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Inertia effects in rheometrical flow systems Part 1: The orthogonal rheometer

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With 6 figures

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Notations

a – distance between axis of upper and lower plate
 $c^* = c' + ic''$ – complex velocity of propagation of shear waves
 h – distance between the plates
 $i = \sqrt{-1}$
 $k^* = k' - ik''$ – complex wave number
 $k = (\rho\omega/2\eta)^{1/2}$
 k_e – wave number in a Hookean material
 $n = 0, 1, 2, \dots$
 t – time
 x, y, z – Cartesian coordinates
 A – area of a plate
 A_+ – displacement amplitude of shear wave in $+z$ direction
 A_- – displacement amplitude of shear wave in $-z$ direction
 E – kinetic energy
 F – force on lower plate
 $G^* = G' + iG''$ – complex shear modulus
 R – radius of the plates
 ΔW – energy dissipated in the sample during one cycle
 δ – loss angle
 Γ – propagation factor
 λ – wave length
 ω – angular velocity
 η – viscosity
 ρ – density
 τ – shear stress
 ξ – particle displacement

1. Introduction

The precursor of the present orthogonal rheometer was introduced by *Kimball and Lovell* (1) as early as 1927. The sample in the form of a cylindrical bar was at one end rotated around its axis by a motor. At the other end the bar, which was in a horizontal position, was loaded with a dead weight connected to the bar with the aid of a ball bearing. As a result of internal friction in the bar the latter end face is twisted with respect to the driven end face. In a semi-empirical way *Kimball and Lovell* (1) came to the right conclusion that the twist

angle was equal to the loss angle $\delta = \arctan G''/G'$ of the rod material. In this apparatus the bar was subjected to bending deformations.

An important improvement was introduced by *Gent* (2). Instead of bending deformations of a bar shear deformations of a pair of disks were used. Moreover *Gent* (2) measured the force necessary to prevent the twisting of the disks. In the present design of commercially available apparatus the same principle is applied, the ball bearing being replaced by an air bearing. According to *Gent* (2) and *Maxwell and Chartoff* (3) the radial and tangential component of the force on this bearing are related to respectively the real and imaginary part of the complex shear modulus of the sample. Theoretical investigations concerning these relations were performed by *Blyler and Kurtz* (4), *Bird and Harris* (5), *Huilgol* (6), *Gordon and Schowalter* (7) and *Yamamoto* (8). The force relations, first given by *Gent* (2), were justified by these authors for materials obeying several constitutive equations (Oldroyd-Maxwell model, Bird-Carreau model, simple fluid etc.) in the limit of small deformations. Moreover *Gordon and Schowalter* (7) showed that for linear viscoelastic materials restriction to small deformations is not necessary. In these studies inertia effects and end effects were neglected. Inertia effects were taken into account by *Abbott and Walters* (9) and by *Goldstein* (10). *Abbott and Walters* (9) not only examined the influence of inertia effects on the plate forces, but also on the locus of the points about which planes with $z = \text{constant}$ ($z = \text{direction of plate axis}$) rotate rigidly. As to the latter problem they were able to find an exact solution for a Newtonian liquid. In the present paper an alternative procedure will be given which is not only marked by its simplicity but

is also applicable to linear viscoelastic materials in general. The plate forces, too, are readily found by this procedure.

2. Flow field in the orthogonal rheometer

2.1. Flow field neglecting inertia effects

In fig. 1a the geometry of the orthogonal rheometer is sketched. The sample in the form of a disk with thickness h is clamped between two circular plates, the axis of these plates being in line. Thereafter the upper plate is shifted with respect to the lower plate over the distance a in the $+y$ direction of the Eulerian coordinate system shown, where it is assumed that the sample sticks to the plates, see fig. 1b. Subsequently the upper plate is rotated clockwise with constant angular velocity ω . The lower plate, the axis of which is supposed to have a frictionless bearing, rotates with the same angular velocity.

Now consider material lines parallel with the plate axis before the shifting of the upper plate. In a rotating coordinate system fixed to a particle on the lower plate the particular material line will describe a cone, see fig. 1c. It is to be noticed that the corresponding particle on the upper plate

in this coordinate system turns a circle counter clockwise with angular velocity ω . In a rotating coordinate system the deformation of the sample therefore can be described as a superposition of two harmonic simple shear deformations polarized perpendicularly to each other, having the same amplitude a , but a phase shift of $\pi/2$ radians. The above applies both to linear and non linear viscoelastic materials (7). There is one special material line in the set chosen, i.e. the line joining the centres of the plates. It is evident that the position of this line is fixed with respect to the Eulerian coordinate system. Also it is readily seen that the horizontal distance r from an arbitrary point in the sample to this line does not change with time ($r = (x_0^2 + y_0^2)^{1/2}$, where x_0 and y_0 are the coordinates in the undeformed state). In the Eulerian coordinate system each plane $z = z_0$ therefore rotates rigidly with angular velocity ω about the point $(0, az_0/h)$.

As in the rotating coordinate system this line describes a cone, see fig. 1c, it is clear that the position of a point of this line with respect to the z -axis is determined by the amplitude of one of the composing harmonic shear deformations at the same z -coordinate.

2.2. Flow field including inertia effects

In accordance with earlier investigations we assume that the fluid is contained between infinite plates, so edge effect will be ignored. Further we assume that the fluid is incompressible. As it is also assumed that the sample sticks to the plates and maintains its cohesion, this means that centripetal forces and gravitational forces do not effect the flow field but merely appear in the pressure term. So we can safely work in a rotating coordinate system. The influence of the inertia is confined to the velocity of the shear waves in the sample, which now will be finite. The problem of finding the locus of the points around which rigid rotation takes place therefore reduces to finding the amplitude of a harmonic shear wave in the sample taking wave propagation into account.

3. Calculation of flow fields including inertia effects

3.1. Flow field of a Newtonian liquid

To ensure that the flow field will be symmetric with respect to the plane $z = h/2$ we start from the situation sketched in fig. 2.

Fig.1. Geometry of the orthogonal rheometer.

- a) Orthogonal rheometer with the sample in the initial (undeformed) state
- b) Orthogonal rheometer during measurements
- c) Motion between fluid particles in a rotating coordinate system fixed to a particle on the bottom plate

Fig. 2. Description of the motion of the fluid in a rotating coordinate system fixed to a particle on the bottom. $\xi(h, t) = a/2 \exp i\omega t$; $\xi(0, t) = -a/2 \exp i\omega t$

For the particle displacement $\xi(z, t)$ we can write:

$$\begin{aligned} \xi(z, t) &= \xi^*(z) \exp i\omega t \\ &= (A_+ \exp -ik^*z + A_- \exp \\ &\quad + ik^*z) \exp i\omega t, \end{aligned} \quad [1]$$

where

$$k^* = k' - ik'' = \frac{\omega}{c^*} = \omega \left(\frac{\rho}{G^*} \right)^{1/2}$$

is the complex wave vector.

$c^* = c' + ic''$ = complex velocity of shear waves in the liquid

$G^* = G' + iG''$ = complex shear modulus of the liquid

ρ = density of the liquid

$ik^* \equiv \Gamma$ is known as the (complex) propagation factor of an attenuated propagating wave.

The boundary conditions are:

$$\begin{aligned} \xi^*(h) &= a/2, \\ \xi^*(0) &= -a/2. \end{aligned} \quad [2]$$

From [1] and [2] it is found:

$$\begin{aligned} \xi^*(z) &= \Xi(z) + i\Pi(z) \\ &= \frac{a}{2} \frac{\sinh ik^*z - \sinh ik^*(h-z)}{\sinh ik^*h}. \end{aligned} \quad [3]$$

Eq. [3] is equivalent with eq. [44] of Abbott and Walters (9). For a Newtonian liquid we have:

$$ik^* = \left(\frac{\rho\omega}{2\eta} \right)^{1/2} (1 + i). \quad [4]$$

So

$$k' = k'' \equiv k = \left(\frac{\rho\omega}{2\eta} \right)^{1/2}, \quad [5]$$

where η = viscosity of the liquid.

From [3], [4] and [5] it follows:

$$\begin{aligned} \Xi(z) + i\Pi(z) &= \frac{a}{2} \frac{\sinh k(1+i)z - \sinh k(1+i)(h-z)}{\sinh k(1+i)h}. \end{aligned} \quad [6]$$

Equating the real and imaginary parts of both sides of [6] leads to:

$$\begin{aligned} \Xi(z) &= \frac{a}{2} [\sinh kz \cos kh + \sinh k(z-h) \\ &\quad \cdot \cos k(z-h)] \sinh kh \cos kh \\ &\quad + [\cosh kz \sin kz + \cosh k(z-h) \\ &\quad \cdot \sin k(z-h)] \cosh kh \sin kh / \\ &\quad [\sinh^2 kh \cos^2 kh + \cosh^2 kh \sin^2 kh]; \end{aligned} \quad [7]$$

$$\begin{aligned} \Pi(z) &= \frac{a}{2} [\cosh kz \sin kz + \cosh k(z-h) \\ &\quad \cdot \sin k(z-h)] \sinh kh \cos kh \\ &\quad - [\sinh kz \cos kz + \sinh k(z-h) \\ &\quad \cdot \cos k(z-h)] \cosh kh \sin kh / \\ &\quad [\sinh^2 kh \cos^2 kh + \cosh^2 kh \sin^2 kh]. \end{aligned} \quad [8]$$

Eqs. [7] and [8] are, apart from the factor ω , identical with the eqs. [21] and [22] of Abbott and Walters (9). They are deduced here in a very simple and direct way. Eqs. [7] and [8] determine the locus of the rotation centres, i.e. any plane $z = \text{constant}$ rotates as if rigid with the angular velocity ω around the point given by [7] and [8].

Eqs. [7] and [8] can be written in a more compact form. From [3] it follows:

$$\begin{aligned} \xi^*(z) &= \frac{a}{2} \frac{2 \cosh ik^*h/2 \sinh ik^*(z-h/2)}{2 \sinh ik^*h/2 \cosh ik^*h/2} \\ &= \frac{a}{2} \frac{\sinh ik^*(z-h/2)}{\sinh ik^*h/2}. \end{aligned} \quad [9]$$

So we find:

$$\begin{aligned} \Xi(z) &= \frac{a}{2} [\sinh k(z-h/2) \sinh kh/2 \\ &\quad \cdot \cos k(z-h/2) \cos kh/2 \\ &\quad + \cosh k(z-h/2) \cosh kh/2 \\ &\quad \cdot \sin k(z-h/2) \sin kh/2] / \\ &\quad [\sinh^2 kh/2 + \sin^2 kh/2]; \end{aligned} \quad [10]$$

$$\begin{aligned} \Psi(z) = & \frac{a}{2} [\sinh kh/2 \cosh k(z - h/2) \\ & \cdot \sin k(z - h/2) \cos kh/2 \\ & - \sinh k(z - h/2) \cosh kh/2 \\ & \cdot \sin kh/2 \cos k(z - h/2)] / \\ & [\sinh^2 kh/2 + \sin^2 kh/2]. \end{aligned} \quad [11]$$

3.2. Flow field of a general linear viscoelastic fluid

For an arbitrary linear viscoelastic liquid we have

$$\begin{aligned} k^* &= \omega \left(\frac{\rho}{G^*} \right)^{1/2} = \omega \left(\frac{\rho}{|G^*|} \right)^{1/2} \exp - i\delta/2 \\ &= \omega \left(\frac{\rho \cos \delta}{G'} \right)^{1/2} (\cos \delta/2 - i \sin \delta/2) \\ &= k'(1 - i \tan \delta/2), \end{aligned} \quad [12]$$

where

$$\begin{aligned} k' &= \frac{\omega c'}{c'^2 + c''^2} = \frac{\omega}{c} = \frac{2\pi}{\lambda} \\ &= \omega \cos \delta/2 \left(\frac{\rho \cos \delta}{G'} \right)^{1/2}, \end{aligned} \quad [13]$$

with c and λ the phase velocity and the wave length respectively (11).

From [9] and [12] it follows

$$\begin{aligned} \Xi(z) = & \frac{a}{2} [\sinh \{k'(z - h/2) \tan \delta/2\} \\ & \cdot \sinh \{k'h/2 \tan \delta/2\} \\ & \cdot \cos k'(z - h/2) \cos k'h/2 \\ & + \cosh \{k'(z - h/2) \tan \delta/2\} \\ & \cdot \cosh \{k'h/2 \tan \delta/2\} \\ & \cdot \sin k'(z - h/2) \sin k'h/2] / \\ & [\sinh^2 \{k'h/2 \tan \delta/2\} + \sin^2 k'h/2]; \end{aligned} \quad [14]$$

$$\begin{aligned} \Psi(z) = & \frac{a}{2} [\sinh \{k'h/2 \tan \delta/2\} \\ & \cdot \cosh \{k'(z - h/2) \tan \delta/2\} \\ & \cdot \sin k'(z - h/2) \cos k'h/2 \\ & - \sinh \{k'(z - h/2) \tan \delta/2\} \\ & \cdot \cosh \{k'h/2 \tan \delta/2\} \\ & \cdot \sin k'h/2 \cos k'(z - h/2)] / \\ & [\sinh^2 \{k'h/2 \tan \delta/2\} + \sin^2 k'h/2]. \end{aligned} \quad [15]$$

In fig. 3 the projections of the centres of rotation on the $x - y$ plane and the $z - y$ plane are shown for various values of $k'h$ and $\tan \delta/2$.

3.3. Flow field of a linear elastic material; the correspondence principle

For a linear elastic material it follows from [9] with $k^* = \omega(\rho'/G)^{1/2} \equiv k_e$:

$$\Xi(z) = \frac{a}{2} \frac{\sin k_e(z - h/2)}{\sin k_e h/2}. \quad [16]$$

On the other hand [16] can be deduced directly from [1] and [2] by putting $k^* = k_e$. The formulae for an arbitrary linear viscoelastic material, including the Newtonian liquid are then found by substituting for k_e the appropriate value of k^* , i.e. substituting G^* for G . This is a consequence of the well-known correspondence principle.

3.4. Flow field of a non-linear viscoelastic material

The flow field of a non-linear viscoelastic material in the orthogonal rheometer was studied by Goldstein (10) using a Bird-Carreau model. He showed that when inertia effects are neglected the kinematics is the same as for a linear viscoelastic material. Nonlinearity does effect the stresses, however.

To calculate the flow field including inertia effects in the way presented before, wave propagation in non-linear viscoelastic materials has to be considered. This is beyond the scope of this paper, however, and will be the subject of a further study.

4. Forces on the plates

As pointed out earlier in this paper the deformation of the sample can be described in a rotating coordinate system as a superposition of two harmonic shear deformations with the same amplitude but polarized perpendicularly to each other and showing a phase shift of $\pi/2$ radians. The resulting deformation is a harmonic shear deformation from which the polarization direction turns counter-clockwise with the angular velocity ω . In case of a linear viscoelastic material this results in a shear stress, the direction of which also turns counter-clockwise with angular velocity ω but advances the shear strain by the angle δ , where $\tan \delta = G''/G'$ if

Fig. 3. Projections of the locus of the centres of rotation on the x - y plane and the z - y plane, respectively, for $k'h = 0, 1, 2$ and 3

Fig. 3a: $\tan \delta/2 = 1$; fig. 3b: $\tan \delta/2 = 0.25$; fig. 3c: $\tan \delta/2 = 0.05$; fig. 3d: $\tan \delta/2 = 0.01$

Mind the change of the scale factor of x/a in the lower parts of figs. 3c and 3d. See note on page 449

inertia effects are neglected. As the rotating coordinate system itself turns clockwise with the angular velocity ω , the shear stress is fixed in the Eulerian coordinate system, see fig. 4. For the determination of the forces on the plates it suffices therefore to calculate the (complex)

amplitude of the shear stress due to one of the composing shear deformations. For the tension on the lower plate we can write:

$$\tau^* = G^* \frac{d\xi^*(z)}{dz} \Big|_{z=0} \quad [17]$$

Fig. 4. Shear stress in the sample at the upper plate in an Eulerian coordinate system, inertia effects being ignored. $|G^*| = G'/\cos \delta$

Substitution of [9] into [17] leads to

$$\tau^* = \frac{a}{2} \frac{G^* i k^*}{\tanh i k^* h/2}. \quad [18]$$

By expanding the hyperbolic function we find:

$$\tau^* = \frac{a G^*}{h} \left(1 - \frac{k^{*2} h^2}{12} - \frac{k^{*4} h^4}{720} \dots \right). \quad [19]$$

The force on the lower plate therefore reads:

$$F_y^* = F_y - i F_x = \frac{a \pi R^2 G^*}{h} \cdot \left(1 - \frac{k^{*2} h^2}{12} - \frac{k^{*4} h^4}{720} \dots \right). \quad [20]^*$$

Substituting $G^* = i \omega \eta^*$ into [20] gives:

$$-F_x - i F_y = \frac{a \pi R^2 \omega \eta^*}{h} \cdot \left(1 - \frac{k^{*2} h^2}{12} - \frac{k^{*4} h^4}{720} \dots \right). \quad [21]$$

Comparing [21] with eq. [50] of Abbott and Walters (9), which reads in our notation

$$F_x - i F_y = \frac{a \pi R^2 \omega \eta^*}{h} \cdot \left(1 + \frac{k^{*2} h^2}{12} - \frac{k^{*4} h^4}{720} \dots \right). \quad [22]$$

It is evident that these equations are not identical. The sign-difference of F_x in both equations is trivial as it is caused by a different choice of the direction of rotation, but this is not so for the difference between the forms in brackets. The same discrepancy appears in comparing our eq. [18] with eq. [49] of these authors, the latter of which can be written in our notation as:

$$\tau^* = \frac{a}{2} \frac{G^* k^*}{\tanh k^* h/2}. \quad [23]$$

Now for an elastic material k^* is real and then [18] can be written as:

$$\tau = \frac{a}{2} \frac{G k_e}{\tan k_e h/2}, \quad [24]$$

from which it is seen that resonance occurs for

$$k_e h/2 = \frac{\pi h}{\lambda} = (2n + 1) \frac{\pi}{2}$$

or

$$h = (2n + 1) \frac{\lambda}{2}$$

as it should be. On the other hand for k^* is real [23] does not show resonance features. It is evident therefore that in the paper of Abbott and Walters (9) α (which is identical with our k^*) is in error by a factor i . In eq. [44] of the latter authors this error has no effect.

For a linear elastic material it follows from [17] and [16]:

$$\begin{aligned} \tau &= \frac{a G k_e}{2 \tan k_e h/2} \\ &= \frac{a G}{h} \left(1 - \frac{k_e^2 h^2}{12} - \frac{k_e^4 h^4}{720} \dots \right). \end{aligned} \quad [25]$$

Eq. [19] is found from [25] by substituting G^* for G . This is another consequence of the correspondence principle.

For $|k^* h| \ll 1$ it follows:

$$\begin{aligned} F_y &= \frac{a \pi R^2}{h} G', \\ F_x &= \frac{a \pi R^2}{h} G''. \end{aligned} \quad [26]$$

5. On the magnitude and phase of τ^*

In a rotating coordinate system one can write for the tension on the lower plate due to one of the composing harmonic shear deformations:

$$\begin{aligned} \tau^* &= \frac{a}{2} G^* i k^* \coth i k^* h/2 \exp i \omega t \\ &= G^* i k^* \coth i k^* h/2 \xi, \end{aligned} \quad [27]$$

where $\xi = \frac{a}{2} \exp i \omega t$ stands for the considered harmonic shear deformation.

*) It should be noted that eq. [20] applies to the situation sketched in fig. 2, i.e. in effect to an experiment with a distance between the plate axis of $a/2$ and a sample thickness of $h/2$. For the same reason in fig. 3 only the parts of the curves between $y/a = 0$ and $y/a = 0.5$ are significant for a real experiment.

Further we have:

$$G^* = G'(1 + i \tan \delta) = \frac{G'}{\cos \delta} \exp i \delta \quad [28]$$

$$k^* = k' \left(1 - i \tan \frac{\delta}{2} \right) = \frac{k'}{\cos \frac{\delta}{2}} \exp -i \delta/2 \quad [29]$$

$$\coth i k^* h/2 = P \exp -i \psi, \quad [30]$$

where

$$P = \frac{\{\sinh^2(kh \tan \delta/2) + \sin^2 kh\}^{1/2}}{2 \left\{ \sinh^2 \left(\frac{kh}{2} \tan \delta/2 \right) + \sin^2 \frac{kh}{2} \right\}} \quad [31]$$

and

$$\tan \psi = \frac{\sin kh}{\sinh(kh \tan \delta/2)}. \quad [32]$$

So we find from [27] to [30] for the angle ϕ between τ^* and ξ :

$$\phi = \pi/2 + \delta - \delta/2 - \arctg \frac{\sin kh}{\sinh(kh \tan \delta/2)}, \quad [33]$$

so

$$\tan \phi = \frac{\sinh(k'h \tan \delta/2) + \sin k'h \cdot \tan \delta/2}{\sin k'h - \sinh(k'h \tan \delta/2) \cdot \tan \delta/2} \quad [34]$$

and the magnitude M of τ^*

$$M = \frac{G'}{\cos \delta} \cdot \frac{k'}{\cos \delta/2} \cdot \frac{a \{\sinh^2(k'h \tan \delta/2) + \sin^2 k'h\}^{1/2}}{4 \left\{ \sinh^2 \left(\frac{k'h}{2} \tan \delta/2 \right) + \sin^2 \frac{k'h}{2} \right\}} \quad [35]$$

From [34] and [35] it follows:

$$F_x = AM \sin \phi = \frac{a G' k' A}{4 \cos \delta} \cdot \frac{\{\sinh(k'h \tan \delta/2) + \sin k'h \cdot \tan \delta/2\}}{\left\{ \sinh^2 \left(\frac{k'h}{2} \tan \delta/2 \right) + \sin^2 \frac{k'h}{2} \right\}} \quad [36]$$

and

$$F_y = AM \cos \phi = \frac{a G' k' A}{4 \cos \delta} \cdot \frac{\{\sin k'h - \sinh(k'h \tan \delta/2) \cdot \tan \delta/2\}}{\left\{ \sinh^2 \left(\frac{k'h}{2} \tan \delta/2 \right) + \sin^2 \frac{k'h}{2} \right\}} \quad [37]$$

In fig. 5

$$\tau_R \equiv \frac{\tau^*}{|G^*|}$$

is shown for various values of $h'k$ with $\tan \delta/2$ as a parameter. The values of $\tan \delta/2$ along the curves are given in the figure. It is seen from this figure that for a constant value of $\tan \delta/2$ the F_y -force is a decreasing function of $k'h$ in the region of $k'h$ considered ($0 \leq k'h \leq 3$). This behaviour is a consequence of resonance phenomena and is most pronounced for small values of $\tan \delta/2$. In particular for $\tan \delta/2 = 0$ it follows from [37] that $F_y = 0$ (resonance) for $k'h = (2n+1)\pi$, while $F_y \rightarrow \infty$ (anti-resonance) for $k'h = 2n\pi$ ($n = 0, 1, 2, \dots$).

Fig. 5. Reduced shear stress τ_R^* on the lower plate as a function of $\tan \delta/2$ and $k'h = 0, 1$ and 3 in a polar diagram. See note on page 449

On the other hand for a constant value of $\tan \delta/2 F_x$ is only a weakly increasing function of $k'h$; especially for low values of $\tan \delta/2 F_x$ is nearly independent of $k'h$ in the region of $k'h$ considered.

For a Newtonian liquid ($\tan \delta/2 = 1$) we find:

$$\tan \phi = \frac{\sinh kh + \sinh kh}{\sinh kh - \sinh kh} \quad [38]$$

and

$$M = G'' k \sqrt{2} \frac{a \{ \sinh^2 kh + \sin^2 kh \}^{1/2}}{4 \left\{ \sinh^2 \frac{kh}{2} + \sin^2 \frac{kh}{2} \right\}}; \quad [39]$$

so

$$F_x = \frac{G'' k a A}{4} \frac{\{ \sinh kh + \sinh kh \}}{\left\{ \sinh^2 \frac{kh}{2} + \sin^2 \frac{kh}{2} \right\}} \quad [40]$$

and

$$F_y = \frac{G'' k a A}{4} \frac{\{ \sinh kh - \sinh kh \}}{\left\{ \sinh^2 \frac{kh}{2} + \sin^2 \frac{kh}{2} \right\}}. \quad [41]$$

From [41] and fig. 5 it is learned that for a Newtonian liquid $F_y \neq 0$. F_y , however, is proportional to the energy stored in the sample, and as no elastic energy can be stored in a Newtonian liquid, F_y is solely determined by the stored kinetic energy. We will come back to this point later on in this paper.

For $k'h \ll 1$ (long wave length approximation) it follows from [34]:

$$\tan \phi = \frac{2 \tan \delta/2}{1 - \tan^2 \delta/2} = \tan \delta, \text{ so } \phi = \delta, \quad [42]$$

and from [35]:

$$M = \frac{G'}{\cos \delta} \cdot \frac{k'}{\cos \delta/2} \cdot \frac{a}{2} \cdot \frac{2 \cos \delta/2}{k'h} = \frac{a G'}{h \cos \delta}, \quad [48]$$

where terms of the order $(k'h)^3$ and higher in the expansions of $\sinh$ and $\sin$ are neglected.

Eq. [26] is readily deduced from [42] and [43], see fig. 4.

6. Energy considerations

6.1. Stored kinetic energy in a Newtonian liquid

As to the inertial force the same reasoning applies as given in 4 with respect to the elastic

force, the only difference being that the inertial force is π radians out of phase with the deformation.

The kinetic energy E is found from

$$\begin{aligned} E &= \int_0^h \frac{1}{2} \rho A \left\{ \frac{\partial}{\partial t} \operatorname{Re}(|\xi^*(z)| \exp i \omega t) \right\}^2 dz \\ &\quad + \int_0^h \frac{1}{2} \rho A \left\{ \frac{\partial}{\partial t} \operatorname{Re}(|\xi^*(z)| \exp i(\omega t + \pi/2)) \right\}^2 dz \\ &= \int_0^h \frac{1}{2} \rho A \omega^2 (|\xi^*(z)|^2 (\sin^2 \omega t + \cos^2 \omega t)) dz \\ &= \frac{1}{2} \rho A \omega^2 \int_0^h (|\xi^*(z)|^2) dz. \end{aligned} \quad [44]$$

Now from [14] and [15] it follows after some manipulation:

$$\begin{aligned} |\xi^*(z)|^2 &= \Xi^2(z) + \Psi^2(z) \\ &= \frac{a^2 \sinh^2 \{ k'(z - h/2) \tan \delta/2 \} + \sin^2 k'(z - h/2)}{4 \sinh^2 (k'h/2 \tan \delta/2) + \sin^2 k'h/2} \end{aligned} \quad [45]$$

Taking into account that for a Newtonian liquid $\tan \delta/2 = 1$ and $k' = k$, substitution of [45] into [44] leads to

$$\begin{aligned} E &= \frac{G'' k A a^2}{8} \frac{\{ \sinh kh - \sinh kh \}}{\{ \sin^2 kh/2 + \sinh^2 kh/2 \}} \\ &= \frac{F_y a}{2}, \end{aligned} \quad [46]$$

where the relations $\omega = G''/\eta$ and $\omega = 2\eta k^2/\rho$ are used. From [46] it follows that the energy stored by the F_y force equals the kinetic energy.

It may be enlightening to deduce eq. [44] working in an Eulerian frame. The kinetic energy of a slice of the sample, bounded by the planes $z = z_0$ and $z = z_0 + dz$, is $\frac{1}{2} \rho A \omega^2 \cdot \{1 + |\xi^*(z)|^2\} dz$. The extra kinetic energy E due to the deviation of the locus of the rotation centres from a straight line therefore amounts to

$$E = \frac{1}{2} \rho A \omega^2 \int_0^h |\xi^*(z)|^2 dz \quad [44]$$

and causes an extra F_y force. The extra F_y force for a general linear viscoelastic liquid (see eq. [20]) can be found in the same way.

6.2. Energy dissipation in a linear viscoelastic liquid

The energy dissipation per cm^3 and per cycle in the sample is a function of z only and amounts to

$$2 \int_0^{2\pi/\omega} \text{Re}(\tau^*(z)) \cdot \text{Re} \left\{ \frac{\partial}{\partial t} \left(\frac{\partial \xi(z, t)}{\partial z} \right) \right\} dt.$$

So the total amount of energy dissipated in the sample during one cycle, ΔW , is:

$$\Delta W = 2A \int_0^{2\pi/\omega} \int_0^h \text{Re}(\tau^*(z)) \cdot \text{Re} \left\{ \frac{\partial^2 \xi(z, t)}{\partial t \partial z} \right\} dt dz. \quad [47]$$

From [9] it is found:

$$\tau^*(z) = \frac{a G^* i k^* \cosh i k^* (z - h/2)}{2 \sinh i k^* h/2}. \quad [48]$$

Introducing

$$G^* = \frac{G'}{\cos \delta} \exp i \delta,$$

$$i k^* = \frac{k'}{\cos \delta/2} \exp i (\pi/2 - \delta/2),$$

$$\cosh i k^* (z - h/2) = [\cosh^2 \{k'(z - h/2) \tan \delta/2\} - \sin^2 k'(z - h/2)]^{1/2} \exp i \phi,$$

$$\sinh i k^* h/2 = [\sinh^2 \{k' h/2 \tan \delta/2\} + \sin^2 k' h/2]^{1/2} \exp i \psi$$

into [48] leads to

$$\begin{aligned} \text{Re}(\tau^*(z)) &= \frac{a G' k'}{2 \cos \delta \cdot \cos \delta/2} \\ &\cdot \left[\frac{\cosh^2 \{k'(z - h/2) \tan \delta/2\} - \sin^2 k'(z - h/2)}{\sinh^2 \{k' h/2 \tan \delta/2\} + \sin^2 k' h/2} \right]^{1/2} \\ &\cdot \sin \{ \omega t + \delta/2 + (\phi - \psi) \}. \end{aligned} \quad [49]$$

In the same way we find:

$$\begin{aligned} \text{Re} \left\{ \frac{\partial^2 \xi(z, t)}{\partial t \partial z} \right\} &= - \frac{a \omega k'}{\cos \delta/2} \\ &\cdot \left[\frac{\cosh^2 \{k'(z - h/2) \tan \delta/2\} - \sin^2 k'(z - h/2)}{\sinh^2 \{k' h/2 \tan \delta/2\} + \sin^2 k' h/2} \right]^{1/2} \\ &\cdot \cos \{ \omega t - \delta/2 + (\phi - \psi) \}. \end{aligned} \quad [50]$$

From [47], [49] and [50] it follows:

$$\begin{aligned} \Delta W &= \frac{A \pi a^2 G' k'}{2 \cos \delta} \\ &\cdot \left[\frac{\sinh(k' h \tan \delta/2) + \sin k' h \tan \delta/2}{\sinh^2(k' h/2 \tan \delta/2) + \sin^2 k' h/2} \right]. \end{aligned} \quad [51]$$

Comparing [36] and [51] it is seen that

$$\Delta W = 2 \pi a F_x. \quad [52]$$

The physical interpretation of [52] is straightforward if one considers a rotating coordinate system, the origin of which is located on the locus of the centres of rotation determined by [10] and [11].

From fig. 6 it follows that the work done by the plate forces in this coordinate system during a cycle is $2 \pi a F_x$.

Fig. 6. Illustration of the calculation of the dissipated energy

The analysis given in this paper of the orthogonal rheometer can be applied to other flow systems as well and the analysis of the balance rheometer along these lines will be given in a forthcoming paper.

Summary

The flow field of a linear viscoelastic material in the orthogonal rheometer, taking fluid inertia into account, has been studied theoretically and an exact solution is given. The flow field of a Newtonian liquid is included in this solution as a special case. The forces on the plates are readily deduced from this solution. The paper concludes with an energy consideration.

Zusammenfassung

Das Strömungsfeld eines linear-viskoelastischen Stoffes im Orthogonal-Rheometer wurde unter Berücksichtigung der Flüssigkeitsträgheit theoretisch untersucht und eine exakte Lösung dafür angegeben. Das Strömungsfeld einer newtonschen Flüssigkeit ergibt sich als Sonderfall dieser Lösung. Die auf die Platten ausgeübten Kräfte lassen sich in einfacher Weise aus dieser Lösung ableiten. Abschließend wird eine Energiebetrachtung angestellt.

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