathscr{F}_2} \subset \left(\mathscr{F}_1^{\perp} \cap \mathscr{F}_2^{\perp}\right)^{\perp} \\ &\mathscr{F}_1 \cap \mathscr{F}_2 \subset \overline{\mathscr{F}_1 \cap \mathscr{F}_2} \subset \left(\mathscr{F}_1^{\perp} + \mathscr{F}_2^{\perp}\right)^{\perp} \end{aligned} \tag{1.11}$$

For a function space $\mathscr{F}$ we define the codistribution $d\mathscr{F}$ as

$$d\mathscr{F}(x) = \operatorname{span}_{\mathbf{R}}\{d\mathscr{F}(x) | F \in \mathscr{F}\}, \ x \in M \tag{1.12}$$

and the distribution $D_{\mathscr{F}}$ as

$$D_{\mathscr{F}}(x) = \operatorname{span}_{\mathbf{R}}\{X_F(x) | F \in \mathscr{F}\}, \ x \in M \tag{1.13}$$

In order to simplify considerably the technical details of the sequel we make the following assumption (also dating back to Lie), which will hold *throughout* this paper.

Assumption 1. Every function space satisfies

Condition A. There exists a number of *independent* functions $F_1, \ldots, F_k$ on M such that $\mathscr{F} = \operatorname{span}\{F_1, \ldots, F_k\}$ (independent means that dim $\operatorname{span}_{\mathbf{R}}\{dF_1(x), \ldots, dF_k(x)\} = k, \forall x$).

Remark. In applications this assumption may not be the most natural one. Instead, one may replace condition A by the weaker

Condition A′. $\dim d\mathscr{F}(x) = \text{constant}, \forall x$

If Condition A′ is satisfied, there exist *locally* k independent functions such that *locally* $\mathscr{F} = \operatorname{span}\{F_1, \ldots, F_k\}$. However this implies that some of the following propositions (especially Lemma 1.2 and its consequences) hold only *locally*.

With every function group satisfying Condition A we can associate a so-called *Poisson structure*. Consider a set of smooth functions w_{ij}, $i, j = 1, \ldots, k$ on $\mathbb{R}^k$.

They define a Poisson structure if (cf. Weinstein (1983))

$$\text{i)} \qquad w_{ij} + w_{ji} = 0 \qquad\qquad i, j = 1, \ldots, k$$

$$\text{ii)} \qquad \sum_{l=1}^{k} \left(w_{lj} \frac{\partial w_{ir}}{\partial x_l} + w_{li} \frac{\partial w_{rj}}{\partial x_l} + w_{lr} \frac{\partial w_{ji}}{\partial x_l} \right) = 0 \qquad i, j, r = 1, \ldots, k$$

By i) the rank of the matrix $(w_{ij}(x))$ is for every x even. If rank $(w_{ij}(x)) = k$, so $k = 2n$, for every x, then we speak of a *symplectic structure*. In fact in this case $\omega := \sum_{i,j=1}^{2n} w^{ij}(x)\, dx_i \wedge dx_j$, where $(w^{ij}(x)) = (w_{ij}(x))^{-1}$, is a symplectic form on $\mathbb{R}^{2n}$ (this follows from i) and ii)).

Given a Poisson structure w_{ij} on $\mathbb{R}^k$ and a function F on $\mathbb{R}^k$ we define the Hamiltonian vectorfield X_F on $\mathbb{R}^k$ by

$$X_F(x) = -\sum_{i,j=1}^{k} w_{ij}(x) \frac{\partial F(x)}{\partial x_j} \frac{\partial}{\partial x_i} \tag{1.14}$$

It is easy to see that in case w_{ij} is a symplectic structure this is just the ordinary definition of a Hamiltonian vectorfield as in (1.1). Moreover we define a Poisson bracket $\{\ ,\ \}_{\mathbb{R}^k}$ corresponding to the Poisson structure w_{ij} on $\mathbb{R}^k$ as follows. Let $F, G : \mathbb{R}^k \to \mathbb{R}$, then

$$\{F, G\}_{\mathbb{R}^k}(x) = \sum_{i,j=1}^{k} w_{ij}(x) \frac{\partial F}{\partial x_i}(x) \frac{\partial G}{\partial x_j}(x) \tag{1.15}$$

Again it is easy to conclude that if w_{ij} is a symplectic structure then this is just the ordinary Poisson bracket (1.2) on $\mathbb{R}^{2n}$. It follows from i) and ii) that $\{\ ,\ \}_{\mathbb{R}^k}$ is anti-symmetric and satisfies the Jacobi-identity

$$\{\{F, G\}_{\mathbb{R}^k}, H\}_{\mathbb{R}^k} + \{\{G, H\}_{\mathbb{R}^k}, F\}_{\mathbb{R}^k} + \{\{H, F\}_{\mathbb{R}^k}, G\}_{\mathbb{R}^k} = 0 \tag{1.16}$$

as the ordinary Poisson bracket.

The connection between Poisson structures and function groups satisfying Condition A is the following. Let $\mathscr{F} = \operatorname{span}\{F_1, \ldots, F_k\}$ be a function group on (M, ω), with F_i independent. Then there exist functions $w_{ij} : \mathbb{R}^k \to \mathbb{R}$, $i, j = 1, \ldots, k$, such that

$$\{F_i, F_j\}_M = w_{ij} \circ (F_1, \ldots, F_k) \tag{1.17}$$

It follows from the properties of the usual Poisson bracket on (M, ω) that the functions w_{ij} satisfy i) and ii). Hence a function group satisfying Condition A defines a Poisson structure! Using the theory of Poisson structures one can prove the following basic theorem on function groups (Lie (1890), Weinstein (1983)).

Theorem 1.1. *Let $\mathscr{F}$ be a function group on (M, ω) satisfying Condition* A *such that $d(\mathscr{F} \cap \mathscr{F}^{\perp})(x)$ has constant dimension. Suppose* $\dim d\mathscr{F} = k$ *and* $\dim d(\mathscr{F}$

$\cap \mathscr{F}^{\perp}) = r$. Then locally there exist canonical coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ for M such that

$$F = \operatorname{span}\{q_1, \dots, q_l, p_1, \dots, p_l, p_{l+1}, \dots, p_{l+r}\} \tag{1.18}$$

with $2l + r = k$.

Remark. Since the above theorem is *local*, it remains valid if we replace Condition A by Condition A′.

We will now derive some propositions which will be useful later on. First we derive some connections between $\mathscr{F}$ and its distribution $D_{\mathscr{F}}$.

Lemma 1.2. *Let $\mathscr{F}$ be a function space satisfying Condition* A *and let $G: M \to \mathbb{R}$ be such that $dG \in d\mathscr{F}$ (i.e. $dG(x) \in d\mathscr{F}(x), \forall x$). Then $G \in \mathscr{F}$. Also if $X_G \in D_{\mathscr{F}}$, then $G \in \mathscr{F}$.*

Proof. By Condition A, $\mathscr{F} = \operatorname{span}\{F_1, \dots, F_k\}$, with F_i independent functions. Denote the map $(F_1, \dots, F_k)$: $M \to \mathbb{R}^k$ by F. Then since $dG \in d\mathscr{F}$, there exists a 1-form α_G on $\mathbb{R}^k$ such that $F^*\alpha_G = dG$. Hence $F^*(d\alpha_G) = d(dG) = 0$ and so $d\alpha_G = 0$. By Poincare's lemma there exists a function $\tilde{G}$: $\mathbb{R}^k \to \mathbb{R}$ such that $\alpha_G = d\tilde{G}$. Therefore $d(F^* \circ \tilde{G}) = dG$, or equivalently, $d(\tilde{G} \circ (F_1, \dots, F_k) - G) = 0$. Since M is connected (and hence pathwise connected) this implies that $G = \tilde{G} \circ (F_1, \dots, F_k) +$constant. Since the constant functions are included in $\mathscr{F}$ we obtain $G \in \mathscr{F}$. Finally, if $X_G \in D_{\mathscr{F}}$, then $dG = -\omega(X_G, -) \in d\mathscr{F}$, and hence $G \in \mathscr{F}$. □

A (general) distribution D on M is called *involutive* if whenever X_1 and X_2 are vectorfields in D (i.e. $X_i(x) \in D(x)$, $\forall x$, $i = 1,2$), then also the Lie bracket $[X_1, X_2]$ is contained in D. We obtain

Proposition 1.3. *Let $\mathscr{F}$ be a function space satisfying Condition* A. *Then $D_{\mathscr{F}}$ is involutive if and only if $\mathscr{F}$ is a function group.*

Proof. Recall the basic identity concerning Lie brackets and Poisson brackets (cf. Abraham & Marsden (1978)): for every F, G: $M \to \mathbb{R}$

$$[X_F, X_G] = X_{\{F,G\}} \tag{1.19}$$

Let $\mathscr{F}$ be a function group and $F_1, F_2 \in \mathscr{F}$. Then $[X_{F_1}, X_{F_2}] = X_{\{F_1, F_2\}} \in D_{\mathscr{F}}$. Since $D_{\mathscr{F}}$ is spanned by Hamiltonian vectorfields this implies that $D_{\mathscr{F}}$ is involutive. Conversely, assume that $D_{\mathscr{F}}$ is involutive, and let $F_1, F_2 \in \mathscr{F}$. Then $X_{\{F_1, F_2\}} = [X_{F_1}, X_{F_2}] \in D_{\mathscr{F}}$, and so by Lemma 1.2 $\{F_1, F_2\} \in \mathscr{F}$. Hence $\mathscr{F}$ is a function group. □

Now we turn attention to the inclusions (1.10) and (1.11).

Proposition 1.4. *Let $\mathscr{F}$ be a function group satisfying Condition* A. *Then*

$$\mathscr{F} = (\mathscr{F}^{\perp})^{\perp} \tag{1.20}$$

Furthermore, let $\mathscr{F}_1, \mathscr{F}_2$ *be function groups satisfying Condition* A. *Assume* $\mathscr{F}_1 + \mathscr{F}_2$ *is also a function group satisfying Condition* A. *Then*

$$\mathscr{F}_1 + \mathscr{F}_2 = \left(\mathscr{F}_1^{\perp} \cap \mathscr{F}_2^{\perp}\right)^{\perp} \tag{1.21}$$

Assume $\mathscr{F}_1^{\perp} + \mathscr{F}_2^{\perp}$ *is a function group satisfying Condition* A. *Then*

$$\mathscr{F}_1 \cap \mathscr{F}_2 = \left(\mathscr{F}_1^{\perp} + \mathscr{F}_2^{\perp}\right)^{\perp} \tag{1.22}$$

Proof. (see also Weinstein (1983), Prop. 7.1). By Condition A, $\mathscr{F} = \operatorname{span}\{F_1,\dots,F_k\}$, with F_i independent. Let $F = (F_1,\dots,F_k)$: $M \to \mathbb{R}^k$, and denote the foliation of M with leaves $F^{-1}(c)$, $c \in \mathbb{R}^k$, by Φ. By Prop. 1.4. $D_{\mathscr{F}}$ is involutive and has constant dimension. Hence $D_{\mathscr{F}}$ integrates to a foliation of M which we denote by $\Phi^{\perp}$. It is easy to see that $\mathscr{F}^{\perp}$ are precisely the functions which are constant along the leaves of $\Phi^{\perp}$ (Notice that $\mathscr{F}^{\perp}$ satisfies Condition A′, but not necessarily Condition A). Moreover $D_{\mathscr{F}}$ is the orthogonal complement of $D_{\mathscr{F}}^{\perp}$ under the symplectic structure ω. Hence $(\mathscr{F}^{\perp})^{\perp} = \mathscr{F}$.

For the proof of (1.20) we denote the foliations corresponding to $\mathscr{F}_1$ and $\mathscr{F}_2$ by Φ_1 and Φ_2, and to $\mathscr{F}_1 + \mathscr{F}_2$ by Φ_{1+2}. By the first part of the proof we get foliations $\Phi_1^{\perp}$, $\Phi_2^{\perp}$ and $\Phi_{1+2}^{\perp}$. It is clear that the leaves of $\Phi_{1+2}^{\perp}$ are exactly intersections of leaves of $\Phi_1^{\perp}$ and $\Phi_2^{\perp}$. By similar reasoning as in the first part one concludes to (1.21). For (1.22) we notice that by (1.21), $\mathscr{F}_1^{\perp} + \mathscr{F}_2^{\perp} = ((\mathscr{F}_1^{\perp})^{\perp} \cap (\mathscr{F}_2^{\perp})^{\perp})^{\perp}$, and so by (1.20), $\mathscr{F}_1^{\perp} + \mathscr{F}_2^{\perp} = (\mathscr{F}_1 \cap \mathscr{F}_2)^{\perp}$, and $\mathscr{F}_1 \cap \mathscr{F}_2 = (\mathscr{F}_1^{\perp} + \mathscr{F}_2^{\perp})^{\perp}$ □

In Proposition 1.2 we saw that to every function group $\mathscr{F}$ satisfying Condition A there corresponds an involutive distribution $D_{\mathscr{F}}$ of constant dimension. The converse question is answered in

Proposition 1.5. *Let D be an involutive distribution of constant dimension. By Frobenius' theorem there exist locally independent functions $K_1,\dots,K_k$ such that* $D(x) = \operatorname{Ker}\operatorname{span}_{\mathbb{R}}\{dK_1(x),\dots,dK_k(x)\}$. Assume *that the functions $K_1,\dots,K_k$ are* globally *defined. Then*:

There exists a function group $\mathscr{F}$ such that $D = D_{\mathscr{F}}$ if and only if $\operatorname{span}\{K_1,\dots,K_k\}$ *is a function group. Moreover if* $\operatorname{span}\{K_1,\dots,K_k\}$ *is a function group, then* $\mathscr{F} = (\operatorname{span}\{K_1,\dots,K_k\})^{\perp}$.

Proof. Let $D = D_{\mathscr{F}}$, with $\mathscr{F}$ a function group. Then for any $F \in \mathscr{F}$

$$\{K_i, F\} = -X_F(K_i) = -dK_i(X_F) = 0,\ i = 1,\dots,k$$

Jacobi's identity then implies

$$\{\{K_i, K_j\}, F\} = d\{K_i, K_j\}(X_F) = 0, \forall F \in \mathscr{F}, i, j = 1,\dots,k$$

Hence $d\{K_i, K_j\}(x) \in \operatorname{span}_{\mathbb{R}}\{dK_1(x),\dots,dK_k(x)\}$, $\forall x$.

By Lemma 1.2 it follows that $\{K_i, K_j\} \in \operatorname{span}\{K_1, \dots, K_k\}$. Hence $\operatorname{span}\{K_1,\dots, K_k\}$ is a function group (even satisfying Condition A). It is clear that $\operatorname{span}\{K_1,\dots, K_k\} \subset \mathscr{F}^{\perp}$. Now let $G \in \mathscr{F}^{\perp}$. Then $\{G, F\} = -dG(X_F) = 0$, $\forall F \in \mathscr{F}$, and hence $\operatorname{Ker}\operatorname{span}\{dK_1,\dots, dK_k\}(x) = D_{\mathscr{F}}(x) \subset \operatorname{Ker} dG(x)$, or equivalently, $dG \in \operatorname{span}\{dK_1, \dots, dK_k\}$. Therefore by Lemma 1.2, $G \in \operatorname{span}\{K_1,\dots, K_k\}$. Hence $\operatorname{span}\{K_1,\dots, K_k\} = \mathscr{F}^{\perp}$. Conversely, suppose that $\operatorname{span}\{K_1,\dots, K_k\}$ is a function group. Since $D = \operatorname{Ker}\operatorname{span}\{dK_1,\dots, dK_k\}$, $\dim D = 2n - k$. Define $\mathscr{F} = (\operatorname{span}\{K_1,\dots, K_k\})^{\perp}$. Since $\operatorname{span}\{K_1,\dots, K_k\}$ is a function group satisfying condition A it follows from Proposition 1.4 that $\dim dF = 2n - k$. Furthermore it is clear that $D_{\mathscr{F}} \subset \operatorname{Ker}\operatorname{span}\{dK_1,\dots, dK_k\} = D$. Since $\dim D_{\mathscr{F}} = \dim d\mathscr{F} = 2n - k$, necessarily $D = D_{\mathscr{F}}$. □

2. Controlled Invariance

Consider an arbitrary affine nonlinear system

$$\dot{x} = A(x) + \sum_{j=1}^{m} u_j B_j(x) \tag{2.1}$$

with $A, B_1, \dots, B_m$ smooth vectorfields. Define the distribution $\mathscr{B} := \operatorname{span}\{B_1,\dots, B_m\}$ by $\mathscr{B}(x) = \operatorname{span}\{B_1(x),\dots, B_m(x)\}$, and define the sum $D_1 + D_2$ of two distributions D_1 and D_2 as the smallest distribution containing D_1 as well as D_2.

An involutive distribution D is *invariant* for (2.1) if

$$\begin{aligned} &\text{i)} \quad [A, X] \subset D \\ &\text{ii)} \quad [B_j, X] \subset D \end{aligned} \qquad \text{for every } X \in D, \quad j = 1,\dots, m$$

and *locally controlled invariant* (l.c.i.) if

$$\begin{aligned} &\text{i)} \quad [A, X] \in D + \mathscr{B} \\ &\text{ii)} \quad [B_j, X] \in D + \mathscr{B} \end{aligned} \qquad \text{for every } X \in D, \; j = 1,\dots, m$$

or more succinctly

$$\begin{aligned} &\text{i)} \quad [A, D] \subset D + \mathscr{B} \\ &\text{ii)} \quad [\mathscr{B}, D] \subset D + \mathscr{B} \end{aligned} \tag{2.2}$$

Usually the following standard assumption is made (see Section 3)

Assumption 2. The distributions D, $\mathscr{B}$ and $D + \mathscr{B}$ all have constant dimension.

Roughly speaking, for a definition of local controlled invariance for Hamiltonian systems, we replace involutive distributions by function groups and Lie brackets by Poisson brackets. First we will state the definitions and develop the theory without making regularity assumptions as in Assumption 2.

Definition 2.1. Consider a Hamiltonian system (1.4) on (M, ω), and let $\mathscr{F}$ be a function group on M. Then $\mathscr{F}$ is *invariant* for (1.4) if

$$\left.\begin{array}{ll} \text{i)} & \{H, F\} \subset \mathscr{F} \\ \text{ii)} & \{C_j, F\} \subset \mathscr{F} \end{array}\right\} \quad \text{for every } F \in \mathscr{F},\ j=1,\dots,m$$

and *locally controlled invariant* if

$$\left.\begin{array}{ll} i) & \{H, F\} \subset \mathscr{F} + \operatorname{span}\{C_1,\dots,C_m\} \\ \text{ii)} & \{C_j, F\} \subset \mathscr{F} + \operatorname{span}\{C_1,\dots,C_m\} \end{array}\right\} \quad \text{for every } F \in \mathscr{F},\ j=1,\dots,m,$$

or more succinctly if we define $\mathscr{C} := \operatorname{span}\{C_1,\dots,C_m\}$

$$\begin{aligned} \{H, \mathscr{F}\} &\subset \mathscr{F} + \mathscr{C} \\ \{\mathscr{C}, \mathscr{F}\} &\subset \mathscr{F} + \mathscr{C} \end{aligned} \tag{2.3}$$

Remark. In the above definition one can replace function groups by function spaces, just like one can take *arbitrary* distributions in (2.2). However if a function space $\mathscr{F}$ satisfies (2.3), then so does the function group $\bar{\mathscr{F}}$. In fact let $F_1,\ F_2 \in \mathscr{F}$. Then by the Jacobi-identity

$$\{H, \{F_1, F_2\}\} = -\{F_1, \{F_2, H\}\} - \{F_2, \{H, F_1\}\} \in \{F_1, \mathscr{F} + \mathscr{C}\}$$
$$+ \{F_2, \mathscr{F} + \mathscr{C}\} \subset \bar{\mathscr{F}} + \mathscr{C}, \text{ and similarly } \{\mathscr{C}, \{F_1, F_2\}\} \subset \bar{\mathscr{F}} + \mathscr{C}.$$

The above definition is justified by the fact that a function group $\mathscr{F}$ is invariant, resp. locally controlled invariant, if and only if its corresponding distribution $D_{\mathscr{F}}$ is invariant, resp. locally controlled invariant:

Proposition 2.2. *Let $\mathscr{F}$ be a function group satisfying Condition* A. *Then $\mathscr{F}$ is an invariant function group if and only if $D_{\mathscr{F}}$ is an invariant distribution. Moreover assume that $\mathscr{F} + \mathscr{C}$ satisfies Condition* A. *Then $\mathscr{F}$ is a l.c.i. function group if and only if $D_{\mathscr{F}}$ is a l.c.i. distribution.*

Proof. Let $\{H, F\} \subset \mathscr{F}$, $\{C_i, F\} \subset F$ for any $F \in \mathscr{F}$. Then $[X_H, X_F] = X_{\{H, F\}} \in D_{\mathscr{F}}$ and $[X_{C_i}, X_F] \in D_{\mathscr{F}}$. So D_F is invariant. Conversely if $[X_H, X_F] = X_{\{H, F\}} \in D_{\mathscr{F}}$ then by Lemma 1.2, $\{H, F\} \in \mathscr{F}$. Similarly for $[X_{C_i}, X_F]$. So $\mathscr{F}$ is invariant. Analogously if $\mathscr{F}$ is l.c.i. then for any $F \in \mathscr{F}$, $[X_H, X_F] \in D_{\mathscr{F}+\mathscr{C}} = D_{\mathscr{F}} + D_{\mathscr{C}}$ and $[X_{C_i}, X_F] \in D_{\mathscr{F}} + D_{\mathscr{C}}$. Therefore $D_{\mathscr{F}}$ is l.c.i. (In this case $\mathscr{B} = D_{\mathscr{C}}$!). Conversely if $[X_H, X_F] = X_{\{H, F\}} \in D_{\mathscr{F}} + D_{\mathscr{C}}$, then by Lemma 1.2, $\{H, F\} \in \mathscr{F} + \mathscr{C}$. Similarly $\{C_i, F\} \in \mathscr{F} + \mathscr{C}$. □

So every l.c.i. function group $\mathscr{F}$ generates a l.c.i. involutive distribution $D_{\mathscr{F}}$. Conversely if $D = \mathrm{Ker}\{dK_1,\ldots,dK_k\}$ is a l.c.i. involutive $(2n-k)$-dimensional distribution, then by Proposition 1.5 $\mathscr{F} = (\mathrm{span}\{K_1,\ldots,K_k\})^{\perp}$ is a l.c.i. function group if $\mathrm{span}\{K_1,\ldots,K_k\}$ is a function group. (Under the assumption that $\mathscr{F} + \mathscr{C}$ satisfies Condition A).

We now set up an algorithm, called the $\mathscr{F}^*$-algorithm, to produce the *maximal* l.c.i. function group *contained* in a given function group. This algorithm is completely similar to the $\mathscr{V}^*$-algorithm in the linear case (Wonham (1979)), and the corresponding algorithm in the nonlinear case (cf. Isidori et al. (1981a)). We notice that the *existence* of a maximal l.c.i. function group contained in a given function group $\mathscr{L}$ is already ensured by the following reasoning. Let $\mathscr{F}_1, \mathscr{F}_2$ be l.c.i. and contained in $\mathscr{L}$. Then by an easy application of the Jacobi-identity $\overline{\mathscr{F}_1 + \mathscr{F}_2}$ is again l.c.i. and contained in $\mathscr{L}$. Introducing a partial ordering on function groups by setting $\mathscr{F}_1 \leqslant \mathscr{F}_2$ if $d\mathscr{F}_1(x) \subset d\mathscr{F}_2(x)$, $\forall x \in M$, and applying Zorn's lemma, this implies that there exists a maximal l.c.i. function group $\mathscr{F}^*$ contained in $\mathscr{L}$.

$\mathscr{F}^*$-algorithm. Define

$$\begin{aligned}\mathscr{F}^1 &= \mathscr{L}\\ \mathscr{F}^{i+1} &= \mathscr{L} \cap (H+\mathscr{C})^{-1}(\mathscr{F}^i + \mathscr{C}) \qquad i = 1,2,\ldots\end{aligned} \tag{2.4}$$

Here $\mathscr{F}^i + \mathscr{C}$ denotes the smallest function space containing $\mathscr{C}$ as well as $\mathscr{F}^i$, and for an arbitrary function space $\mathscr{F}$,

$$(H+\mathscr{C})^{-1}(\mathscr{F}) := \{G \in C^\infty(M) \mid \{H,G\} \subset \mathscr{F} \text{ and } \{\mathscr{C},G\} \subset \mathscr{F}\}$$

Proposition 2.3. *Let $\mathscr{F}^i$ be defined by* (2.4). *Then*

a. $\mathscr{F}^{i+1} \subset \mathscr{F}^i$, $i = 1,2,\ldots$
b. *$\mathscr{F}^i$ is a function group, $i = 1,2,\ldots$*
c. *If for a certain k $\mathscr{F}^{k+1} = \mathscr{F}^k$, then $\mathscr{F}^{k+1} = \mathscr{F}^k$, $\forall l$*

Proof. a. By induction. It is clear that $\mathscr{F}^2 \subset \mathscr{F}^1$. Now assume $\mathscr{F}^i \subset \mathscr{F}^{i-1}$. Let $F \in \mathscr{F}^{i+1}$. Then $F \in \mathscr{L}$ and $\{H, F\} \subset \mathscr{F}^i + \mathscr{C} \subset \mathscr{F}^{i-1} + \mathscr{C}$, as well as $\{\mathscr{C}, F\} \subset \mathscr{F}^i + \mathscr{C} \subset \mathscr{F}^{i-1} + \mathscr{C}$. Hence $F \in \mathscr{F}^i$.

b. By induction. By definition $\mathscr{F}^1 = \mathscr{L}$ is a function group. Let $\mathscr{F}^i$ be a function group, and take F_1, $F_2 \in \mathscr{F}^{i+1}$.

Then

$$\begin{aligned}\{H,\{F_1,F_2\}\} &= -\{F_1,\{F_2,H\}\} - \{F_2,\{H,F_1\}\}\\ &\in \{F_1, \mathscr{F}^i + \mathscr{C}\} + \{F_2, \mathscr{F}^i + \mathscr{C}\} \subset \mathscr{F}^i + \mathscr{C},\end{aligned}$$

since F_1, $F_2 \in \mathscr{F}^{i+1} \subset \mathscr{F}^i$ and $\mathscr{F}^i$ is a function group.

Analogously $\{\mathscr{C},\{F_1,F_2\}\}\subset\mathscr{F}^i+C$. Hence $\mathscr{F}^i$ is closed under Poisson bracket. Now take a smooth function G: $\mathbb{R}^2\to\mathbb{R}$. Then

$$\{H,G(F_1,F_2)\} = \frac{\partial G}{\partial x_1}(F_1,F_2)\{H,F_1\}+\frac{\partial G}{\partial x_2}(F_1,F_2)\{H,F_2\}$$

$$\subset \mathscr{F}^{i+1}.\mathscr{F}^i\subset\mathscr{F}^i,$$

since $\frac{\partial G}{\partial x_i}(F_1,F_2)\in\mathscr{F}^{i+1}\subset\mathscr{F}^i$.

Analogously, $\{\mathscr{C},G(F_1,F_2)\}\subset\mathscr{F}^i$. Hence $\mathscr{F}^i$ is a function group.

c. Immediate. □

If there exists a k such that $\mathscr{F}^{k+1}=\mathscr{F}^k$ then by c the algorithm ends in k steps and we denote $\mathscr{F}^*:=\mathscr{F}^k$. It follows that $\{H,\mathscr{F}^*\}\subset\mathscr{F}^*+\mathscr{C}$, $\{\mathscr{C},\mathscr{F}^*\}\subset\mathscr{F}^*+\mathscr{C}$, so $\mathscr{F}^*$ is a l.c.i. function group contained in $\mathscr{L}$. Furthermore

Proposition 2.4. *If the algorithm* (2.4) *ends in k steps, then* $\mathscr{F}^*=\mathscr{F}^k$ *is the* maximal *locally controlled invariant function group contained in* $\mathscr{L}$.

Proof. Let $\mathscr{F}$ be a l.c.i. function group contained in $\mathscr{L}$. By induction we will prove $\mathscr{F}\subset\mathscr{F}^i$, $i=1,2,\ldots$, and hence $\mathscr{F}\subset\mathscr{F}^*$. By assumption $\mathscr{F}\subset\mathscr{F}^1$. Suppose $\mathscr{F}\subset\mathscr{F}^i$. Then $\{H,\mathscr{F}\}\subset\mathscr{F}+\mathscr{C}\subset\mathscr{F}^i+\mathscr{C}$, and $\{\mathscr{C},\mathscr{F}\}\subset\mathscr{F}+\mathscr{C}\subset\mathscr{F}^i+\mathscr{C}$. Hence $F\subset F^{i+1}$. □

Just as in the case of l.c.i. distributions (see Assumption 2) we will make some regularity assumptions which will hold throughout the paper. First of all we assume that the l.c.i. function groups $\mathscr{F}$ satisfy Condition A (Assumption 1). Furthermore in Definition 2.1 we make the following

Assumption 3. $C_1,\ldots,C_m$ are independent functions, so $\mathscr{C}$ is a function space satisfying Condition A. Furthermore $\mathscr{F}+\mathscr{C}$ satisfies Condition A.

With respec? to the $\mathscr{F}^*$-algorithm (2.4) we state

Assumption 4. The function groups $\mathscr{F}^i$ and the function spaces $\mathscr{F}^i+\mathscr{C}$ all satisfy Condition A.

Under Assumption 4 it is easy to conclude that the $\mathscr{F}^*$-algorithm ends in a finite number of steps, since $\dim\mathscr{F}^i(x)$ is a non-increasing function of i and $\dim M$ is finite. (For this we may in fact replace Condition A by the weaker Condition A'.)

Let now $\mathscr{F}^*$ be the maximal l.c.i. function group contained in $\mathscr{L}$. Then by Proposition 2.2 we know that $D_{\mathscr{F}^*}$ is a l.c.i. distribution contained in $\operatorname{Ker} d\mathscr{L}^{\perp}$. In general however, $D_{\mathscr{F}^*}$ is *not* the maximal l.c.i. distribution contained in

$\operatorname{Ker} d\mathscr{L}^{\perp}$. This is only true if the maximal l.c.i. distribution contained in $\operatorname{Ker} d\mathscr{L}^{\perp}$, denoted by D^* is of the form $D_{\mathscr{F}}$ for a certain function group $\mathscr{F}$ (see Proposition 1.5.). In fact if $D^* = D_{\mathscr{F}}$, then $\mathscr{F} = \mathscr{F}^*$. Furthermore notice that if we write $D^* = \operatorname{Ker} d\mathscr{K}$, then we always have $\mathscr{F}^* \subset \mathscr{K}^{\perp}$, since by Proposition 2.2 $D_{\mathscr{F}^*}$ is a l.c.i. distribution and so $D_{\mathscr{F}^*} \subset D^* \subset \ker d\mathscr{K}$.

Example 2.5. Consider a Hamiltonian system on $(\mathbb{R}^4, dp_1 \wedge dq_1 + dp_2 \wedge dq_2)$ with $H(q,p) = \frac{1}{2}e^{q_2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}q_1^2 + \frac{1}{2}q_2^2$ and $C(q,p) = q_1$. We want to compute the maximal l.c.i. function group $\mathscr{F}^*$ contained in $\mathscr{C}^{\perp} = \operatorname{span}\{q_1, q_2, p_2\}$, and the maximal l.c.i. distribution D^* contained in $\operatorname{Ker} dC = \ker dq_1$. Since $\{H, C\} = e^{q_2}p_1$, and $\{C, \{H, C\}\} = -e^{q_2}$ and so $X_C(X_H(C)) \neq 0$ it follows from Isidori et al. (1981a), that $D^* = \operatorname{Ker}\operatorname{span}\{dq_1, d(e^{q_2}p_1)\}$

In fact

$$D^* = \operatorname{span}\left\{\frac{\partial}{\partial p_2}, p_1\frac{\partial}{\partial p_1} - \frac{\partial}{\partial q_2}\right\}$$

Denote $\mathscr{K} = \operatorname{span}\{q_1, e^{q_2}p_1\}$, so $D^* = \operatorname{Ker} d\mathscr{K}$. Then $\mathscr{F}^* \subset \mathscr{K}^{\perp}$. Now by Proposition 1.4 $(\mathscr{K})^{\perp} = (\bar{\mathscr{K}})^{\perp} = (\operatorname{span}\{q_1, q_2, p_1\})^{\perp} = \operatorname{span}\{q_2\}$. However $\{H, q_2\} = p_2 \notin \operatorname{span}\{q_2\} + \operatorname{span}\{q_1\}$. Hence $\mathscr{K}^{\perp}$ is not a l.c.i. function group, and so $\mathscr{F}^*$ contains only the constant functions: $\mathscr{F}^* = \mathbb{R}$.

Remark. Local controlled invariance of function groups has an interesting *global* aspect. Let $\mathscr{F}$ be a l.c.i. function group satisfying Condition A. Then $\mathscr{F}$ defines a Poisson structure on $\mathbb{R}^k$, with $k = \dim d\mathscr{F}$. Now *assume* there exists a global basis $x_1, \dots, x_k$ of $\mathbb{R}^k$ in which the Poisson structure is linear, i.e. the functions w_{ij} take the form $w_{ij}(x) = \sum_{r=1}^{k} c_{ijr}x_r$. Then $\mathbb{R}^k$ can be interpreted as the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ with structure coefficients c_{ijr} (see Weinstein (1983)). Denote the corresponding Lie group by G, then it follows that $D_{\mathscr{F}}$ is generated by a symplectic action of G on M, and that the induced map $G \to \mathbb{R}^k$ is the *momentum mapping* of this action. In Nijmeijer & van der Schaft (1984a) such a l.c.i. distribution generated by a group action is called a *partial symmetry*.

3. Hamiltonian Feedback

As is well-known (Isidori et al. (1981b), Nijmeijer (1981)) the conditions of local controlled invariance of a distribution as stated in (2.2) are (under Assumption 2) equivalent to the *local* existence of a feedback which makes this distribution *invariant* (hence the name local controlled invariance). Precisely, let D be an involutive distribution for a nonlinear system (2.1) such that Assumption 2 is satisfied. Then D satisfies (2.2) if and only if there locally exists a feedback $u = \alpha(x) + \beta(x)v$, with α: $M \to \mathbb{R}^m$, β: $M \to \mathbb{R}^{m\times m}$ smooth maps with $\det\beta(x)$

$\neq 0$, and $v = (v_1, \ldots, v_m)$ the new input vector, such that

$$\left[A + \sum_{j=1}^{m} \alpha_j B_j, D\right] \subset D$$
$$\left[\sum_{j=1}^{m} \beta_{ij} B_j, D\right] \subset D \qquad i = 1, \ldots, m \tag{3.1}$$

I.e., D is invariant for the feedback transformed system

$$\dot{x} = A(x) + \sum_{j=1}^{m} \alpha_j(x) B_j(x) + \sum_{j=1}^{m} v_j \left(\sum_{i=1}^{m} \beta_{ji}(x) B_i(x) \right) \tag{3.2}$$

Let now $\mathscr{F}$ be a l.c.i. function group for the Hamiltonian system (1.4) such that Assumptions 1 and 3 are satisfied. Then by Proposition 2.2 $D_{\mathscr{F}}$ is a l.c.i. distribution and hence there exists locally a feedback $u = \alpha(x) + \beta(x) v$ such that $D_{\mathscr{F}}$ is invariant for

$$\dot{x} = X_H(x) - \sum_{j=1}^{m} \alpha_j(x) X_{C_j}(x) - \sum_{j=1}^{m} v_j \left(\sum_{i=1}^{m} \beta_{ji}(x) X_{C_i}(x) \right) \tag{3.3}$$

However, in general the transformed system (3.3) is *not* Hamiltonian anymore.

Theorem 3.1. (van der Schaft (1981, 1983b)). *Denote* $C := (C_1, \ldots, C_m)$: $M \to Y$ *for the Hamiltonian system* (1.4). *Then the system* (3.3) *is again Hamiltonian if and only if the feedback* $u = \alpha(x) + \beta(x) v$ *satisfies*

i) *There exists a function* P: $Y \to \mathbb{R}$ *such that* $\sum_{j=1}^{m} \alpha_j(x) X_{C_j}(x) = X_{P \circ C}(x)$

ii) *There exists a regular mapping* $(R_1, \ldots, R_m)$: $Y \to Y$ *such that* $\sum_{i=1}^{m} \beta_{ji}(x) X_{C_i}(x) = X_{R_j \circ C}(x)$ $j = 1, \ldots, m$

Such a feedback is called a Hamiltonian feedback $(P, R_1, \ldots, R_m)$. *The feedback transformed system is the Hamiltonian system*

$$\dot{x} = X_{H - P \circ C}(x) - \sum_{j=1}^{m} v_j X_{R_j \circ C}$$
$$y_j' = R_j \circ C(x) \tag{3.4}$$

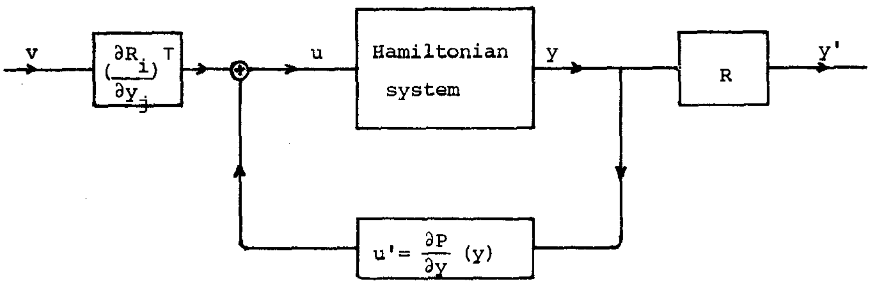
Remark. Hamiltonian feedback has a direct physical interpretation. Part i) corresponds to the addition of a static Hamiltonian compensator

$$u_j = \frac{\partial P}{\partial y_j}(y) \qquad j = 1, \ldots, m, \tag{3.5}$$

or said otherwise, the addition of an extra "potential energy" $P \circ C(x)$. In Part ii) we *transform* the outputs via the (coordinate) transformation $R := (R_1, \ldots, R_m)$: $Y \to Y$ and the inputs in a corresponding way via

$$(v_1, \ldots, v_m) = (u_1, \ldots, u_m)\left(\frac{\partial R_i}{\partial y_j}(y)\right)^{-1} \tag{3.6}$$

(The total induced mapping from T^*Y to itself is a canonical transformation.) Pictorially



(3.7)

Furthermore Hamiltonian feedback is necessarily *output* feedback.

If there exists a Hamiltonian feedback $P, R_1, \ldots, R_m)$ that makes $D_{\mathscr{F}}$ invariant then

$$\begin{aligned} &[X_{H-P\circ C}, D_{\mathscr{F}}] \subset D_{\mathscr{F}} \\ &\left[X_{R_j\circ C}, D_{\mathscr{F}}\right] \subset D_{\mathscr{F}} \qquad j = 1, \ldots, m \end{aligned} \tag{3.8}$$

or equivalently since $\mathscr{F}$ satisfies Condition A (see Prop. 2.2) $\{H - P \circ C, \mathscr{F}\} \subset \mathscr{F}$, $\{R_j \circ C, \mathscr{F}\} \subset \mathscr{F}, j = 1, \ldots, m$. Hence $\mathscr{F}$ is invariant w.r.t. the feedback transformed Hamiltonian system.

Definition 3.2. A l.c.i. function group $\mathscr{F}$ is called (*locally*) *Hamiltonian controlled invariant* ((l.)h.c.i.) if there exists (locally on Y) a Hamiltonian feedback $(P, R_1, \ldots, R_m)$ such that

$$\begin{aligned} &\{H - P \circ C, \mathscr{F}\} \subset \mathscr{F} \\ &\{R_j \circ C, \mathscr{F}\} \subset \mathscr{F} \qquad j = 1, \ldots, m. \end{aligned} \tag{3.9}$$

The central problem of this paper is now the following. Given a l.c.i. function group $\mathscr{F}$, what additional conditions does $\mathscr{F}$ have to satisfy in order that $\mathscr{F}$ is locally Hamiltonian controlled invariant (l.h.c.i.). One additional condition is that $\mathscr{F}$ also has to be *conditioned invariant*, a concept which is treated in the next section.

4. Conditioned Invariance

In this section we define conditioned invariance for Hamiltonian systems and treat its duality with controlled invariance.

Definition 4.1. Let (1.4) be a Hamiltonian system, and let $\mathscr{S}$ be a function group on M. Then $\mathscr{S}$ is *conditioned invariant* if

$$\{H, \mathscr{S} \cap \mathscr{C}^{\perp}\} \subset \mathscr{S} \tag{4.1}$$

The relation of Definition 4.1. with the usual definition of a conditioned invariant (or "(h, f)"-invariant) distribution is the following. An involutive distribution D for a general nonlinear system (2.1) is conditioned invariant if (cf. Isidori et al. (1981a), Nijmeijer & van der Schaft (1982a))

$$[A, D \cap \ker d\mathscr{C}] \subset D \tag{4.2.a}$$

$$[B_j, D \cap \ker d\mathscr{C}] \subset D \qquad j = 1,\ldots, m \tag{4.2.b}$$

In the case of a Hamiltonian system $A = X_H$ and $B_j = X_{C_j}$, $j = 1,\ldots, m$. Now let D satisfying (4.2) for a Hamiltonian system be generated by a function group $\mathscr{S}$, thus $D = D_{\mathscr{S}}$. Then (4.2.a) yields $X_{\{H, \mathscr{S} \cap \mathscr{C}^{\perp}\}} = [X_H, D_{\mathscr{S}} \cap D_{\mathscr{C}^{\perp}}] \subset [X_H, D_{\mathscr{S}} \cap \ker d\mathscr{C}] \subset D_{\mathscr{S}}$, since $D_{\mathscr{C}^{\perp}} \subset \ker d\mathscr{C}$. So by Lemma 1.2 $\{H, \mathscr{S} \cap \mathscr{C}^{\perp}\} \subset \mathscr{S}$ (if $\mathscr{S}$ satisfies Condition A). Therefore, $\mathscr{S}$ is conditioned invariant. Moreover if $\mathscr{C}$ is a function group satisfying Condition A then by Proposition 1.5 $\ker d\mathscr{C} = D_{\mathscr{C}^{\perp}}$ and it is easy to see that (4.1) is actually *equivalent* to (4.2.a). Furthermore in this case (4.2.b) is automatically satisfied since $[X_{C_j}, D_{\mathscr{S}} \cap D_{\mathscr{C}^{\perp}}] \subset [X_{C_j}, D_{\mathscr{C}^{\perp}}] = 0$. Therefore in general the notion of a conditioned invariant function group is slightly weaker than that of a conditioned invariant distribution, but if $\mathscr{C}$ is a function group the notions are equivalent (under the standard regularity assumptions).

We will now set up (completely similarly to the linear case) an algorithm to compute the *minimal* conditioned invariant function group containing a given function group, called the $\mathscr{S}^*$-algorithm. The *existence* of such a minimal function group is already ensured by the following argument. Let $\mathscr{S}^1, \mathscr{S}^2$ be conditioned invariant function groups containing a function group $\mathscr{N}$. Then $\mathscr{S}^1 \cap \mathscr{S}^2$ is again a conditioned invariant function group containing $\mathscr{N}$. Hence by Zorn's lemma there exists a minimal one.

$\mathscr{S}^*$-algorithm. Define for a given function group $\mathscr{N}$

$$\begin{aligned} &\mathscr{S}^1 = \mathscr{N} \\ &\mathscr{S}^{i+1} = \mathscr{N} + \{H, \bar{\mathscr{S}}^i \cap \mathscr{C}^{\perp}\} \qquad i = 1,2,\ldots \end{aligned} \tag{4.3}$$

where $\bar{\mathscr{S}}^i$ denotes the closure of $\mathscr{S}^i$ under Poisson bracket and $\mathscr{N} + \{H, \bar{\mathscr{S}}^i \cap \mathscr{C}^{\perp}\}$ is the minimal function space containing $\mathscr{N}$ as well as $\{H, \bar{\mathscr{S}}^i \cap \mathscr{C}^{\perp}\}$ (so by definition the $\mathscr{S}^i$ are function spaces).

Proposition 4.2. *Let* $\mathscr{S}^i$ *be defined by* (4.3), *then*

a. $\mathscr{S}^i \subset \mathscr{S}^{i+1}$, $i=1,2,\ldots$

b. *If for a certain* k, $\mathscr{S}^k = \mathscr{S}^{k+1}$, *then* $\mathscr{S}^k = \mathscr{S}^{k+l}$, $\forall l$.

Proof. a. By induction. $\mathscr{S}^1 \subset \mathscr{S}^2$ is clear. Assume $\mathscr{S}^{i-1} \subset \mathscr{S}^i$ and let $S^i \in \mathscr{S}^i$. Then there exists an $N \in \mathscr{N}$ and $S^{i-1} \in \bar{\mathscr{S}}^{i-1} \cap \mathscr{C}^\perp$ and a function $G: \mathbb{R}^2 \to \mathbb{R}$ such that $S^i = G(N, \{H, S^{i-1}\})$. Since by assumption $S^{i-1} \in \bar{\mathscr{S}}^{i-1} \subset \bar{\mathscr{S}}^i$, it follows that $S^i \in \mathscr{S}^{i+1}$.

b. Immediate. □

If there exists a k such that $\mathscr{S}^k = \mathscr{S}^{k+1}$ the algorithm ends in k steps by part b., and we denote $\mathscr{S}^* = \bar{\mathscr{S}}^k$. It follows that $\{H, \bar{\mathscr{S}}^k \cap \mathscr{C}^\perp\} \subset \mathscr{S}^k \subset \bar{\mathscr{S}}^k$, and so $\mathscr{S}^*$ is a conditioned invariant function group containing N. Furthermore

Proposition 4.3. $\mathscr{S}^*$ *is the minimal conditioned invariant function group containing* $\mathscr{N}$.

Proof. Let $\mathscr{S}$ be a conditioned invariant function group with $\mathscr{N} \subset \mathscr{S}$. By induction we prove $\mathscr{S}^i \subset \mathscr{S}$, $\forall i$. Assume $\mathscr{S}^i \subset \bar{\mathscr{S}}^i \subset \mathscr{S}$. Then $\{H, \bar{\mathscr{S}}^i \cap C^\perp\} \subset \{H, \mathscr{S} \cap \mathscr{C}^\perp\} \subset \mathscr{S}$, and so $\mathscr{S}^{i+1} \subset \mathscr{S}$. □

Again we will make the regularity assumption that the function groups $\mathscr{S}$ in Definition 4 satisfy Condition A. With respect to the $\mathscr{S}^*$-algorithm we assume that the function groups $\bar{\mathscr{S}}^i$ all satisfy Condition A. Then it is clear by dimensionality arguments that the $\mathscr{S}^*$-algorithm always ends in a finite number of steps.

We now consider the *duality* between (local) controlled invariance and conditioned invariance for Hamiltonian systems.

Proposition 4.4. *Let* $\mathscr{F}$ *be a l.c.i. function group. Then* $\mathscr{F}^\perp$ *is a conditioned invariant function group. Conversely let* $\mathscr{S}$ *be a conditioned invariant function group.* Assume *that* $\mathscr{C}$ *is a function group* and *that* $\mathscr{S}^\perp + \mathscr{C}$ *is a function group. Furthermore let* $\mathscr{S}, \mathscr{S}^\perp, \mathscr{C}, \mathscr{S}^\perp + \mathscr{C}$ *satisfy Condition* A. *Then* $\mathscr{S}^\perp$ *is a l.c.i. function group.*

Proof. Let $\mathscr{F}$ be l.c.i. Take an $F \in \mathscr{F}$ and an $F^\perp \in \mathscr{F}^\perp \cap \mathscr{C}^\perp$. Then $\{\{H, F^\perp\}, F\} = -\{\{F^\perp, F\}, H\} - \{\{F, H\}, F^\perp\} \in \{\mathscr{F} + \mathscr{C}, F^\perp\} = 0$, since $F^\perp \in \mathscr{F}^\perp \cap \mathscr{C}^\perp \subset (\mathscr{F} + \mathscr{C})^\perp$ (see 1.11). Hence $\{H, \mathscr{F}^\perp \cap \mathscr{C}^\perp\} \subset \mathscr{F}^\perp$, and $\mathscr{F}^\perp$ is conditioned invariant.

Conversely let $\mathscr{S}$ be conditioned invariant. Take an $S \in \mathscr{S} \cap \mathscr{C}^\perp$ and $S^\perp \in \mathscr{S}^\perp$. Then $\{\{H, S^\perp\}, S\} = -\{\{S^\perp, S\}, H\} - \{\{S, H\}, S^\perp\} \in \{\mathscr{S}, S^\perp\} = 0$. Hence $\{H, \mathscr{S}^\perp\} \subset (\mathscr{S} \cap \mathscr{C}^\perp)^\perp = \mathscr{S}^\perp + \mathscr{C}^{\perp\perp} = \mathscr{S}^\perp + \mathscr{C}$, by Proposition 1.4. Furthermore for any $C \in \mathscr{C}$, $\{\{C, S^\perp\}, S\} = -\{\{S^\perp, S\}, C\} - \{\{S, C\}, S^\perp\} = 0$ and so $\{\mathscr{C}, \mathscr{S}^\perp\} \subset (\mathscr{S} \cap \mathscr{C}^\perp)^\perp = \mathscr{S}^\perp + \mathscr{C}$. □

Remark. We may also compare the $\mathscr{F}^*$-algorithm with the $\mathscr{S}^*$ algorithm. Let $\mathscr{L}^\perp = \mathscr{N}$ and define $\mathscr{F}^i$ and $\mathscr{S}^i$ according to (2.4), resp. (4.3). Then

a. $\mathscr{S}^i \subset \bar{\mathscr{S}}^i \subset (\mathscr{F}^i)^\perp \quad i=1,2,\ldots$

b. Assume that $\mathscr{C}$ and $\bar{\mathscr{S}}^i + \mathscr{C}$, $i=1,2,\ldots$ are function groups. Furthermore assume that $\bar{\mathscr{S}}^i, (\bar{\mathscr{S}}^i)^\perp, \mathscr{C}, (\bar{\mathscr{S}}^i)^\perp + \mathscr{C}$ all satisfy Condition A. Then $\bar{\mathscr{S}}^i = (\mathscr{F}^i)^\perp$, $i = 1,2,\ldots$, and so $\mathscr{S}^* = (\mathscr{F}^*)^\perp$.

Proof. a. By induction. Since $\mathscr{L}^\perp = \mathscr{N}$, $\mathscr{S}^1 = (\mathscr{F}^1)^\perp$. Assume $\mathscr{S}^i \subset (\mathscr{F}^i)^\perp$. Take an $F \in \mathscr{F}^{i+1}$, i.e. $F \in \mathscr{L}^\perp$ and $\{H, F\} \in F^i + \mathscr{C}$, $\{\mathscr{C}, F\} \subset \mathscr{F}^i + \mathscr{C}$. Then $\{\{H, \bar{\mathscr{S}}^i \cap \mathscr{C}\}, F\} = -\{\{\bar{\mathscr{S}}^i \cap \mathscr{C}^\perp, F\}, H\} - \{\{F, H\}, \bar{\mathscr{S}}^i \cap \mathscr{C}^\perp\}$.

Now $F \in \mathscr{F}^{i+1} \subset \mathscr{F}^i$, and hence by assumption $\{F, \bar{\mathscr{S}}^i\} = 0$. Furthermore

$$\{\{H, F\}, \bar{\mathscr{S}}^i \cap \mathscr{C}^\perp\} \subset \{\mathscr{F}^i + \mathscr{C}, \bar{\mathscr{S}}^i \cap \mathscr{C}^\perp\} = 0, \quad \text{since } \bar{\mathscr{S}}^i \subset (\mathscr{F}^i)^\perp$$

Hence $\mathscr{S}^{i+1} \subset (\mathscr{F}^{i+1})^\perp$.

b. By induction. Assume $\mathscr{F}^i = (\bar{\mathscr{S}}^i)^\perp$. Then $\bar{\mathscr{S}}^i \subset (\bar{\mathscr{S}}^i)^{\perp\perp} = (\mathscr{F}^i)^\perp$ and therefore by part a. $\bar{\mathscr{S}}^{i+1} \subset (\mathscr{F}^{i+1})^\perp$. Now take an $S^\perp \in (\bar{\mathscr{S}}^{i+1})^\perp$. Then $S^\perp \in \mathscr{L}^\perp$ and $S^\perp \in \{H, \bar{\mathscr{S}}^i \cap \mathscr{C}^\perp\}^\perp$. This last inclusion means that for any $S^i \in \bar{\mathscr{S}}^i \cap \mathscr{C}^\perp$, $\{S^\perp, \{H, S^i\}\} = 0$. Now

$$\{S^\perp, \{H, S^i\}\} = -\{H, \{S^i, S^\perp\}\} - \{S^i, \{S^\perp, H\}\}$$

and since $\mathscr{S}^i \in \bar{\mathscr{S}}^i \subset \bar{\mathscr{S}}^{i+1}$ this implies that $\{\{H, S^\perp\}, S^i\} = 0$ for any $S^i \in \bar{\mathscr{S}}^i \cap \mathscr{C}^\perp$.

Hence $\{H, (\bar{\mathscr{S}}^{i+1})^\perp\} \subset (\bar{\mathscr{S}}^i \cap \mathscr{C}^\perp)^\perp = (\bar{\mathscr{S}}^i)^\perp + \mathscr{C}^\perp$, by Proposition 1.4. Analogously $\{\mathscr{C}, (\bar{\mathscr{S}}^{i+1})^\perp\} \subset (\bar{\mathscr{S}}^i)^\perp + \mathscr{C}$. By assumption $(\bar{\mathscr{S}}^i)^\perp = \mathscr{F}^i$ and hence $(\bar{\mathscr{S}}^{i+1})^\perp \subset \mathscr{F}^{i+1}$. Together, for any i we have $\mathscr{F}^i = (\mathscr{F}^i)^{\perp\perp} \subset (\bar{\mathscr{S}}^i)^\perp \subset \mathscr{F}^i$, and hence $(\bar{\mathscr{S}}^i)^\perp = \mathscr{F}^i$. □

Example 4.5. We continue Example 2.5 and calculate the minimal conditioned invariant function group containing $\mathscr{C} = \text{span}\{q_1\}$.

$\mathscr{S}^1 = \text{span}\{q_1\}$

$\mathscr{S}^2 = \text{span}\{q_1\} + \{H, \mathscr{C} \cap \mathscr{C}^\perp\} = \text{span}\{q_1, e^{q_2} p_1\}$

Hence $\bar{\mathscr{S}}^2 = \text{span}\{q_1, q_2, p_1\}$ and so

$\mathscr{S}^3 = \text{span}\{q_1\} + \text{span}\{e^{q_2} p_1, p_2, q_1\}$ and

$$\bar{\mathscr{S}}^3 = \text{span}\{q_1, q_2, p_1, p_2\} = C^\infty(\mathbb{R}^4).$$

Therefore $\mathscr{S}^* = \bar{\mathscr{S}}^3 = C^\infty(\mathbb{R}^4)$ and $\mathscr{F}^*$ (see Example 2.5) $= (\mathscr{S}^*)^\perp = \mathbb{R}$, in agreement with the above remark.

5. Invariance by Hamiltonian Feedback

We return to the central problem of finding conditions in order that a function group $\mathscr{F}$ for a Hamiltonian system (1.4) is locally Hamiltonian controlled invariant, i.e., in order that there locally exist functions $P, R_1, \ldots, R_m$ on Y such that

$$\begin{aligned}&\{H - P\circ C, \mathscr{F}\} \subset \mathscr{F}\\ &\{R_j\circ C, \mathscr{F}\} \subset \mathscr{F} \qquad j = 1,\dots,m\end{aligned} \tag{5.1}$$

We noticed already that $\mathscr{F}$ has to be at least locally controlled invariant. Furthermore let $\mathscr{F}$ satisfy (5.1). Then

$$\{H, \mathscr{F}\cap\mathscr{C}^{\perp}\} = \{H - P\circ C, \mathscr{F}\cap\mathscr{C}^{\perp}\} \subset \mathscr{F} \tag{5.2}$$

Hence $\mathscr{F}$ also has to be conditioned invariant. Moreover if (5.1) is satisfied it follows from the Jacobi-identity that $\{H - P\circ C, \mathscr{F}^{\perp}\} \subset \mathscr{F}^{\perp}$. Hence $\{H, \mathscr{F}^{\perp}\} \subset \mathscr{F}^{\perp} + \mathscr{C}$ and as in (5.2) $\{H, \mathscr{F}^{\perp}\cap\mathscr{C}^{\perp}\} \subset \mathscr{F}^{\perp}$. Similarly $\{R_j\circ C, \mathscr{F}^{\perp}\} \subset \mathscr{F}^{\perp}$ and so $\{C, F^{\perp}\} \subset \mathscr{F}^{\perp} + \mathscr{C}$. Therefore if $\mathscr{F}$ is l.h.c.i. then $\mathscr{F}$ as well as $\mathscr{F}^{\perp}$ have to be locally controlled *and* conditioned invariant. However in Proposition 4.4. we already derived that if $\mathscr{F}$ is l.c.i. then $\mathscr{F}^{\perp}$ is conditioned invariant, and hence if $\mathscr{F}^{\perp}$ is l.c.i. then $\mathscr{F} = (\mathscr{F}^{\perp})^{\perp}$ is conditioned invariant (see Prop. 1.4). So a necessary condition for $\mathscr{F}$ to be l.h.c.i. is that $\mathscr{F}$ and $\mathscr{F}^{\perp}$ are both locally controlled invariant.

One may suspect that this condition (maybe under some additional integrability and regularity conditions) is also *sufficient* for $\mathscr{F}$ to be locally Hamiltonian controlled invariant. However this is *not* true as already shown by the *linear* case (see van der Schaft (1983a), Nijmeijer & van der Schaft (1984b)). In this case $\mathscr{F}$ is spanned by linear functions on $\mathbb{R}^{2n}$ and $D_{\mathscr{F}}$ corresponds to a linear subspace $\mathscr{V}$ of $\mathbb{R}^{2n}$. Moreover $\mathscr{F}$ is l.c.i. if and only if $\mathscr{F}$ is conditioned invariant. So the condition that $\mathscr{F}$ and $\mathscr{F}^{\perp}$ are l.c.i. is equivalent to $\mathscr{F}$ being controlled and conditioned invariant. This implies that there exists *output* feedback $u = Ky$ which makes $\mathscr{V}$ invariant. Now output feedback $u = Ky$ is Hamiltonian feedback if and only if $K = K^T$. However in general K *cannot* be taken to be symmetric. Only in case $\mathscr{F}$ is *Lagrangian* or *symplectic* this is always possible.

Definition 5.1. Let $\mathscr{F}$ be a function group. $\mathscr{F}$ is *Lagrangian* if $\mathscr{F}^{\perp} = \mathscr{F}$ and *symplectic* if $\mathscr{F}^{\perp}\cap\mathscr{F} = \mathbb{R}$. Furthermore $\mathscr{F}$ is *coisotropic* if $\mathscr{F}^{\perp} \subset \mathscr{F}$.

Remark. If $\mathscr{F}$ is coisotropic then $\dim d\mathscr{F}(x) \geqslant \frac{1}{2}\dim M$. Furthermore a coisotropic $\mathscr{F}$ is Lagrangian if and only if $\dim d\mathscr{F}(x) = \frac{1}{2}\dim M$.

Therefore also in the nonlinear case for *arbitrary* $\mathscr{F}$ the condition that $\mathscr{F}$ and $\mathscr{F}^{\perp}$ are l.c.i. may only imply that $D_{\mathscr{F}}$ can be made invariant by *output* feedback, not necessarily Hamiltonian feedback. In fact in the nonlinear case there is an extra complication because controlled and conditioned invariance is not enough for the existence of output feedback. We need an extra *integrability* condition as shown in Nijmeijer & van der Schaft (1982a).

In this section we shall show that also in the nonlinear case a Lagrangian function group $\mathscr{F}$ which is l.c.i. (and hence also $\mathscr{F}^{\perp} = \mathscr{F}$ is l.c.i.) can be made invariant by Hamiltonian feedback, *provided* the extra integrability condition for the existence of output feedback is satisfied. Moreover for Hamiltonian systems this condition can be stated in a much more concrete way than for general nonlinear systems. Preliminary investigations (Nijmeijer & van der Schaft (1984c)) suggest that also *symplectic* nonlinear functions groups $\mathscr{F}$ are l.h.c.i. if $\mathscr{F}$ and $\mathscr{F}^{\perp}$ are l.c.i., again provided the extra integrability condition is satisfied. Furthermore in the next section we shall show how *coisotropic* l.c.i. function groups may be made invariant by *dynamic* Hamiltonian feedback, by *lifting* the function group to a Lagrangian l.c.i. function group for an augmented Hamiltonian system.

Remark 1. In all these three cases ($\mathscr{F} = \mathscr{F}^{\perp}$, $\mathscr{F}^{\perp} \subset \mathscr{F}$, $\mathscr{F} \cap \mathscr{F}^{\perp} = \mathbb{R}$) $\mathscr{F}$ satisfies the conditions of Theorem 1.1, so we may derive a local normal form for $\mathscr{F}$.

Remark 2. We have argued that only under severe conditions a l.c.i. function group $\mathscr{F}$ is also l.h.c.i. So in general there only exists a non-Hamiltonian feedback $u = \alpha(x) + \beta(x)\, v$ which makes $D_{\mathscr{F}}$ invariant. Although the resulting transformed system $\dot{x} = X_H - \sum_{j=1}^{m} \alpha_j X_{C_j} + \sum_{i=1}^{m} v_i \sum_{j=1}^{m} \beta_{ij} X_{C_j}$ is therefore not Hamiltonian it can be made again Hamiltonian by the addition of *output injection*, i.e.

$$\begin{aligned}\dot{x} &= X_H - \sum_{j=1}^{m} \alpha_j X_{C_j} - \sum_{j=1}^{m} C_j X_{\alpha_j} + \sum_{i=1}^{m} v_i \left(\sum_{i=1}^{m} \beta_{ij} X_{C_j} + \sum_{j=1}^{m} C_j X_{\beta_{ij}} \right) \\ &= X_H - \sum_{j=1}^{m} \alpha_j C_j + \sum_{i=1}^{m} v_i X_{(\sum_{j=1}^{m} \beta_{ij} C_j)}\end{aligned}$$

This is investigated in van der Schaft (1985), and gives a hint to handle l.c.i. function groups which are not l.h.c.i. in a "Hamiltonian way".

So let $\mathscr{F}$ be a l.c.i. Lagrangian function group on (M, ω), satisfying Condition A. Then there exist n independent functions $F_1, \ldots, F_n$ such that $\mathscr{F} = \operatorname{span}\{F_1, \ldots, F_n\}$. As we saw in Section 1 F defines a Poisson structure on $\mathbb{R}^n$. In this case because $\mathscr{F} = \mathscr{F}^{\perp}$ the Poisson structure is identically zero, i.e. $\{G_1, G_2\}_{\mathbb{R}^n} = 0$, $\forall G_1, G_2\colon \mathbb{R}^n \to \mathbb{R}$. However the following construction works for arbitrary $\mathscr{F}$. Define $F\colon M \to \mathbb{R}^n$ as $F = (F_1, \ldots, F_n)$ and let $z = (z_1, \ldots, z_n)$ be coordinates for $\mathbb{R}^n$ such that $z_i \circ F = F_i$, $i = 1, \ldots, n$. Let furthermore $C := (C_1, \ldots, C_m)\colon M \to Y$ and let $y = (y_1, \ldots, y_m)$ be (local) coordinates for Y such that $y_j \circ C = C_j$, $j = 1, \ldots, m$. Denote by $(y, u) = (y_1, \ldots, y_m, u_1, \ldots, u_m)$ the corresponding natural coordinates for T^*Y. T^*Y has the canonically defined Poisson (in fact symplectic) structure

$$\{G_1(y,u), G_2(y,u)\}_{T^*Y} = \sum_{j=1}^{m} \left(\frac{\partial G_1}{\partial u_j} \frac{\partial G_2}{\partial y_j} - \frac{\partial G_1}{\partial y_j} \frac{\partial G_2}{\partial u_j} \right) \tag{5.3}$$

for $G_1, G_2\colon T^*Y \to \mathbb{R}$. Therefore we can give $T^*Y \times \mathbb{R}^n$ the product structure of the Poisson structures on T^*Y respectively $\mathbb{R}^n$:

$$\begin{aligned}&\{G_1(y,u,z), G_2(y,u,z)\}_{T^*Y \times \mathbb{R}^n} \\ &\quad = \{G_1(y,u,z), G_2(y,u,z)\}_{T^*Y} + \{G_1(y,u,z), G_2(y,u,z)\}_{\mathbb{R}^n} \\ &\quad = \{G_1(y,u,z), G_2(y,u,z)\}_{T^*Y}\end{aligned} \tag{5.4}$$

since the Poisson structure on $\mathbb{R}^n$ is zero. (Notice that for computing $\{G_1(y,u,z), G_2(y,u,z)\}_{T^*Y}$ we treat z as a parameter). Since $\mathscr{F}$ is l.c.i.

$$\{H, F_i\} \subset \mathscr{F} + \mathscr{C}, \{C_j, F_i\} \subset \mathscr{F} + \mathscr{C} \tag{5.5}$$

and hence under Assumption 3 there exist smooth functions $K_j^i(y,z), V^i(y,z)$ such that

$$\begin{aligned} \{H, F_i\}(x) &= V^i(C(x), F(x)) & i &= 1,\dots,n \\ \{C_j, F_i\}(x) &= -K_j^i(C(x), F(x)) & i &= 1,\dots,n \\ & & j &= 1,\dots,m \end{aligned} \tag{5.6}$$

or equivalently

$$\left\{H - \sum_{j=1}^{m} u_j C_j, F_i\right\}_M (x) = F_i^e(C(x), u, F(x)) \qquad i = 1,\dots,n \tag{5.7}$$

with F_i^e: $T^*Y \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$F_i^e(y,u,z) = V^i(y,z) + \sum_{j=1}^{m} u_j K_j^i(y,z) \qquad i = 1,\dots,n \tag{5.8}$$

We now state the main theorem.

Theorem 5.2. *Let* (1.4) *be a Hamiltonian system, and let* $\mathscr{F} = \text{span}\{F_1,\dots,F_n\}$ *be a Lagrangian function group on* M. *Suppose*

1. $\mathscr{F}$ *is l.c.i., so there exist functions* $F_1^e,\dots,F_n^e$: $T^*Y \times \mathbb{R}^n \to \mathbb{R}$ *such that* (5.7) *holds.*

Moreover suppose $\mathscr{F}$ *satisfies the integrability condition*:

2. *The function space* $\text{span}\{F_1^e,\dots,F_n^e\}$ *on* $T^*Y \times \mathbb{R}^n$ *"projects" to a function space on* T^*Y, *i.e. there exist functions* $G_1^e,\dots,G_n^e$: $T^*Y \to \mathbb{R}$ *such that*

$$\begin{aligned} &\underset{i=1,\dots,n}{\text{span}} \left\{ \sum_{j=1}^{m} \left(\frac{\partial F_i^e}{\partial y_j} dy_j + \frac{\partial F_i^e}{\partial u_j} du_j \right) \right\} \\ &= \underset{i=1,\dots,n}{\text{span}} \left\{ \sum_{j=1}^{m} \left(\frac{\partial G_i^e}{\partial y_j} dy_j + \frac{\partial G_i^e}{\partial u_j} du_j \right) \right\} = \underset{i=1,\dots,n}{\text{span}} \{dG_i^e\} \end{aligned} \tag{5.9}$$

in every point (y,u,z) *of* $T^*Y \times \mathbb{R}^n$.

Furthermore suppose the following regularity assumptions are satisfied.

3. *The function spaces* $\mathscr{F}$, $\mathscr{C}$ *and* $\mathscr{F}+\mathscr{C}$ *satisfy Condition* A.

4. *The function spaces* $\mathscr{F}^e := \operatorname{span}\{F_1^e,\dots,F_n^e\}$ *on* $T^*Y\times\mathbb{R}^n$, *and* $\mathscr{G}^e := \operatorname{span}\{G_1^e,\dots,G_n^e\}$ *on* T^*Y *satisfy Condition* A.

5. *The codistribution* $\underset{i=1,\dots,n}{\operatorname{span}}\left\{\dfrac{\partial F_i^e}{\partial u_j}du_j\right\}$ *on* $T^*Y\times\mathbb{R}^n$ *has constant dimension.*

Then there exist locally on Y *functions* $P, R_1,\dots,R_m$ *with* $\det\left(\dfrac{\partial R_i}{\partial y_j}\right)\neq 0$, *such that* (5.1) *holds. Conversely, if* (5.1) *holds, then* $\mathscr{F}$ *satisfies* 1. *and* 2.

Remark. Since the theorem is essentially local in nature, we may replace Condition A by Condition A′.

Proof. Let $\mathscr{F}$ satisfy 1 up till 5. First we will prove that $\mathscr{F}^e$ as well as $\mathscr{G}^e$ are function *groups*, such that $\{F_i^e, F_j^e\}_{T^*Y\times\mathbf{R}^n} = \{G_i^e, G_j^e\}_{T^*Y} = 0$, $i, j = 1,\dots,n$.

Consider F_1, $F_2\in\mathscr{F}$. By application of the Jacobi-identity in every $x=(q,p)\in M$ (see also van der Schaft (1983c))

$$\begin{aligned}
&\left\{H-\sum_{j=1}^m u_jC_j, \{F_1,F_2\}\right\}(x)\\
&= \left\{\left\{H-\sum_{j=1}^m u_jC_j, F_1\right\}, F_2\right\}(x) - \left\{\left\{H-\sum_{j=1}^m u_jC_j, F_2\right\}, F_1\right\}(x)\\
&= \{F_1^e(C(x),u,F(x)), F_2(x)\} - \{F_2^e(C(x),u,F(x)), F_1(x)\}\\
&= \sum_{j=1}^m \frac{\partial F_1^e}{\partial y_j}\{C_j,F_2\} + \sum_{i=1}^n \frac{\partial F_1^e}{\partial z_i}\{F_i,F_2\}\\
&\quad - \sum_{j=1}^m \frac{\partial F_2^e}{\partial y_j}\{C_j,F_1\} - \sum_{i=1}^n \frac{\partial F_2^e}{\partial z_i}\{F_i,F_1\}\\
&= \sum_{j=1}^m \frac{\partial F_1^e}{\partial y_j}\{C_j,F_2\} - \sum_{j=1}^m \frac{\partial F_2^e}{\partial y_j}\{C_j,F_1\},
\end{aligned}$$

since $\mathscr{F}$ is Lagrangian.

Now $F_i^e = \sum_{j=1}^m u_jK_j^i(y,z) + V^i(y,z)$, $i=1,\dots,n$. Hence

$$\frac{\partial F_i^e}{\partial y_r} = \sum_{j=1}^m u_j\frac{\partial K_j^i}{\partial y_r} + \frac{\partial V^i}{\partial y_r}$$

and

$$\{C_j, F_i\} = -K_j^i = -\frac{\partial F_i^e}{\partial u_j}.$$

Therefore

$$\left\{ H - \sum_{j=1}^{m} u_j C_j, \{F_1, F_2\} \right\}(x)$$

$$= \sum_{j=1}^{m} \left(\frac{\partial F_1^e}{\partial u_j} \frac{\partial F_2^e}{\partial y_j} - \frac{\partial F_1^e}{\partial y_j} \frac{\partial F_2^e}{\partial u_j} \right)(C(x), u, F(x))$$

$$= \{F_1^e, F_2^e\}_{T^*Y \times \mathbb{R}^n}. \tag{5.10}$$

Hence the mapping $F_i \to F_i^e$ from $\mathscr{F}$ to $\mathscr{F}^e$ is an algebra morphism with respect to the Poisson bracket on M, respectively on $T^*Y \times \mathbb{R}^n$. Therefore $\mathscr{F}^e$ is a function group, and since $\mathscr{F}$ is Lagrangian

$$\{F_i^e, F_j^e\}_{T^*Y \times \mathbb{R}^n} = 0, \text{ for any } F_i^e, F_j^e \in \mathscr{F}^e.$$

Since the Poisson structure on $\mathbb{R}^n$ is zero it follows that also $\mathscr{G}^e$ is a function group with $\{G_i^e, G_j^e\}_{T^*Y} = 0$.

Now turn attention to the functions F_i^e. By condition 5. the rank of the matrix

$$\begin{pmatrix} K_1^1 & \cdots & K_1^n \\ \vdots & & \vdots \\ K_m^1 & \cdots & K_m^n \end{pmatrix}$$

is constant, say r, in every point (y, z). Hence we may take $F_1, \dots, F_n$ spanning $\mathscr{F}^e$ in such a way that $K_1^i(y, z) = \cdots = K_m^i(y, z) = 0$, $\forall (y, z)$, $i = r+1, \dots, n$. Then, equivalently $\{C_j, F_i\}$ is identically zero, $j = 1, \dots, m$, $i = r+1, \dots, n$, and hence $F_i \in \mathscr{C}^\perp$, $i = r+1, \dots, n$. Since $\mathscr{F}$ is conditioned invariant this implies that $\{H, F_i\} \subset \mathscr{F}$, $i = r+1, \dots, n$. Hence V^i can be taken as a function of z only, $i = r+1, \dots, n$. Since the F_i^e are affine in u, the G_i^e can be also taken affine in u, i.e.

$$G_i^e(y, u) = \sum_{j=1}^{m} u_j L_j^i(y) + W^i(y) \qquad i = 1, \dots, n \tag{5.11}$$

Furthermore by the above reasoning we may take k independent functions $G_1^e, \dots, G_k^e$ of the form (5.11) and spanning $\mathscr{G}^e$ such that

$$\dim \operatorname{span} \left\{ \begin{pmatrix} L_1^1(y) \\ \vdots \\ L_m^1(y) \end{pmatrix}, \dots, \begin{pmatrix} L_1^k(y) \\ \vdots \\ L_m^k(y) \end{pmatrix} \right\} = k$$

It follows that the Hamiltonian vectorfields $X_{G_1^e}, \dots, X_{G_k^e}$ on T^*Y are independent and project to k independent vectorfields on Y. In fact, denote the projection from T^*Y to Y by π, then

$$\pi_* X_{G_i^e} = \sum_{j=1}^{m} L_j^i(y) \frac{\partial}{\partial y_j} \qquad i = 1, \dots, k \tag{5.12}$$

Since $\{G_i^e, G_j^e\}_{T^*Y} = 0$, it follows that

$$\left[\pi_* X_{G_i^e}, \pi_* X_{G_j^e}\right] = 0 \qquad i, j = 1, \dots, k \tag{5.13}$$

Hence there exists a local coordinate transformation $(R_1(y), \dots, R_m(y)) = (y_1', \dots, y_m')$ on Y, with $\det\left(\frac{\partial R_i}{\partial y_j}\right) \neq 0$, such that $\pi_* X_{G_i^e} = \frac{\partial}{\partial y_i'}$, $i = 1, \dots, k$.

Since $\{G_i^e, G_j^e\} = 0$ and because of (5.12) we can even take (local) canonical coordinates $(y', v) = (y_1', \dots, y_m', v_1, \dots, v_m)$ for T^*Y such that

$$G_i^e = v_i \qquad i = 1, \dots, k \tag{5.14}$$

(This follows from Darboux's theorem. See for similar arguments van der Schaft (1981, Th. 3.3), (1983c, Th. 2.7)). The submanifold $v_1 = \cdots = v_m = 0$ is a Lagrangian submanifold of T^*Y, and therefore has a generating function $P(y_1, \dots, y_m)$, i.e.

$$\begin{aligned} &\{(y_1', \dots, y_m', v_1, \dots, v_m) | v_1 = \cdots = v_m = 0\} \\ &\quad = \left\{(y_1, \dots, y_m, u_1, \dots, u_m) | u_j = \frac{\partial P}{\partial y_j}(y), j = 1, \dots, m\right\} \end{aligned} \tag{5.15}$$

In the new canonical coordinates (y', v) for T^*Y the F_i^e are of the form

$$F_i^e(y', v, z) = \sum_{j=1}^{k} v_j K_j'^i(z) + V'^i(z), \tag{5.16}$$

since F_i^e is affine in v and by condition 2

$$\begin{aligned} \underset{i=1,\dots,n}{\operatorname{span}} \left\{ \sum_{j=1}^{m} \left(\frac{\partial F_i^e}{\partial y_j'} dy_j' + \frac{\partial F_i^e}{\partial v_j} dv_j \right) \right\} &= \underset{i=1,\dots,n}{\operatorname{span}} \{dG_i^e\} \\ &= \operatorname{span}\{dv_1, \dots, dv_k\} \end{aligned}$$

in every point (y', v, z). However since $F_i^e(y', v, z)$ is determined by

$$\left\{H - P \circ C - \sum_{j=1}^{m} v_j C_j', F_i\right\}(x) = F_i^e(C'(x), v, F(x)) \tag{5.17}$$

where $C' = (C_1', \dots, C_m')$ in the new coordinates $(y_1', \dots, y_m')$ for Y, it follows from (5.16) that

$$\begin{aligned} &\{H - P \circ C, F_i\}(x) = V'^i(F(x)) && i = 1, \dots, n \\ &\{C_j', F_i\}(x) = -K_j'^i(F(x)) && i = 1, \dots, n \\ & && j = 1, \dots, m \end{aligned} \tag{5.18}$$

or equivalently

$$\begin{aligned}&\{H-P\circ C,\mathscr{F}\}\subset\mathscr{F}\\&\{R_j\circ C,\mathscr{F}\}\subset\mathscr{F}\qquad j=1,\dots,m\end{aligned}\tag{5.19}$$

as was to be proved.

Conversely, suppose that (5.19) = (5.1) holds. Then

$$\{C_j,F_i\}=\{(R^{-1})_j\circ R\circ C,F_i\}=\sum_{k=1}^{m}\frac{\partial(R^{-1})j}{\partial y_k'}(R\circ C)\{F_k\circ C,F_i\}$$

with R: $Y\to Y$ defined by $R=(R_1,\dots,R_m)$. Since $\{R_k\circ C,F_i\}=-K_k'^i\in\mathscr{F}$ it follows that $\{C_j,F_i\}\in\mathscr{F}+\mathscr{C}$, $\forall i,j$. Also, since $\{H-P\circ C,F_i\}=V'^i\in\mathscr{F}$ we have

$$\{H,F_i\}=\{H-P\circ C,F_i\}+\{P\circ C,F_i\}=V'^i+\sum_{j=1}^{m}\frac{\partial(P\circ R^{-1})}{\partial y_j'}(R\circ C),$$

and

$$\{R_j\circ C,F_i\}=V'^i+\sum_{j=1}^{m}\frac{\partial(P\circ R^{-1})}{\partial y_j'}(R\circ C)K_j'^i\in\mathscr{F}+\mathscr{C}.$$

Hence for $i=1,\dots,n$

$$\begin{aligned}F_i^e(y,u,z)&=V'^i(z)+\sum_{j=1}^{m}\frac{\partial(P\circ R^{-1})}{\partial y_j'}(R(y))K_j'^i(z)\\&\quad+\sum_{j=1}^{m}u_j\left(\sum_{k=1}^{m}\frac{\partial(R^{-1})j}{\partial y_k'}(R(y))K_j'^i(z)\right)\end{aligned}\tag{5.20}$$

and we may define $G_i^e(y,u)$, $i=1,\dots,n$ as

$$G_i^e(y,u,z)=\sum_{j=1}^{m}\frac{\partial(P\circ R^{-1})}{\partial y_j'}(R(y))+\sum_{j=1}^{m}u_j\left(\sum_{k=1}^{m}\frac{\partial(R^{-1})_j}{\partial y_k'}(R(y))\right)\tag{5.21}$$

Then it is clear that condition 2 is satisfied. □

Remark 1. Condition 2 is a direct specialization of the integrability condition for output feedback derived in Nijmeijer & van der Schaft (1982a, Theorem 3.1, 3.2, condition iii) to Hamiltonian systems. In this reference one seeks for conditions to make a distribution D for a general nonlinear system $\dot{x}=f(x,u)$, $y=h(x)$ invariant by means of *output* feedback. One defines the codistribution P on M by $\operatorname{Ker}P=D$. Then the prolonged codistribution $\dot{P}$ on TM has to be such that the codistribution $\operatorname{span}\{P, f^*\dot{P}\} \cap \operatorname{span}\{dh, du\}$ is involutive. In the Hamiltonian case $D = D_{\mathscr{F}}$, and if $\mathscr{F}$ is Lagrangian $P = d\mathscr{F}$. Furthermore since $f(x,u) = X_H(x) - \sum_{j=1}^{m} u_j X_{C_j}(x)$ we have for $\dot{F}_i \in \dot{\mathscr{F}}$

$$f^*\dot{F}_i = \dot{F}_i \circ \left(X_H(x) - \sum_{j=1}^{m} u_j X_{C_j}(x) \right) = \left(X_H - \sum_{j=1}^{m} u_j X_{C_j} \right)(F_i)(x)$$

$$= \left\{ H - \sum_{j=1}^{m} u_j C_j, F_i \right\}(x) = F_i^e(C(x), u, F(x))$$

Therefore

$$\operatorname{span}\{P, f^*\dot{P}\} \cap \operatorname{span}\{dh, du\} = \operatorname{span}\left\{ d\mathscr{F}, d\left\{ H - \sum_{j=1}^{m} u_j C_j, F_i \right\} \right\}$$

$$\cap \operatorname{span}\{dh, du\}$$

$$= \operatorname{span}\{dz, d\mathscr{F}^e\} \cap \operatorname{span}\{dy, du\}.$$

Now the distribution $\operatorname{span}\{d\mathscr{F}^e, dz\} \cap \operatorname{span}\{dy, du\}$ on $T^*Y \times \mathbb{R}^n$ is involutive *if and only if* condition 2 is satisfied. This follows from the following Lemma which we state without proof.

Lemma 5.3. *Let $x = (x_1, \ldots, x_n)$ and let $f_1(x), \ldots, f_k(x)$ be functions. Then for k, $r \leqslant n$: $\operatorname{span}\{df_1(x), \ldots, df_k(x), dx_{r+1}, \ldots, dx_m\} \cap \operatorname{span}\{dx_1, \ldots, dx_r\}$ is involutive, if and only if there exist functions $g_1(x_1, \ldots, x_r), \ldots, g_k(x_1, \ldots, x_r)$ such that*

$$\operatorname*{span}_{i=1,\ldots,k} \left\{ \sum_{l=1}^{r} \frac{\partial f_i}{\partial x_l} dx_l \right\} = \operatorname*{span}_{i=1,\ldots,k} \left\{ \sum_{l=1}^{r} \frac{\partial g_i}{\partial x_l} dx_l \right\}.$$

Remark 2. Condition 2 can be equivalently formulated in the following way. Denote the projection from $T^*Y \times \mathbb{R}^n$ to T^*Y by p. Then condition 2 is equivalent to: $D_{\mathscr{F}^e} = \operatorname{span}\{X_{F_1^e}, \ldots, X_{F_n^e}\}$ (distribution on $T^*Y \times \mathbb{R}^n$!) projects under p to a distribution on T^*Y. In fact $p_* D_{\mathscr{F}^e}$ should equal $D_{\mathscr{G}^e} = \operatorname{span}\{X_{G_1^e}, \ldots, X_{G_n^e}\}$ (a distribution on T^*Y). Now we know from van der Schaft (1982a) that $D_{\mathscr{F}^e}$ projects to a distribution on T^*Y if and only if

$$\left[\frac{\partial}{\partial z_i}, X_{F_j} \right] \subset \operatorname{span}\left\{ \frac{\partial}{\partial z_i}, \ldots, \frac{\partial}{\partial z_m} \right\} + D_{\mathscr{F}^e} \qquad \begin{array}{l} i = 1, \ldots, n \\ j = 1, \ldots, n \end{array} \tag{5.22}$$

Let us give a simple (mathematical) example of the above theorem:

Example 5.4. Consider the Hamiltonian system on

$$M = \{(q_1, q_2, p_1, p_2) \mid p_1 + p_2 \neq 0\}$$

with

$$
\begin{aligned}
H &= \frac{1}{(p_1+p_2)} + \frac{1}{4}(q_1-q_2)^2 + (p_1+p_2)\sin q_1 \cos q_2 \\
&\quad + (p_1+p_2)\cos q_1 \sin q_2 \\
C_1 &= (p_1+p_2)\cos\tfrac{1}{2}(q_1-q_2) \\
C_2 &= (p_1+p_2)\sin\tfrac{1}{2}(q_1-q_2)
\end{aligned}
$$

and output manifold $Y = \mathbb{R}^2\backslash\{0\}$. Then $\mathscr{F} := \text{span}\{q_1+q_2, p_1-p_2\}$ is a Lagrangian function group. Choose new canonical coordinates $P_1 = q_1+q_2$, $P_2 = p_1-p_2$, $Q_1 = -\frac{1}{2}(p_1+p_2)$, $Q_2 = \frac{1}{2}(q_1-q_2)$. It easily follows that $\mathscr{F}$ is l.c.i. Furthermore $\mathscr{F}$ is also l.h.c.i. as can be seen as follows. First apply Hamiltonian feedback $V_1(y_1, y_2) = (y_1^2+y_2^2)^{-1/2}$. Then $V_1 \circ C = \dfrac{1}{p_1+p_2}$. Furthermore, since $C_1 = -2Q_1\cos Q_2$, $C_2 = -2Q_1 \sin Q_2$, there exist R_1 and R_2 on Y such that $R_1 \circ C = Q$, $R_2 \circ C = Q_2$. Then it is clear that $\{R_1 \circ C, \mathscr{F}\} \subset \mathscr{F}, \{R_2 \circ C, \mathscr{F}\} \subset \mathscr{F}$. Finally apply feedback $V_2(y_1', y_2') = (y_2')^2$ then $\{H - V_1(C_1, C_2) - V_2(R_1 \circ C, R_2 \circ C), \mathscr{F}\} \subset \mathscr{F}$, so $\mathscr{F}$ is made invariant by Hamiltonian feedback.

If there exists a Lagrangian function group $\mathscr{F}$ satisfying (5.1) we obtain the following *normal form* for the feedback transformed Hamiltonian system. By Theorem 1.1. there exists local canonical coordinates $(q_1,\dots,q_n,p_1,\dots,p_n)$ such that $\mathscr{F} = \text{span}\{p_1,\dots,p_n\}$. Then (5.1) implies

$$
\frac{\partial(H - P\circ C)}{\partial q_i} \in \mathscr{F}, \ \frac{\partial(R_j \circ C)}{\partial q_i} \in \mathscr{F} \qquad \begin{matrix} i=1,\dots,n \\ j=1,\dots,m \end{matrix} \tag{5.23}
$$

Hence $H - P \circ C$ and $R_j \circ C$ are of the form

$$
\begin{aligned}
H - P\circ C &= \sum_{i=1}^{n} h_i(p) q_i + h(p) \\
R_j \circ C &= \sum_{i=1}^{n} c_i^j(p) q_i + c^j(p) \qquad j = 1,\dots,m
\end{aligned} \tag{5.24}
$$

and the feedback transformed system equals

$$
\begin{aligned}
\dot{q}_i &= \sum_{k=1}^{n} \frac{\partial h_k(p)}{\partial p_i} q_k + \frac{\partial h(p)}{\partial p_i} - \sum_{j=1}^{m} v_j \sum_{k=1}^{n} \frac{\partial c_k^j(p)}{\partial p_i} q_k + \frac{\partial c^j(p)}{\partial p_i} \\
\dot{p}_i &= -h_i(p) + \sum_{j=1}^{m} v_j c_i^j(p) \qquad i = 1,\dots,n
\end{aligned} \tag{5.25}
$$

Notice furthermore the close relationship with the notion of *complete integrability* of a Hamiltonian vectorfield X_H. X_H is completely integrable if there exist n independent functions $F_1,\dots,F_n$ such that $\{H, F_i\} = 0$ and $\{F_i, F_j\} = 0$, $i, j =$

$1, \dots, n$. It follows that $\mathscr{F} := \operatorname{span}\{F_1, \dots, F_n\}$ is a Lagrangian function group satisfying $\{H, \mathscr{F}\} = 0$.

We remark that the construction of *action-angle* coordinates used in the context of such completely integrable Hamiltonian vectorfields can be immediately applied to Lagrangian invariant function groups. This yields a sort of *global* interpretation of the local normal form (5.25) (see Abraham & Marsden (1978)).

Remark. Although we have confined ourselves to *affine* Hamiltonian systems (1.4), the developed theory of local controlled invariance can be extended to general Hamiltonian systems given by a generating function $H(q, p, u)$, as treated in Brockett (1977), van der Schaft (1982a, b, 1983b). In this case a function group $\mathscr{F} = \operatorname{span}\{F_1, \dots, F_k\}$ is called l.c.i. if there exist functions F_i^e on $T^*Y \times \mathbb{R}^k$ such that

$$\{H(q, p, u), F_i\} = F_i^e\left(-\frac{\partial H}{\partial u}(q, p, u), u, F(q, p)\right) \qquad i = 1, \dots, k$$

where $F := (F_1, \dots, F_k)$. So the only difference with (5.7) is that the F_i^e need no longer to be affine in u. Furthermore if $\mathscr{F}$ is Lagrangian it can be proved (cf. van der Schaft (1984b), Theorem 5) that the mapping $F_i \to F_i^e$ is a Poisson algebra morphism, and so $\{F_i, F_j\}_{T^*Y \times \mathbb{R}^k} = 0$, $\forall i, j$, as in Theorem 5.2.

6. Invariance by Dynamic Hamiltonian Feedback

In the previous section we have seen that under an integrability condition and some regularity assumptions a *Lagrangian* function group $\mathscr{F}$ locally can be made invariant by Hamiltonian feedback if $\mathscr{F}$ is locally controlled invariant. In this section we will show how we can extend this procedure to *coisotropic* function groups. The trick is to *augment* the state space of the system and to *lift* the coisotropic function group to a *Lagrangian* function group on the augmented state space. Hamiltonian feedback for this augmented system corresponds to *dynamic* Hamiltonian feedback for the original system.

Definition 6.1. Let $\mathbb{R}^{2l}$ be endowed with its natural symplectic form. We define the *auxiliary Hamiltonian system* on $\mathbb{R}^{2l}$ as

$$\begin{aligned} \dot{\xi}_i &= -\nu_i & i &= 1, \dots, l \\ \dot{\xi}_i &= \nu_i & i &= l+1, \dots, 2l \\ \eta_i &= \xi_i & i &= 1, \dots, 2l \end{aligned} \qquad (6.1)$$

where $\xi = (\xi_1, \dots, \xi_{2l})$ are the standard coordinates for the state space $M^{au} := \mathbb{R}^{2l}$, $\nu = (\nu_1, \dots, \nu_{2l})$ are the standard coordinates for the input space $U^{au} := \mathbb{R}^{2l}$ and $\eta = (\eta_1, \dots, \eta_{2l})$ are the standard coordinates for the output space $Y^{au} := \mathbb{R}^{2l}$.

Given a Hamiltonian system (1.4) with state space (M, ω) and output space Y the augmented Hamiltonian system is defined as the *product* Hamiltonian system of (1.4) and (6.1):

$$\begin{aligned} \dot{x} &= X_H(x) - \sum_{j=1}^{m} u_j X_{C_j}(x) \\ \dot{\xi}_i &= -\nu_i & i &= 1,\dots,l \\ \dot{\xi}_i &= \nu_i & i &= l+1,\dots,2l \\ y_j &= C_j(x) & j &= 1,\dots,m \\ \eta_i &= \xi_i & i &= 1,\dots,2l \end{aligned} \tag{6.2}$$

with state space $M^a = M \times M^{au}$, output space $Y^a = Y \times Y^{au}$, and input-output space $T^*Y^a = T^*Y \times Y^{au} \times U^{au}$. Denoting $x^a = (x, \xi)$, $y^a = (y, \eta)$, $u^a = (u, \nu)$, the augmented Hamiltonian system has an internal energy $H^a(x, \xi) = H(x)$ and output maps

$$\begin{aligned} C_j^a(x,\xi) &= C_j(x) & j &= 1,\dots,m \\ C_j^a(x,\xi) &= \xi_i & j &= m+1,\dots,m+2l \end{aligned}$$

Dynamic Hamiltonian feedback for (1.4) is *static* Hamiltonian feedback for the augmented system (6.2), i.e. there exist functions $P^a(y,\eta)$, $R_1^a(y,\eta)$, $\dots, R_{m+2l}^a(y,\eta)$, with $(R_1^a,\dots,R_{m+2l}^a)$: $Y \times \mathbb{R}^{2l} \to Y \times \mathbb{R}^{2l}$ a diffeomorphism, such that the augmented system is transformed into

$$\dot{x}^a = X_{H^a - P^a \circ C^a}(x^a) - \sum_{j=1}^{m} u_j X_{R_j^a \circ C^a}(x^a) - \sum_{j=m+1}^{m+2l} \nu_j X_{R_j^a \circ C^a}(x^a) \tag{6.3}$$

Remark. Of course, the auxiliary system (6.1) corresponds (modulo minus signs) to $2l$ independent integrators.

The main theorem of this section reads as follows

Theorem 6.2. *Let* (1.4) *be a Hamiltonian system on* (M, ω), *and let* $\mathcal{F}$ *be a coisotropic function group on* M, *satisfying Condition* A. *Thus* $\mathcal{F} = \text{span}\{F_1,\dots,F_k\}$, F_i *independent*, $k \geqslant n$. *Suppose* $\mathcal{F}$ *satisfies*

1) $\mathcal{F}$ *is locally controlled invariant, so there exist functions* $F_1^e,\dots,F_k^e$: $T^*Y \times \mathbb{R}^k \to \mathbb{R}$ *such that*

$$\left\{H - \sum_{j=1}^{m} u_j C_j, F_i\right\}(x) = F_i^e(C(x), u, F(x)), \qquad i = 1,\dots,k$$

(*again we denote* $F = (F_1,\dots,F_k)$: $M \to \mathbb{R}^k$)

2) *The function space* $\mathcal{F}^e := \text{span}\{F_1^e,\dots,F_k^e\}$ *on* $T^*Y \times \mathbb{R}^k$ "*projects*" *to a function space on* T^*Y, *i.e. there are functions* $G_1^e,\dots,G_k^e$: $T^*Y \to \mathbb{R}$ *such that*

$$\operatorname*{span}_{i=1,\dots,k}\left\{\sum_{j=1}^{m} \frac{\partial F_i^e}{\partial y_j} dy_j + \frac{\partial F_i^e}{\partial u_j} du_j\right\} = \operatorname*{span}_{i=1,\dots,k}\{dG_i^e\} \tag{6.4}$$

in every point (y, u, z) *of* $T^*Y \times \mathbb{R}^k$

3) *The function spaces* $\mathscr{C}$, $\mathscr{F} + \mathscr{C}$ *satisfy Condition* A

4) *The function spaces* $\mathscr{F}^e$ *and* $\mathscr{G}^e := \mathrm{span}\{G_1^e, \dots, G_k^e\}$ *satisfy Condition* A.

5) *The codistribution* $\underset{i=1,\dots,k}{\mathrm{span}} \left\{ \frac{\partial F_i^e}{\partial u_j} du_j \right\}$ *on* $T^*Y \times \mathbb{R}^k$ *has constant dimension.*

Then, there exists locally a Lagrangian function group $\mathscr{F}_{lift}$ *on the augmented space* $M \times \mathbb{R}^{2l}$, *with* $l = k - n$, *such that if the projection* $M \times \mathbb{R}^{2l} \to M$ *is denoted by* π, *then*

$$\pi_* D_{\mathscr{F}_{\text{lift}}} = D_{\mathscr{F}}, D_{\mathscr{F}_{\text{lift}}} \cap (TM \times 0) = D_{\mathscr{F}^\perp}$$
$$\dim d\mathscr{F}_{\text{lift}} = \dim d\mathscr{F} \tag{6.5}$$

($TM \times 0$ *is the distribution on* $M \times \mathbb{R}^{2l}$ *with zero-components in the* $\mathbb{R}^{2l}$*-direction*). $\mathscr{F}_{\text{lift}}$ *is called the* lift *of* $\mathscr{F}$. *Moreover there exists locally a dynamic Hamiltonian feedback* $(P^a(y, \xi), R_1^a(y, \xi), \dots, R_{m+2l}^a(y, \xi))$, *with* $\xi \in \mathbb{R}^{2l}$ *such that*

$$\{H^a - P^a \circ C^a, \mathscr{F}_{\text{lift}}\} \subset \mathscr{F}_{\text{lift}}$$
$$\{R_j^a \circ C, \mathscr{F}_{\text{lift}}\} \subset \mathscr{F}_{\text{lift}} \qquad j = 1, \dots, m + 2l \tag{6.6}$$

Conversely, if (6.6) *holds, then conditions* 1 *and* 2 *are satisfied.*

In order to construct the function group $\mathscr{F}_{\text{lift}}$ we need the following lemma, which follows from Theorem 1.1. but also can be proved directly.

Lemma 6.3. *Let* $\mathscr{F}$ *be a coisotropic function group on* (M, ω), *with* $\dim d\mathscr{F}(x) = k \geqslant n = \frac{1}{2}\dim M, \forall x$. *Then there exist local canonical coordinates* $(q_1, \dots, q_n, p_1, \dots, p_n)$ *for M such that locally* $\mathscr{F} = \mathrm{span}\{q_1, \dots, q_{k-n}, p_1, \dots, p_n\}$.

Proof. Consider the function group $\mathscr{F}^\perp$. Take $2n - k$ independent functions $p_{k-n+1}, \dots, p_n$ such that $\mathscr{F}^\perp = \mathrm{span}\{p_{k-n+1}, \dots, p_n\}$. Since $\mathscr{F}^\perp \subset \mathscr{F}$ we have $\{p_i, p_j\} = 0$ for every $i, j = k - n + 1, \dots, n$.

Hence $p_{k-n+1}, \dots, p_n$ are a set of partial canonical coordinates. By Darboux's theorem (see the proof in Weinstein (1983)) we can extend the set $(p_{k-n+1}, \dots, p_n)$ to a set $(q, p) = (q_1, \dots, q_n, p_1, \dots, p_n)$ of canonical coordinates. Now consider a function $F(q, p)$ contained in $\mathscr{F}$. Since $\mathscr{F}^\perp = \mathrm{span}\{p_{k-n+1}, \dots, p_n\}$ we have $0 = -\{F, p_i\} = \frac{\partial F}{\partial q_i}$, $i = k - n + 1, \dots, n$. Hence F is only a function of $(q_1, \dots, q_{k-n}, p_1, \dots, p_n)$. Since $\dim d\mathscr{F} = k$ it follows that $\mathscr{F} = \mathrm{span}\{q_1, \dots, q_{k-n}, p_1, \dots, p_n\}$. □

Proof of Theorem 6.2. Let $\mathscr{F}$ be a coisotropic function group for which conditions 1 up till 5 are satisfied. Choose local canonical coordinates for $\mathscr{F}$ as in Lemma 6.3., i.e. $\mathscr{F} = \mathrm{span}\{q_1, \dots, q_l, p_1, \dots, p_n\}$, with $l = k - n$. The state space of the auxiliary Hamiltonian system will be $\mathbb{R}^{2l}$ with its usual symplectic form. Define locally the lifted function group $\mathscr{F}_{\text{lift}}$ on $M \times \mathbb{R}^{2l}$ as

$$\mathscr{F}_{\text{lift}} = \mathrm{span}\{q_1 + \xi_{l+1}, \dots, q_l + \xi_{2l}, p_1 + \xi_1, \dots, p_l + \xi_l, p_{l+1}, \dots, p_n\} \tag{6.7}$$

with $\xi = (\xi_1, \ldots, \xi_{2l}) \in \mathbb{R}^{2l}$. Since

$$\{q_i + \xi_{l+i}, p_i + \xi_i\}_{M \times \mathbb{R}^{2l}} = \{q_i, p_i\}_M + \{\xi_{l+i}, \xi_i\}_{\mathbb{R}^{2l}} = -1 + 1 = 0,$$

$i = 1, \ldots, l$, it follows that for any $F_1, F_2 \in \mathscr{F}_{\text{lift}}$

$$\{F_1, F_2\}_{M \times \mathbb{R}^{2l}} = 0 \tag{6.8}$$

Since $\dim d\mathscr{F}_{\text{lift}} = n + l = \frac{1}{2}\dim(M \times \mathbb{R}^{2l})$ this implies that $\mathscr{F}_{\text{lift}}$ is a *Lagrangian* function group on $M \times \mathbb{R}^{2l}$ (Of course $\mathscr{F}_{\text{lift}}$ is only defined in a neighborhood of every point in M). Since $\mathscr{F}$ is locally controlled invariant it follows that

$$\begin{aligned} \{H^a, \mathscr{F}_{\text{lift}}\} &\subset \mathscr{F}_{\text{lift}} + \mathscr{C}^a \\ \{\mathscr{C}^a, \mathscr{F}_{\text{lift}}\} &\subset \mathscr{F}_{\text{lift}} + \mathscr{C}^a \end{aligned} \tag{6.9}$$

Hence, $\mathscr{F}_{\text{lift}}$ is locally controlled invariant for the augmented Hamiltonian system. Furthermore we have

$$\begin{aligned} &\left\{H^a - \sum_{j=1}^m u_j C_j^a - \sum_{k=1}^{2l} v_k C_k^a, q_i + \xi_{l+i}\right\} \\ &= \left\{H - \sum_{j=1}^m u_j C_j - \sum_{k=1}^{2l} v_k \xi_k, q_i + \xi_{l+i}\right\} \\ &= \left\{H - \sum_{j=1}^m u_j C_j, q_i\right\} - \sum_{k=1}^{2l} v_k\{\xi_k, \xi_{l+i}\} \\ &= \left\{H - \sum_{j=1}^m u_j C_j, q_i\right\} + v_i \qquad i = 1, \ldots, l \end{aligned} \tag{6.10}$$

and

$$\begin{aligned} \left\{H^a - \sum_{j=1}^m u_j C_j^a - \sum_{k=1}^{2l} v_k C_k^a, p_i + \xi_i\right\} &= \left\{H - \sum_{j=1}^m u_j C_j, p_i\right\} - \sum_{k=1}^{2l} v_k\{\xi_k, \xi_i\} \\ &= \left\{H - \sum_{j=1}^m u_j C_j, p_i\right\} - v_{l+i} \\ &\qquad i = 1, \ldots, l \end{aligned} \tag{6.11}$$

and

$$\left\{H^a - \sum_{j=1}^m u_j C_j^a - \sum_{k=1}^{2l} v_k C_k^a, p_{l+i}\right\} = \left\{H - \sum_{j=1}^m u_j C_j, p_{l+i}\right\} \qquad i = 1, \ldots, n-l \tag{6.12}$$

The functions $\{H - \sum_{j=1}^{m} u_j C_j, q_i\}$, $i = 1,\dots,l$ and $\{H - \sum_{j=1}^{m} u_j C_j, p_i\}$, $i = 1,\dots,n$ are all contained in $\mathscr{F}^e$. It is now easily seen that conditions 2, 3, 4, 5 imply conditions 2, 3, 4, 5 of Theorem 5.2. for the *augmented* Hamiltonian system. Hence by Theorem 5.2. there exist (locally on $Y \times \mathbb{R}^{2l}$) functions $P^a(y,\eta), R_1^a(y,\eta),\dots, R_{m+2l}^a(y,\eta)$, where the Jacobian of $(R_1^a,\dots,R_{m+2l}^a)$ has full rank, such that (6.6) is satisfied. The proof of the converse statement follows the same lines as the proof of the converse statement in Theorem 5.2. □

Example 6.4. Consider the Hamiltonian system from Example 2.5 with $H(q,p) = \frac{1}{2}e^{q_2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}q_1^2 + \frac{1}{2}q_2^2, C(q,p) = q_1$. Consider the velocity $\dot{q}_1 = \dfrac{\partial H}{\partial p_1} = e^{q_2}p_1$. Define $\mathscr{F} := (\text{span}\{e^{q_2}p_1\})^\perp = \text{span}\{q_2, p_2 + q_1 p_1, p_1\}$. Since $\text{span}\{e^{q_2}p_1\} \subset \mathscr{F}$, $\mathscr{F}$ is coisotropic. Furthermore since $\mathscr{F} + \text{span}\{q_1\} = \text{span}\{q_1, q_2, p_1, p_2\}$, $\mathscr{F}$ is l.c.i. Let us therefore try to construct a dynamic Hamiltonian feedback which makes $\mathscr{F}$ invariant. First we have to choose canonical coordinates (Q_1, Q_2, P_1, P_2) such that $\mathscr{F} = \text{span}\{Q_1, P_1, P_2\}$. Since the Poisson structure (1.17) associated with $\mathscr{F}$ is

$$(w_{ij}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -p_1 \\ 0 & p_1 & 0 \end{pmatrix}$$

we take $Q_1 = q_2$, $P_1 = p_2 + q_1 p_1$, $P_2 = \log p_1 + q_2$, $Q_2 = q_1 p_1$ (assume $p_1 > 0$). The augmented state space will be Q_1, Q_2, P_1, P_2, ξ_1, ξ_2 and $\mathscr{F}_{\text{lift}} = \text{span}\{Q_1 + \xi_2, P_1 + \xi_1, P_2\}$. However although $\mathscr{F}_{\text{lift}}$ is l.c.i. and Lagrangian, it *cannot* be made invariant by Hamiltonian feedback. This is because the integrability condition 2 is not satisfied (because of the $q_1 p_1$-term in $\mathscr{F}$, while $C = q_1$)

7. Application to Disturbance Decoupling

Probably the easiest application of controlled invariance is the problem of disturbance decoupling. Consider a Hamiltonian system (1.4) with additional disturbances

$$\begin{aligned} \dot{x} &= X_H(x) - \sum_{j=1}^{m} u_j X_{C_j}(x) - \sum_{k=1}^{l} d_k E_k(x) \\ y_j &= C_j(x) \qquad j = 1,\dots,m \end{aligned} \tag{7.1}$$

The disturbances $d_1,\dots,d_l$ enter the equations via the (known) vectorfields $E_1,\dots,E_l$. Suppose one has furthermore a set of functions of the state

$$z_j = G_j(x) \qquad j = 1,\dots,r \tag{7.2}$$

where $z_1,\dots,z_r$ are the so-called *to-be-regulated* variables.

One now wishes to apply Hamiltonian feedback to (7.1) such that *after* feedback the disturbances do not influence the to-be-regulated variables. This can be done, using Theorems 5.2 and 6.2, if there exists a locally controlled invariant function group $\mathscr{F}$, contained in the function group $\mathscr{G}^{\perp} = (\text{span}\{G_1,\ldots,G_r\})^{\perp}$, which is Lagrangian or coisotropic, and such that the distribution span$\{E_1,\ldots,E_l\}$ is contained in $D_{\mathscr{F}^{\perp}}$ (Recall the construction of $\mathscr{F}_{\text{lift}}$ in (6.5)!). In case $\mathscr{F}$ is Lagrangian, static Hamiltonian feedback is sufficient; otherwise dynamic Hamiltonian feedback is needed. Notice that a necessary condition for the existence of such an $\mathscr{F}$ is that $\mathscr{G}^{\perp}$ itself has to be coisotropic. Indeed let $\mathscr{F} \subset \mathscr{G}^{\perp}$ with $\mathscr{F}^{\perp} \subset \mathscr{F}$, then $\mathscr{G} \subset (\mathscr{G}^{\perp})^{\perp} \subset \mathscr{F}^{\perp} \subset \mathscr{F} \subset \mathscr{G}^{\perp}$.

A computational procedure for solving the disturbance decoupling problem by Hamiltonian feedback may be the following:

1. Compute the maximal l.c.i. function group $\mathscr{F}^*$ contained in $\mathscr{G}^{\perp}$, using the $\mathscr{F}^*$-algorithm (2.4).
2. Check if span$\{E_1,\ldots,E_l\}$ is contained in $D_{\mathscr{F}^{*\perp}}$.
3. Check if $\mathscr{F}^*$ is Lagrangian or coisotropic.
4. Check if $\mathscr{F}^*$ satisfies conditions 2,3,4,5 of Theorem 5.2 or 6.2.

Remark. It maybe easier to use the $\mathscr{S}^*$-algorithm (4.3) to compute the minimal conditioned invariant function group $\mathscr{S}^*$ containing $\mathscr{G}$. Then one may check if $(\mathscr{S}^*)^{\perp}$ is locally controlled invariant (see Proposition 4.4.)

A particularly nice situation arises if the disturbance vectorfields E_k are given as the Hamiltonian vectorfields of the functions G_j, i.e.

$$E_k(x) = X_{G_k}(x),\ k = 1,\ldots,r \tag{7.3}$$

In this case if a function group $\mathscr{F}$ satisfies $\mathscr{F} \subset \mathscr{G}^{\perp}$, then automatically $\mathscr{G} \subset (\mathscr{G}^{\perp})^{\perp} \subset \mathscr{F}^{\perp}$, and so span$\{E_1,\ldots,E_r\} = D_{\mathscr{G}} \subset D_{\mathscr{F}}^{\perp}$. Hence if $\mathscr{F}$ is coisotropic we are done (apart from conditions 2,3,4,5 of Theorem 6.2). In the *linear* case one can even prove that in this situation disturbance decoupling by *general* dynamic feedback is possible if and only if there exists a coisotropic $\mathscr{F}$ with $\mathscr{F} \subset \mathscr{G}^{\perp}$ and so if and only if disturbance decoupling by dynamic *Hamiltonian* feedback is possible.

8. Conclusion

The notion of a locally controlled invariant distribution is specialized to Hamiltonian systems by introducing the concept of a locally controlled invariant function group. It is shown that l.c.i. Lagrangian function groups can be made invariant by Hamiltonian feedback provided an extra integrability condition necessary for the existence of output feedback, is satisfied. This also appears to be the case for symplectic functions groups. Other classes of function groups $\mathscr{F}$ with $\mathscr{F}$ and $\mathscr{F}^{\perp}$ l.c.i. in general *cannot* be made invariant by Hamiltonian feedback. However one expects that a "generalized type" of Hamiltonian feedback might work. This is strongly related to the nature of the mapping $F_i \to F_i^e$ in (5.7), which in this paper is only investigated in the (easy) Lagrangian case.

Furthermore, it is shown that coisotropic l.c.i. function groups which satisfy the integrability condition can be made invariant by *dynamic* Hamiltonian feedback.

The developed notion of controlled invariant function groups is only a first, although basic, step in a geometric theory of Hamiltonian control systems. For instance it seems fruitful to combine this notion with the stabilization procedure for Hamiltonian systems as proposed in van der Schaft (1984b).

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