HIGHER ORDER SUPERSYMMETRIES AND FERMIONIC CONSERVATION LAWS OF THE SUPERSYMMETRIC EXTENSION OF THE KdV EQUATION

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By the introduction of nonlocal bosonic and fermionic variables we construct a recursion symmetry of the super KdV equation, leading to a hierarchy of bosonic symmetries and one of fermionic symmetries. The hierarchies of bosonic and fermionic conservation laws arise in a natural way in the construction.

1. Introduction

In two recent papers [1,2] we studied symmetries and nonlocal symmetries of the super KdV and super mKdV equation [3], and we derived a nonlocal higher order symmetry leading to the recursion operator for higher order symmetries in a straightforward way, not relying on the bi-hamiltonian structure of the equations. All symmetries discussed there were even [4] or bosonic symmetries. These equations however, are not supersymmetric in the sense that they admit an odd [4] or fermionic symmetry. Often fermionic or odd symmetries are called supersymmetries. However, there is known a supersymmetric extension of the KdV equation by Manin and Radul [5] which admits a supersymmetry. This supersymmetric extension admits an infinite hierarchy of even higher order or generalised symmetries and conserved quantities. No recursion operator for symmetries is known in this case [6]. Moreover nothing is known about the existence of fermionic conservation laws besides the only one arising from the supersymmetry.

In this paper we shall construct a nonlocal symmetry of the supersymmetric KdV equation, which together with the already known supersymmetry generates a graded Lie algebra of symmetries, comprising a hierarchy of bosonic higher order symmetries and a hierarchy of nonlocal higher order supersymmetries. The familiar supersymmetry is just

the first member of the hierarchy. Moreover, higher order even and odd conserved quantities arise in a natural and elegant way in the construction of the infinite dimensional graded Lie algebra. The construction of higher order even symmetries is given in section 2, while the construction of the above mentioned nonlocal symmetry together with the graded Lie algebra structure is given in section 3.

2. Higher order symmetries of the supersymmetric KdV equation

The existence of higher order even [4] symmetries of the supersymmetric extension of the KdV equation

$$u_t = -u_{xxx} + 6uu_x \quad (2.1)$$

shall be discussed. We start at the supersymmetric extension given by Mathieu [6], i.e.

$$\begin{aligned} u_t &= -u_3 + 6uu_1 - a\varphi\varphi_2, \\ \varphi_t &= -\varphi_3 + (6-a)\varphi_1 u + a\varphi u_1. \end{aligned} \quad (2.2)$$

In (2.2) integer indices refer to differentiation with respect to x , i.e. $u_3 = u_{xxx}$; x, t, u are even, while φ is odd [4]; the parameter a is real. Taking $\varphi \equiv 0$ we get (2.1). Classical higher order symmetries are defined on the infinite jet bundle $J^\infty(x, t; u, \varphi)$ [7] and satisfy the symmetry condition

$$\mathcal{L}_V(D^\infty I) \subset D^\infty I, \quad (2.3)$$

where $D^\infty I$ is the infinite prolongation of the exterior differential system I , describing the partial differential equation, by means of the action of the total partial derivative vector fields D_x, D_t defined by

$$\begin{aligned} D_x &= \partial_x + u_x \partial_u + \varphi_x \partial_\varphi + \dots, \\ D_t &= \partial_t + u_t \partial_u + \varphi_t \partial_\varphi + \dots \end{aligned} \quad (2.4)$$

In (2.3) $\mathcal{L}_V$ denotes the Lie derivative with respect to the vector field V . Due to the fact that (2.4) satisfies (2.3) in an obvious way, the search for higher order symmetries (generalised symmetries) can be restricted to vertical vector fields; i.e. the components ∂_x, ∂_t are taken to be zero.

The vertical vector fields are proved to have the following representation,

$$V = f \partial_u + g \partial_\varphi + (D_x f) \partial_{u_1} + (D_x g) \partial_{\varphi_1} + \dots, \quad (2.5)$$

so we are only interested in the defining function f, g of the vector field. The functions f and g are assumed to depend on a finite number of independent variables of the infinite jet bundle. In the graded case [4] at hand (2.2) we proceed in a similar way, keeping in mind the left module structure of vector fields. We restrict our search for higher order symmetries at this moment to *even* vector fields; moreover our search is for vector fields V whose defining functions f and g (2.5) depend on $x, t, u, \varphi, u_1, \varphi_1, \dots, u_5, \varphi_5$, the other components being obtained by prolongation (2.4), (2.5). More specifically

$$\begin{aligned} f &= f_1 + f_2 \varphi \varphi_1 + f_3 \varphi \varphi_2 + f_4 \varphi \varphi_3 + f_5 \varphi \varphi_4 + f_6 \varphi_1 \varphi_2 + f_7 \varphi_1 \varphi_3, \\ g &= g_1 \varphi + g_2 \varphi_1 + g_3 \varphi_2 + g_4 \varphi_3 + g_5 \varphi_4 + g_6 \varphi_5, \end{aligned} \quad (2.6)$$

whereas in (2.6) $f_1, \dots, f_7, g_1, \dots, g_6$ are dependent on the *even* variables $x, t, u, \dots, u_5$. Formula (2.6) is motivated by the standard grading in the classical sense of (2.2),

$$\begin{aligned} \deg(x) &= -1, & \deg(u) &= 2, \\ \deg(t) &= -3, & \deg(\varphi) &= \frac{3}{2}, \end{aligned} \quad (2.7)$$

and the results obtained for other cases [1].

The vector field V (2.5), (2.6) has to satisfy the symmetry condition (2.3) which is equivalent to

$$\mathcal{L}_V(u_t + u_3 - 6uu_1 + a\varphi\varphi_2) \doteq 0,$$

$$\mathcal{L}_V(\varphi_t + \varphi_3 - (6-a)\varphi_1 u - a\varphi u_1) \doteq 0, \quad (2.8)$$

where " $\doteq 0$ " should be understood as equal to zero on the submanifold of the infinite jet bundle $J^\infty(x, t; u, \varphi)$ defined by (2.2) and its differential consequences.

Conditions (2.8) lead to an overdetermined system of partial differential equations for the functions $f_1, \dots, f_7, g_1, \dots, g_6$. In order to obtain this overdetermined system we made use of the graded differential geometry package described in ref. [8]. Solving this overdetermined system of partial differential equations we obtain the following result:

Theorem 2.1. If $a=3$ there are four vector fields satisfying (2.8) under conditions (2.6), i.e.

$$\begin{aligned} X_1 &= u_1 \partial_u + \varphi_1 \partial_\varphi, \\ X_3 &= (u_3 - 6u_1 u + 3\varphi\varphi_2) \partial_u + (\varphi_3 - 3\varphi_1 u - 3\varphi u_1) \partial_\varphi, \\ X_5 &= -(u_5 - 10u_3 u - 20u_2 u_1 + 30u_1 u^2 + 5\varphi\varphi_4 \\ &\quad + 5\varphi_1 \varphi_3 - 20u\varphi\varphi_2 - 20u_1 \varphi\varphi_1) \partial_u - (\varphi_5 - 5u\varphi_3 \\ &\quad - 10u_1 \varphi_2 - 10u_2 \varphi_1 + 10u^2 \varphi_1 + 20u_1 u \varphi - 5u_3 \varphi) \partial_\varphi, \\ V &= -3tX_3 + xX_1 + 2u\partial_u + \frac{3}{2}\varphi\partial_\varphi. \end{aligned} \quad (2.9)$$

If $a \neq 3$ then X_5 is *not* a symmetry of (2.2).

Next, our search is for *odd* vector fields (2.5) satisfying (2.8); the assumption on the functions f and g is

$$\begin{aligned} f &= f_1 \varphi + f_2 \varphi_1 + f_3 \varphi_2 + f_4 \varphi_3 + f_5 \varphi_4 + f_6 \varphi_5 + f_7 \varphi_6, \\ g &= g_1 + g_2 \varphi \varphi_1 + g_3 \varphi \varphi_2 + g_4 \varphi \varphi_3 + g_5 \varphi \varphi_4 + g_6 \varphi \varphi_5 \\ &\quad + g_7 \varphi_1 \varphi_2 + g_8 \varphi_1 \varphi_3 + g_9 \varphi_1 \varphi_4 + g_{10} \varphi_2 \varphi_3, \end{aligned} \quad (2.10)$$

where $f_1, \dots, f_7, g_1, \dots, g_{10}$ are dependent on $x, t, u, \dots, u_5$. Solving the overdetermined system of partial differential equations resulting from conditions (2.8), (2.10) leads to:

Theorem 2.2. There exists only one *odd* vector field satisfying (2.10) and the symmetry condition (2.8), i.e.

$$Y_{1/2} = \varphi_1 \partial_u + u \partial_\varphi. \quad (2.11)$$

In order to obtain the Lenard recursion operator we did proceed in a way similar to that discussed in refs. [1,2], but unfortunately we were not successful. We

shall discuss recursion formulas for symmetries in the next section.

3. Nonlocal symmetries and hierarchies of symmetries and conserved quantities

By the introduction of nonlocal variables [9] we derive a nonlocal symmetry for the supersymmetric KdV equation ($a=3$),

$$\begin{aligned} u_t &= -u_3 + 6u_1u - 3\phi\phi_2, \\ \phi_t &= -\phi_3 + 3\phi_1u + 3\phi u_1, \end{aligned} \quad (3.1)$$

which together with the supersymmetry $Y_{1/2}$ generates two hierarchies of symmetries. The even and odd conserved quantities arise in a natural way.

We first start with the observation that

$$q_{1/2} = \int_{-\infty}^x \phi \, dx \quad (3.2a)$$

is a potential of (3.1), i.e.

$$(q_{1/2})_x = \phi, \quad (q_{1/2})_t = -\phi_2 + 3\phi u. \quad (3.2b)$$

The quantity $Q_{1/2}$ defined by

$$Q_{1/2} = \int_{-\infty}^{\infty} \phi \, dx \quad (3.3)$$

is a conserved quantity of the supersymmetric KdV equation (2.2), (3.1). We now make the following observation:

Theorem 3.1. The nonlocal vector field Z_1 , defined by

$$Z_1 = (q_{1/2}\phi_1)\partial_u + (q_{1/2}u - \phi_1)\partial_\phi \quad (3.4)$$

is a nonlocal symmetry of (3.1). Moreover there is no nonlocal symmetry, linear with respect to $q_{1/2}$ which satisfies (2.2) for $a \neq 3$.

The vector field Z_1 together with the vector field $Y_{1/2}$ plays a fundamental role in the graded algebra of symmetries of (3.1). From now on, for obvious reasons, we shall restrict ourselves to (3.1).

Note. All odd variables $\phi_0\phi_1\dots q_{1/2}$ etc. are of degree $n/2$ where n is odd (2.7). The vector fields Z_1 and $Y_{1/2}$ are even and odd respectively.

We now want to compute the graded Lie algebra with Z_1 and $Y_{1/2}$ as seed elements, so the graded Lie bracket has to be calculated. In order to do so we have to prolong the vector field $Y_{1/2}$ towards the nonlocal variable $q_{1/2}$, or by just writing $Y_{1/2}$ for this prolongation to calculate the component $\partial_{q_{1/2}}$. The coefficient $Y_{1/2}^{q_{1/2}}$ has to be such that the vector field $Y_{1/2}$ leaves invariant eqs. (3.2b), i.e. the Lie derivative of (3.2b) with respect to $Y_{1/2}$ is zero; i.e.

$$D_x(Y_{1/2}^{q_{1/2}}) - u = 0, \quad (3.5)$$

$$D_t(Y_{1/2}^{q_{1/2}}) + u_2 - 3u^2 + 3\phi\phi_1 = 0. \quad (3.6)$$

By (3.5), (3.6) we are led in a natural and elegant way to the introduction of the *even* nonlocal variable p_1 , defined by

$$p_1 = \int_{-\infty}^x u \, dx, \quad (3.7a)$$

satisfying

$$(p_1)_x = u, \quad (p_1)_t = -u_2 + 3u^2 - 3\phi\phi_1, \quad (3.7b)$$

i.e. p_1 is a potential of the supersymmetric KdV equation (3.1); the integrability conditions being satisfied; conserved quantity P_1 . Now the vector field $Y_{1/2}$ is given by

$$Y_{1/2} = \phi_1\partial_u + u\partial_\phi + p_1\partial_{q_{1/2}} + \dots \quad (3.8)$$

Computation of the commutator of Z_1 and $Y_{1/2}$ leads to a new vector field $Y_{3/2}$, given by

$$\begin{aligned} Y_{3/2} = [Z_1, Y_{1/2}] &= (2q_{1/2}u_1 - p_1\phi_1 + u\phi - \phi_2)\partial_u \\ &+ (2q_{1/2}\phi_1 - p_1u + u_1)\partial_\phi + \dots \end{aligned} \quad (3.9)$$

The vector field $Y_{3/2}$ is a new *nonlocal odd symmetry* of (3.1). Note that, as a polynomial in $q_{1/2}$ and p_1 the coefficients in (3.9) are just the symmetries $2X_1$ and $-Y_{1/2}$ respectively.

We now proceed by induction. In order to compute the graded Lie bracket $[Z_1, Y_{3/2}]$ we first have to compute the prolongation of Z_1 towards the nonlocal variables p_1 and $q_{1/2}$, i.e. the ∂_{p_1} and $\partial_{q_{1/2}}$ components of the vector field, again denoted by Z_1 . An easy calculation, the invariance of (3.7b) by Z_1 , shows that $Z_1^{p_1}$ has to satisfy

$$D_x(Z_1^{p_1}) = q_{1/2}\phi_1, \quad (3.10)$$

from which we have

$$Z_1^{p_1} = q_{1/2} \varphi. \quad (3.11)$$

The $\partial_{q_{1/2}}$ component $Z_1^{q_{1/2}}$ however has to satisfy the invariance condition of (3.2b) by Z_1 , i.e.

$$D_x(Z_1^{q_{1/2}}) - q_{1/2} u + \varphi_1 = 0,$$

from which

$$Z_1^{q_{1/2}} = q_{1/2} p_1 - \varphi - \int_{-\infty}^x p_1 \varphi \, dx. \quad (3.12)$$

So prolongation of Z_1 towards the nonlocal variable $q_{1/2}$ requires the formal introduction of the *odd nonlocal variable* $q_{3/2}$ defined by

$$q_{3/2} = \int_{-\infty}^x p_1 \varphi \, dx, \quad (3.13a)$$

where

$$(q_{3/2})_x = p_1 \varphi, \quad (q_{3/2})_t = p_1 (-\varphi_2 + 3u\varphi) - u_1 \varphi + u\varphi_1, \quad (3.13b)$$

the integrability condition on (3.13b) being satisfied, which is easy to check; so $q_{3/2}$ is an *odd nonlocal potential* of (3.1), $Q_{3/2}$ being the associated *odd and nonlocal* conserved quantity. The vector field Z_1 is now given by

$$\begin{aligned} Z_1 &= q_{1/2} \varphi_1 \partial_u + (q_{1/2} u - \varphi_1) \partial_\varphi + (q_{1/2} p_1 \\ &\quad - \varphi - q_{3/2}) \partial_{q_{1/2}} + q_{1/2} \varphi \partial_{p_1} \\ &= q_{1/2} Y_{1/2} - \varphi_1 \partial_\varphi - (\varphi + q_{3/2}) \partial_{q_{1/2}}. \end{aligned} \quad (3.14)$$

The prolongation of the vector field $Y_{3/2}$ towards the nonlocal variables $q_{1/2}$, $p_{1/2}$ are calculated from the invariance of (3.2b), (3.7b) resulting in

$$\begin{aligned} Y_{3/2}^{q_{1/2}} &= 2q_{1/2} \varphi - \frac{1}{2} (p_1)^2 u + u, \\ Y_{3/2}^{p_1} &= 2q_{1/2} u - p_1 \varphi - \varphi_1. \end{aligned} \quad (3.15)$$

Computation of the graded Lie bracket $[Z_1, Y_{3/2}]$ leads to

$$\begin{aligned} Y_{5/2} &= [-2q_{3/2} u_1 + \frac{1}{2} (p_1)^2 \varphi_1 + p_1 (\varphi_2 - u\varphi) - 4u_1 \varphi \\ &\quad - 3u\varphi_1 + \varphi_3] \partial_u + [-2q_{3/2} \varphi_1 + \frac{1}{2} (p_1)^2 u \\ &\quad + p_1 (-u_1) + u_2 - 2u^2 + \varphi\varphi_1] \partial_\varphi + [-2q_{3/2} u \\ &\quad + \frac{1}{2} (p_1)^2 \varphi + p_1 \varphi_1 - 4u\varphi + \varphi_2] \partial_{p_1} + [-2q_{3/2} p_1 \varphi \\ &\quad + \frac{1}{8} (p_1)^4 - (p_1)^2 u + p_1 u_1 - \frac{1}{2} u^2 - \varphi\varphi_1] \partial_{q_{3/2}}, \end{aligned} \quad (3.16)$$

whereas the ∂_{p_1} , $\partial_{q_{3/2}}$ components of $Y_{5/2}$ are obtained by the invariance of (3.7b), (3.13b). In order to obtain the prolongation of $Y_{5/2}$ towards the nonlocal variable $q_{1/2}$ we have to require the invariance of (3.2b), which results in the condition

$$\begin{aligned} D_x(Y_{5/2}^{q_{1/2}}) &= -2q_{3/2} \varphi_1 + \frac{1}{2} (p_1)^2 u - p_1 u_1 + u_2 \\ &\quad - 2u^2 + \varphi\varphi_1, \end{aligned} \quad (3.17)$$

from which we have

$$\begin{aligned} Y_{5/2}^{q_{1/2}} &= \frac{1}{6} (p_1)^3 - p_1 u + u_1 - 2q_{3/2} \varphi \\ &\quad + \int_{-\infty}^x [u^2 + 2(p_1 \varphi) \varphi + \varphi\varphi_1 - 2u^2] \, dx \\ &= \frac{1}{6} (p_1)^3 - p_1 u + u_1 - 2q_{3/2} \varphi - \int_{-\infty}^x (u^2 - \varphi\varphi_1) \, dx. \end{aligned} \quad (3.18)$$

So expression (3.18) requires in an elegant way the introduction of p_3 , defined by

$$p_3 = \int_{-\infty}^x (u^2 - \varphi\varphi_1) \, dx, \quad (3.19a)$$

where

$$\begin{aligned} (p_3)_x &= u^2 - \varphi\varphi_1, \\ (p_3)_t &= 4u^3 - 2u_2 u + u_1^2 - 9u\varphi\varphi_1 + \varphi\varphi_3 - 2\varphi_1 \varphi_2. \end{aligned} \quad (3.19b)$$

p_3 is a well-known potential, P_3 being the associated conserved quantity.

The commutator $Y_{7/2} = [Z_1, Y_{5/2}]$ requires the prolongation of the vector field Z_1 towards the nonlocal variable $q_{3/2}$, obtained by the invariance of (3.13b) by Z_1 , so

$$D_x(Z_1^{q_{3/2}}) = Z_1(p_1 \varphi) = (q_{1/2} \varphi) \varphi + p_1 (q_{1/2} u - \varphi_1). \quad (3.20)$$

Integration of (3.20) leads to

$$Z_1^{q_{3/2}} = \frac{1}{2} (p_1)^2 q_{1/2} - p_1 \varphi - \int_{-\infty}^x [\frac{1}{2} (p_1)^2 \varphi - u\varphi] \, dx. \quad (3.21)$$

The new *odd nonlocal variable* $q_{5/2}$ is formally defined by

$$q_{5/2} = \int_{-\infty}^x [\frac{1}{2}(p_1)^2\varphi - u\varphi] dx. \quad (3.22a)$$

$q_{5/2}$ is a nonlocal odd potential of the supersymmetric KdV equation (3.1),

$$\begin{aligned} (q_{5/2})_x &= \frac{1}{2}(p_1)^2\varphi - u\varphi, \\ (q_{5/2})_t &= \frac{1}{2}(p_1)^2(-\varphi_2 + 3u\varphi) + p_1(-u\varphi + u\varphi_1) \\ &\quad + u_2\varphi - u_1\varphi_1 - 4u^2\varphi + u\varphi_2. \end{aligned} \quad (3.22b)$$

Proceeding in this way, we obtain a hierarchy of nonlocal supersymmetries by induction,

$$Y_{n+1/2} = [Z_1, Y_{n-1/2}], \quad n \in \mathbb{N}. \quad (3.23)$$

The higher order *even* potentials $p_1, p_3, \dots$ arise in a natural way in the prolongation of the vector fields $Y_{2n+1/2}$ towards the nonlocal variable $q_{1/2}$, where the *nonlocal* superpotentials $q_{1/2}, q_{3/2}, q_{5/2}, \dots$ are obtained in the prolongation of the recursion symmetry Z_1 . A formal proof of these facts is beyond the scope of the presentation of these results and will be discussed elsewhere using the hamiltonian structure.

To obtain the graded Lie algebra structure of symmetries we calculate the graded Lie bracket of the vector fields derived so far. The result is remarkable and fascinating,

$$\begin{aligned} [Y_{1/2}, Y_{1/2}] &= 2X_1, \quad [Y_{3/2}, Y_{3/2}] = 2X_3, \\ [Y_{5/2}, Y_{5/2}] &= 2X_5, \end{aligned} \quad (3.24)$$

so the “squares” of the supersymmetries $Y_{1/2}, Y_{3/2}, Y_{5/2}$ are just the “classical” symmetries $2X_1, 2X_3, 2X_5$ obtained in the previous section (2.9)! The other commutators are

$$\begin{aligned} [Y_{1/2}, Y_{3/2}] &= 0, \quad [Y_{1/2}, Y_{5/2}] = -2X_3, \\ [Y_{3/2}, Y_{5/2}] &= 0, \end{aligned} \quad (3.25)$$

while

$$\begin{aligned} [X_1, X_3] &= [X_1, X_5] = [X_3, X_5] = 0, \\ [Z_1, X_1] &= [Z_1, X_3] = [Z_1, X_5] = 0, \\ [Y_{n+1/2}, X_{2m+1}] &= 0, \quad n=0, 1, 2, \dots; m=0, 1, 2. \end{aligned} \quad (3.26)$$

The results are illustrated in fig. 1. We conjecture that in this way we obtain an infinite hierarchy of *nonlocal supersymmetries* ($Y_{n+1/2}; n \in \mathbb{N}$) and an infinite

Fig. 1. The graded Lie algebra.

hierarchy of ordinary even higher order symmetries ($X_{2n+1}, n \in \mathbb{N}$), while the even and odd conserved quantities $P_{2n+1}, Q_{n+1/2}$ ($n \in \mathbb{N}$) are obtained by prolongation of the vector fields $Y_{n+1/2}$ and Z_1 respectively.

We finish this section with the following lemma on the graded Lie algebra structure:

Lemma 1. Let X_{2n+1} ($n \in \mathbb{N}$) be defined by

$$X_{2n+1} = \frac{1}{2}[Y_{n+1/2}, Y_{n+1/2}] \quad (3.27)$$

and assume

$$[Z_1, X_{2n+1}] = 0, \quad n \in \mathbb{N}, \quad (3.28)$$

then

$$\begin{aligned} [Y_{n+1/2}, Y_{m+1/2}] &= (-1)^{m-n} 2X_{n+m+1}, \quad m-n \text{ even}, \\ &= 0, \quad m-n \text{ odd}, \end{aligned} \quad (3.29)$$

$$[Y_{n+1/2}, X_{2m+1}] = 0, \quad n, m \in \mathbb{N}, \quad (3.30)$$

$$[X_{2n+1}, X_{2m+1}] = 0, \quad n, m \in \mathbb{N}. \quad (3.31)$$

The proof of lemma 1 is straightforward [10].

4. Conclusion

By using the concept of nonlocal symmetries we obtain hierarchies of symmetries and supersymmetries for the supersymmetric KdV equation. The bosonic and fermionic conserved quantities arise in an elegant way. We took great advantage of the devel-

oped software to handle graded differential geometry [8]

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