INFORMATION SCIENCES **15**, 1–29 (1978)

Fuzzy Random Variables—I. Definitions and Theorems

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Communicated by L. A. Zadeh

ABSTRACT

Fuzziness is discussed in the context of multivalued logic, and a corresponding view of fuzzy sets is given. Fuzzy random variables are introduced as random variables whose values are not real but fuzzy numbers, and subsequently redefined as a particular kind of fuzzy set. Expectations of fuzzy random variables, characteristic functions of fuzzy events, probabilities connected to fuzzy random variables, and conditional expectations and probabilities relating to fuzzy random variables are defined as images of the fuzzy set representing the fuzzy random variable under appropriate mappings. Several theorems, some of which relate to independent fuzzy random variables, are proved.

1. INTRODUCTION

This paper is the first of a series of reports on fuzzy random variables. Fuzzy random variables are random variables whose values are not real, but fuzzy numbers. To illustrate this, we give an example of a simple type of fuzzy random variable. Consider an opinion poll, during which a number of individuals are questioned on their opinion concerning the weather in Europe in a particular summer. The responses are classified into three categories, respectively characterized as "very warm", "warm", and "no opinion". Table 1 summarizes the results. Randomness occurs because it is not known which response may be expected from any given individual. Once the response is available, there still is uncertainty about the precise meaning of the response. The latter uncertainty will be characterized by fuzziness, in the sense that each of the responses very warm, warm, and no opinion will be represented by a fuzzy set (in particular by a special type of fuzzy set called fuzzy number).

 0020-0255/78/0015-0001$175

TABLE 1
Results of a Questionnaire

Fraction of respondents	Response
0.4	Very warm
0.5	Warm
0.1	No opinion

In the present paper, following Gaines [1] and Bellman and Giertz [2], the notion of fuzziness is framed in the context of multivalued logic. This view is briefly discussed in Sec. 2 of the paper. Section 3 describes a corresponding perspective on fuzzy sets, as well as a simple but powerful theorem. Also a number of basic definitions are given. In Sec. 4, fuzzy random variables are formally introduced, and subsequently redefined as a particular kind of fuzzy set. In the following sections, the notions of the expectation of a fuzzy random variable, probabilities connected with fuzzy random variables, characteristic functions of fuzzy events, and conditional expectations and probabilities connected with fuzzy random variables are introduced. In each case, these entities are defined as images of the fuzzy set representing the fuzzy random variable under an appropriate mapping. Several theorems, some of them relating to independent fuzzy random variables, are proved.

The way the theory is developed makes the distinction between fuzziness and randomness very clear. Randomness is caused by some chance mechanism, whereas fuzziness is brought about by dimness of perception. The theory is not directly related to other work in this area. The notion of a fuzzy event as it emerges in the present paper is more complex than that of Zadeh [3] (see also Negoita and Ralescu [4]). The probability measure underlying the randomness of fuzzy random variables as defined in the present paper is not fuzzy, in contrast to the linguistic probabilities that are treated in another paper by Zadeh [5]. Fuzzy probabilities arise in the present paper as a result of certain fuzzy events relating to fuzzy random variables (see Sec. 7).

The basic techniques used in the present paper will be further developed in Part II of the paper, where algorithms are given for the evaluation of expectations and probabilities connected to discrete fuzzy random variables. These algorithms will make it possible to work out some examples in detail and develop a feeling for the properties and nature of fuzzy random variables. Possible applications relate to the statistical analysis of imprecise data as well as the solution of certain decision problems.

In the text, literature references are given as needed. Basic references for probability theory and stochastic processes are Loève [6] and Doob [7].

2. FUZZY LOGIC

In this section we give a brief discussion of fuzzy logic as a special multivalued logic. To this end, consider a complete distributive lattice L of statements [8]. The lattice is defined by a set of statements P, and the binary operations $\wedge$ ("and") and $\vee$ ("or"). The connectives possess the usual properties of idempotency, commutativity, associativity, and the absorption identity [8]. Distributivity is also assumed, and moreover the lattice is supposed to be complete, which means that statements such as

$$\bigwedge_{\lambda \in \Lambda} a_\lambda, \qquad \bigvee_{\lambda \in \Lambda} a_\lambda, \tag{2.1}$$

with $a_\lambda \in P$ for each $\lambda \in \Lambda$, are well defined and contained in P. Here a_λ, $\lambda \in \Lambda$, is an indexed subset of statements. It is noted that we do not introduce negation as a logical operation, since it will not be needed in the sequel.

On P we define a function $t: P \to [0,1]$, where for given $a \in P$, the number $t(a)$ is referred to as the "truth value" of the statement a. In usual mathematical logic, $t(a)$ assumes one of the two values 0 ("false") or 1 ("true"). In multivalued logic, of which we are considering a particular instance, $t(a)$ may assume intermediate values. Bellman and Giertz [2] impose the following requirements on the function t:

(a) There exist functions f and g, both mapping $[0,1] \times [0,1]$ into $[0,1]$, such that

$$\begin{aligned} t(a \wedge b) &= f(t(a), t(b)), \\ t(a \vee b) &= g(t(a), t(b)) \end{aligned} \tag{2.2}$$

for all $a \in P$, $b \in P$.

(b) $f(x,y)$ and $g(x,y)$ are continuous and nondecreasing in x.

(c) $f(x,x)$ and $g(x,x)$ are strictly increasing in x.

(d) $f(x,y) \leqslant \min(x,y)$ and $g(x,y) \geqslant \max(x,y)$ for all $x \in [0,1]$, $y \in [0,1]$.

(e) $f(1,1) = 1$ and $g(0,0) = 0$.

Property (a) expresses that the truth value of a compound statement is uniquely determined by the truth values of the component statements; this is called strict truth functionality in multivalued logic. Property (b) expresses that $t(a \wedge b)$ and $t(a \vee b)$ do not become less true if a is changed so that its truth value increases, and that moreover the dependence is continuous. Property (c) is equally plausible. Property (d) requires that the truth value of the statement

$a \wedge b$ cannot be greater than the individual truth values of a and b; similarly, the truth value of the statement $a \vee b$ is required to be at least as large as the individual truth values of a and b. Finally, property (e) expresses that if a and b are both completely true, then also $a \wedge b$ is completely true, while if a and b are both completely false, then $a \vee b$ is also completely false.

Bellman and Giertz [2] prove that (a) through (e) imply that the functions f and g are uniquely given by

$$f(x,y) = \min(x,y), \qquad g(x,y) = \max(x,y). \tag{2.3}$$

We shall refer to the corresponding multivalued logic as fuzzy logic [1]. It is the logic introduced by Zadeh in his work on fuzzy sets. We observe that if the truth function is restricted to assume the values 0 and 1 only, we obtain the usual Boolean logic (except for the definition of negation, which is missing here).

The interval $[0,1]$ together with the min and max operations forms another complete distributive lattice. Hence, t is a homomorphism. It follows that for any subset of statements $a_\lambda \in P$, $\lambda \in \Lambda$, with Λ an index set,

$$t\Big(\bigwedge_{\lambda \in \Lambda} a_\lambda\Big) = \inf_{\lambda \in \Lambda} t(a_\lambda), \qquad t\Big(\bigvee_{\lambda \in \Lambda} a_\lambda\Big) = \sup_{\lambda \in \Lambda} t(a_\lambda). \tag{2.4}$$

It will be helpful to use the existential quantifier $\exists$ and the universal quantifier $\forall$. We define the following equivalences:

$$\bigwedge_{\lambda \in \Lambda} a_\lambda = ((\forall \lambda \in \Lambda) a_\lambda), \qquad \bigvee_{\lambda \in \Lambda} a_\lambda = ((\exists \lambda \in \Lambda) a_\lambda). \tag{2.5}$$

3. FUZZY SETS

In this section we explain our view of fuzzy sets. A fuzzy set $\alpha = (X, \mu, a)$ will be defined by the following elements. X is an ordinary set, called the *basic space*. The *membership function* μ is a map $X \to [0,1]$. Finally $a: X \to P$, with P the "universe of discourse" introduced in the preceding section, assigns a proposition $a(x)$ to each element $x \in X$. The corresponding value $\mu(x)$ of the membership function is the truth value of the proposition $a(x)$, i.e., $\mu(x) = t(a(x))$. Thus, a fuzzy set is an indexed set of statements together with the truth value of each member statement. This view is slightly but not essentially different from the usual introduction to fuzzy sets (see e.g. [4] or [9]).

Given a fuzzy set $\alpha=(X,\mu,a)$, let ϕ be a function mapping $X\to Y$, where Y is another ordinary set. Let us consider for any $y\in Y$ the following statement: there exists an element $x\in X$ such that $a(x)$ holds and $\phi(x)=y$. This statement can be written as

$$b(y)\stackrel{\text{df}}{=}\bigvee_{x\in X}\left[a(x)\wedge(\phi(x)=y)\right], \tag{3.1}$$

where we include the statement $(\phi(x)=y)$ in P. The truth value of the compound statement $b(y)$ is by the properties of the truth function

$$\begin{aligned} t(b(y)) &= t\Big(\bigvee_{x\in X}\left[a(x)\wedge(\phi(x)=y)\right]\Big)\\ &= \sup_{x\in X} t\big(a(x)\wedge(\phi(x)=y)\big)\\ &= \sup_{x\in X}\min\left[t(a(x)),t((\phi(x)=y))\right]\\ &= \sup_{x\in X:\phi(x)=y}\mu(x), \end{aligned} \tag{3.2}$$

since $t\big((\phi(x)=y)\big)=1$ if $\phi(x)=y$ and $t\big((\phi(x)=y)\big)=0$ if $\phi(x)\neq y$. Now, defining $t(b(y))=\nu(y)$, we have thus obtained a new fuzzy set (Y,ν,b), which is called the *image* of the fuzzy set $\alpha=(X,\mu,a)$ in Y under the mapping ϕ. We shall denote this image as $\phi(\alpha)$.

THEOREM 3.1. *Let $\alpha=(X,\mu,a)$ be a fuzzy set, and suppose that ϕ and ψ are functions respectively mapping $\phi:X\to Y$, and $\psi:Y\to Z$. Then $\psi(\phi(\alpha))=(\psi\circ\phi)(\alpha)$, i.e., the image of $\phi(\alpha)$ in Z under ψ is the same as the image of α in Z under the composite mapping $\psi\circ\phi:X\to Z$, where $(\psi\circ\phi)(x)=\psi(\phi(x))$.*

Proof. Let us denote $\phi(\alpha)=(Y,\nu,b)$ and $\psi(\phi(\alpha))=(Z,\pi,c)$. We then have for any $z\in Z$

$$\begin{aligned} c(z) &= \big((\exists y\in Y)\ \psi(y)=z,\ b(y)\big)\\ &= \big((\exists y\in Y)\ \psi(y)=z,\ ((\exists x\in X)\ \phi(x)=y,\ a(x))\big)\\ &= \big((\exists y\in Y,\ \exists x\in X)\ \psi(y)=z,\ \phi(x)=y,\ a(x)\big)\\ &= \big((\exists x\in X)\ \psi(\phi(x))=z,\ a(x)\big), \end{aligned} \tag{3.3}$$

which proves that $\psi(\phi(\alpha))=(Z,\pi,c)$ is indeed the image of $\alpha=(X,\mu,a)$ in Z under the mapping $\psi\circ\phi$. ∎

This theorem will turn out to be quite helpful in the sequel. We conclude this section with a few additional definitions. If $\alpha=(X,\mu,a)$ and $\beta=(Y,\nu,b)$ are two fuzzy sets, then we define the product fuzzy set $\alpha\times\beta$ as the fuzzy set $(X\times Y,\mu\times\nu,a\wedge b)$, where $X\times Y$ is the Cartesian product of the basic spaces X and Y, $(\mu\times\nu)(x,y)=\min[\mu(x),\nu(y)]$, and $(a\wedge b)(x,y)=a(x)\wedge b(y)$. A fuzzy set $\alpha=(R,\mu,a)$ defined on the real line R such that μ is piecewise continuous will be called a *fuzzy number*. If α and β are fuzzy numbers, expressions such as $\alpha+\beta$ and $\alpha\beta$ will denote the images in R of the product fuzzy set $\alpha\times\beta$ under the mappings $(x,y)\mapsto x+y$ and $(x,y)\mapsto xy$, respectively. A fuzzy number such that the set $\{x\in R\,|\,\mu(x)\geqslant a\}$ is convex for each $a\in[0,1]$ is called *unimodal*. The membership function μ of a unimodal fuzzy number is also called unimodal. A fuzzy set (X,μ,a), such that there exists an element $x\in X$ such that $\mu(x)=1$, is called *normal*.

4. FUZZY RANDOM VARIABLES

The notion of a fuzzy random variable will be introduced as follows. Let $(\Omega,\mathcal{F},\mathcal{P})$ be a probability triple. Suppose that U is a random variable defined on this triple. Assume now that we perceive this random variable through a set of windows W_i, $i\in J$, with J a finite or countable set, each representing an interval of the real line, such that $W_i\cap W_j=\varnothing$ for $i\neq j$, and $\cup_{i\in J}W_i=R$. "Perceiving" the random variable through these windows means that for each ω we can only establish whether $U(\omega)\in W_i$ for some $i\in J$.

Let us define the function $I_i:R\to[0,1]$ as the characteristic function of the set W_i. Also let S be the space of all piecewise continuous functions mapping $R\to[0,1]$. We then define the perception of the random variable U, as described above, as the mapping $X:\Omega\to S$ given by

$$\omega\xrightarrow{X}X_\omega,$$

with $X_\omega=I_i$ if and only if $U(\omega)\in W_i$. This means that we associate with each $\omega\in\Omega$ not a real number $U(\omega)$, as in the case of an ordinary random variable, but a characteristic function X_ω, which is an element of S.

The map $X:\Omega\to S$ described above characterizes a special type of fuzzy random variable. The random variable U, of which this fuzzy random variable is a perception, is called an *original* of the fuzzy random variable. We note that corresponding to a given fuzzy random variable there may exist many originals.

At this point we generalize and define a fuzzy random variable as a map $\xi:\Omega\to F$, where F is the set of all fuzzy numbers. Denote the image of ω in F under ξ as $\xi(\omega)=(R,X_\omega,a_\omega)$, with $X_\omega\in S$ and $a_\omega:R\to P$. The map $X:\Omega\to S$, specified by

$$\omega\xrightarrow{X}X_\omega,$$

is required to be such that for each $\mu\in(0,1]$ both U_μ^* and U_μ^{**}, defined by

$$\begin{aligned}U_\mu^*(\omega)&=\inf\{x\in R\,|\,X_\omega(x)\geqslant\mu\},\\ U_\mu^{**}(\omega)&=\sup\{x\in R\,|\,X_\omega(x)\geqslant\mu\},\end{aligned}\tag{4.1}$$

are finite real-valued random variables defined on $(\Omega,\mathfrak{F},\mathfrak{P})$ satisfying

$$(\forall\omega\in\Omega)\qquad X_\omega\big(U_\mu^*(\omega)\big)\geqslant\mu,\quad X_\omega\big(U_\mu^{**}(\omega)\big)\geqslant\mu.\tag{4.2}$$

Finally, for each $\omega\in\Omega$ and each $x\in R$, $a_\omega(x)$ is the statement

$$a_\omega(x)=(\text{the original assumes the value } x \text{ at the point } \omega),\tag{4.3}$$

where we refer to the original random variable of which ξ is a fuzzy perception.

The requirement that for each $\mu\in(0,1]$ the quantities U_μ^* and U_μ^{**} are random variables imposes a measurability condition on the map ξ. The condition that for each $\mu\in(0,1]$ both U_μ^* and U_μ^{**} are finite random variables constitutes a restriction that from a practical point of view is not very serious. The same observation applies to the requirement imposed by (4.2).

The picture that has just been sketched of a fuzzy random variable is complicated by the disturbing thought that the probability space $(\Omega,\mathfrak{F},\mathfrak{P})$ on which a fuzzy random variable manifests itself may be a *reduction* of a richer probability space $(\tilde\Omega,\tilde{\mathfrak{F}},\tilde{\mathfrak{P}})$, the details of which are lost by the fuzzy perception. A reduction of $(\tilde\Omega,\tilde{\mathfrak{F}},\tilde{\mathfrak{P}})$ to $(\Omega,\mathfrak{F},\mathfrak{P})$ is obtained as follows. Let $\tilde{\mathfrak{A}}$ be a sub-sigma-algebra of $\tilde{\mathfrak{F}}$. Define an equivalence relation $\sim$ on $\tilde\Omega$ as follows:

$$\tilde\omega_1\sim\tilde\omega_2\quad\Leftrightarrow\quad\big[(\forall\tilde A\in\tilde{\mathfrak{A}})\;\tilde\omega_1\in\tilde A\;\Leftrightarrow\;\tilde\omega_2\in\tilde A\big].\tag{4.4}$$

Then we take Ω as the quotient set of $\tilde\Omega$ under the equivalence relation $\sim$, i.e., $\Omega=\tilde\Omega/\sim$. To define $\mathfrak{F}$ and $\mathfrak{P}$, let c be the canonical projection $c:\tilde\Omega\to\Omega$. Then we define $\mathfrak{F}$ as the class of subsets of Ω whose inverse images under c belong to $\tilde{\mathfrak{F}}$. The probability measure $\mathfrak{P}$ is defined as follows: if $A\in\mathfrak{F}$, then $\mathfrak{P}(A)=\tilde{\mathfrak{P}}\big(c^{-1}(A)\big)$. We shall call $(\Omega,\mathfrak{F},\mathfrak{P})$ the reduction of $(\tilde\omega,\tilde{\mathfrak{F}},\tilde{\mathfrak{P}})$ generated by $\tilde{\mathfrak{A}}\subset\tilde{\mathfrak{F}}$.

Clearly, $(\Omega, \mathcal{F}, \mathcal{P})$ is a space with less detail than $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}})$. $(\Omega, \mathcal{F}, \mathcal{P})$ may still be used to carry certain random variables originally defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}})$. Let $\tilde{U}$ be any $\tilde{\mathcal{Q}}$-measurable random variable defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}})$. We first prove that for any $\omega \in \Omega$, $\tilde{U}$ is constant on the set $c^{-1}(\omega)$. Suppose $\tilde{\omega}_1 \in c^{-1}(\omega)$ and $\tilde{\omega}_2 \in c^{-1}(\omega)$. Then $c(\tilde{\omega}_1) = c(\tilde{\omega}_2)$ and hence $\tilde{\omega}_1 \sim \tilde{\omega}_2$. Suppose $\tilde{U}(\tilde{\omega}_1) = x$. Then $\tilde{\omega}_1 \in \{\tilde{\omega} | \tilde{U}(\tilde{\omega}) = x\}$, and hence by the equivalence of $\tilde{\omega}_1$ and $\tilde{\omega}_2$ it follows that $\tilde{\omega}_2 \in \{\tilde{\omega} | \tilde{U}(\tilde{\omega}) = x\}$. As a result $\tilde{U}(\omega_2) = x = \tilde{U}(\tilde{\omega}_1)$, which proves that $\tilde{U}$ is constant on $c^{-1}(\omega)$.

We now define the random variable U on $(\Omega, \mathcal{F}, \mathcal{P})$ by $U(\omega) = \tilde{U}(c^{-1}(\omega))$. It is easy to verify that $\tilde{U}$ and U have the same probability distributions. Given any finite set of $\tilde{\mathcal{Q}}$-measurable random variables $\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_n$, all defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}})$, we may obtain a set of random variables $U_1, U_2, \ldots, U_n$, all defined on $(\Omega, \mathcal{F}, \mathcal{P})$ and having the same joint distribution as $\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_n$.

We thus see that given a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, there may exist a richer probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}})$ of which $(\Omega, \mathcal{F}, \mathcal{P})$ is a reduction. To construct such a richer space, we first assume that $(\Omega, \mathcal{F}, \mathcal{P})$ is minimal in the sense that it is its own reduction with respect to $\mathcal{F}$. We then introduce an auxiliary probability space $(\Omega', \mathcal{F}', \mathcal{P}')$. For later purposes it will be enough if we choose it rich enough so that it can carry any finite set of random variables with given joint distribution. We now let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}}) = (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathcal{P} \otimes \mathcal{P}')$, where $\mathcal{F} \otimes \mathcal{F}'$ is the smallest sigma algebra including all sets of the form $A \times A'$ with $A \in \mathcal{F}$ and $A' \in \mathcal{F}'$, and where $\mathcal{P} \otimes \mathcal{P}'$ is the product measure on $\mathcal{F} \otimes \mathcal{F}'$. It is not difficult to see that $(\Omega, \mathcal{F}, \mathcal{P})$ is the reduction of $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}})$ with respect to the sigma algebra $\tilde{\mathcal{Q}} \subset \tilde{\mathcal{F}}$ consisting of all cylinder sets of the form $\tilde{A} = A \times \Omega'$ with $A \in \mathcal{F}$.

Let ξ be a fuzzy random variable such that

$$\omega \xrightarrow{\xi} (R, X_\omega, a_\omega).$$

Denote by $\sigma(X)$ the sigma algebra of subsets of Ω generated by the random variables U_μ^*, $\mu \in (0,1]$, and U_μ^{**}, $\mu \in (0,1]$, as defined in (4.1). For brevity we refer to $\sigma(X)$ as the sigma algebra generated by X. We now account for the fact that the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ on which ξ manifests itself may be a reduction of a richer probability space by allowing any original of which ξ is a fuzzy perception to be a random variable on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}})$ rather than $(\Omega, \mathcal{F}, \mathcal{P})$. For later purposes the construction of $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}})$ as given above will be sufficient. We shall not admit all random variables defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}})$ as originals, however. Any original $\tilde{U}$ of ξ will be required to be measurable with respect to $\sigma(X) \otimes \mathcal{F}'$, meaning that in the Ω-direction $\tilde{U}$ has to be consistent with the available fuzzy information.

Thus the set $\tilde{\mathcal{X}}$ of all possible originals of ξ is defined as the set of all random variables defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}})$ that are $\sigma(X) \otimes \mathcal{F}'$-measurable. For any $\tilde{U} \in \tilde{\mathcal{X}}$, the acceptability that it is an original is given by the truth value of the statement $b(\tilde{U})$, where

$$b(\tilde{U})=(\tilde{U}\text{ is an original of }\xi)$$

$$=\bigwedge_{\substack{\omega\in\Omega\\ \omega'\in\Omega'}}(\text{the original assumes the value }\tilde{U}(\omega,\omega')\text{ at the point }(\omega,\omega'))$$

$$=\bigwedge_{\substack{\omega\in\Omega\\ \omega'\in\Omega'}}a_\omega(\tilde{U}(\omega,\omega')). \tag{4.5}$$

Hence, the acceptability that $\tilde{U}$ is an original is given by

$$t(b(\tilde{U}))=\inf_{\substack{\omega\in\Omega\\ \omega'\in\Omega'}}t\big(a_\omega(\tilde{U}(\omega,\omega'))\big)$$

$$=\inf_{\substack{\omega\in\Omega\\ \omega'\in\Omega'}}X_\omega(\tilde{U}(\omega,\omega')). \tag{4.6}$$

We have thus defined a fuzzy set $(\tilde{\mathfrak{X}},t(b(\cdot)),b)$ consisting of all possible originals of the fuzzy random variable in question. With a slight abuse of notation this fuzzy set will henceforth be indicated as $\mathbf{X}=(\tilde{\mathfrak{X}},X)$. Properties of fuzzy random variables such as its expectation and probabilities in connection with it will be defined as images of this fuzzy set under certain mappings. This is the subject of the next sections.

Because of the central role played by the fuzzy set $\mathbf{X}=(\tilde{\mathfrak{X}},X)$, we adopt in the following the convention of calling $\mathbf{X}$ a fuzzy random variable, as an alternative to calling the map ξ a fuzzy random variable.

In the sequel we sometimes have occasion to work with fuzzy random variables of the form $\mathbf{X}=(\mathfrak{X},X)$, with $\mathfrak{X}$ the set of all $\sigma(X)$-measurable random variables defined on $(\Omega,\mathfrak{F},\mathfrak{P})$. Such fuzzy random variables are called *reduced* fuzzy random variables.

A fuzzy random variable is called normal if for each $\omega\in\Omega$ there exists an $x\in R$ such that $X_\omega(x)=1$. In the sequel all fuzzy random variables will be assumed to be normal.

5. EXPECTATION OF FUZZY RANDOM VARIABLES

In this section we define the notion of the expectation of a fuzzy random variable $\mathbf{X}$, and discuss some of its properties. The expectation of $\mathbf{X}$ is defined as the fuzzy number $E\mathbf{X}$, which is the image of the fuzzy set $\mathbf{X}=(\tilde{\mathfrak{X}},X)$ under the mapping $E:\tilde{\mathcal{X}}\to R$ such that

$$\tilde{U} \overset{E}{\mapsto} E\tilde{U}.$$

In the latter expression, E indicates the usual mathematical expectation.

Denoting the membership function of the fuzzy number $E\mathbf{X}$ as $(E\mathbf{X})$ (a notation that will frequently be used), we may explicitly write $E\mathbf{X}=(R,(E\mathbf{X}))$, where

$$(E\mathbf{X})(x)=\sup_{\tilde{U}\in\tilde{\mathcal{X}}:E\tilde{U}=x}\ \inf_{\substack{\omega\in\Omega\\ \omega'\in\Omega'}} X_\omega(\tilde{U}(\omega,\omega')),\qquad x\in R. \tag{5.1}$$

A fuzzy random variable $\mathbf{X}$ is called *unimodal* if for each $\omega\in\Omega$, the membership function X_ω is unimodal. The following theorem shows that if $\mathbf{X}$ is unimodal, it is not necessary to extend $(\Omega,\mathcal{F},\mathcal{P})$ to $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathcal{P}})$ for the determination of $E\mathbf{X}$. Let $\mathcal{X}$ be the set of $\sigma(X)$-measurable random variables defined on $(\Omega,\mathcal{F},\mathcal{P})$.

THEOREM 5.1. *If* $\mathbf{X}$ *is unimodal, then*

$$(E\mathbf{X})(x)=\sup_{U\in\mathcal{X}:EU=x}\ \inf_{\omega\in\Omega} X_\omega(U(\omega)),\qquad x\in R. \tag{5.2}$$

Proof. Let $\varepsilon>0$ be an arbitrary positive real number. Then there always exists a random variable $\tilde{U}^*\in\tilde{\mathcal{X}}$ with $E\tilde{U}^*=x$ that achieves the supremum in (5.1) with an accuracy ε, i.e.,

$$(E\mathbf{X})(x)=\inf_{\substack{\omega\in\Omega\\ \omega'\in\Omega'}} X_\omega(\tilde{U}^*(\omega,\omega'))+\varepsilon. \tag{5.3}$$

Define the random variable

$$V^*(\omega)=\int\tilde{U}^*(\omega,\omega')\,d\mathcal{P}'(\omega'),\qquad \omega\in\Omega. \tag{5.4}$$

Clearly $V^*\in\mathcal{X}$ and $EV^*=x$. By the unimodality of X_ω we have

$$\inf_{\omega'\in\Omega'} X_\omega(\tilde{U}^*(\omega,\omega'))\leqslant X_\omega(V^*(\omega))\qquad\text{for all}\quad \omega\in\Omega. \tag{5.5}$$

It follows that

$$(E\mathbf{X})(x) \leqslant \inf_{\omega\in\Omega} X_\omega(V^*(\omega)) + \varepsilon \leqslant \sup_{V\in\mathcal{X}:EV=x} \inf_{\omega\in\Omega} X_\omega(V(\omega)) + \varepsilon. \tag{5.6}$$

On the other hand, by restricting $\tilde{U}$ in (5.1) to be such that $\tilde{U}(\omega,\omega') = V(\omega)$ with $V\in\mathcal{X}$, we clearly have

$$(E\mathbf{X})(x) \geqslant \sup_{V\in\mathcal{X}:EV=x} \inf_{\omega\in\Omega} X_\omega(V(\omega)). \tag{5.7}$$

Combining, it follows that

$$\sup_{V\in\mathcal{X}:EV=x} \inf_{\omega\in\Omega} X_\omega(V(\omega)) \leqslant (E\mathbf{X})(x) \leqslant \sup_{V\in\mathcal{X}:EV=x} \inf_{\omega\in\Omega} X_\omega(V(\omega)) + \varepsilon. \tag{5.8}$$

Since ε may be chosen arbitrarily small, the proof of the theorem follows. ■

The following theorem shows that when determining expectations of fuzzy random variables, there is no loss of generality in restricting to unimodal fuzzy random variables. For a given fuzzy random variable $\mathbf{X}$, define for each $\omega\in\Omega$,

$$\overline{X}_\omega(x) = \sup_{u,v\in R:u\leqslant x\leqslant v} \min[X_\omega(u), X_\omega(v)], \qquad x\in R. \tag{5.9}$$

It is not difficult to see that $\overline{X}_\omega$ is unimodal, and that if X_ω is unimodal, then $\overline{X}_\omega = X_\omega$. Let us define the reduced fuzzy random variable $\overline{\mathbf{X}} = (\mathcal{X}, \overline{X})$.

THEOREM 5.2 $E\mathbf{X} = E\overline{\mathbf{X}}$.

Proof. Define for each $\mu\in[0,1]$ the following subsets of R:

$$\begin{aligned} C_\mu &= \left\{x\in R \mid (\exists \tilde{U}\in\tilde{\mathcal{X}})\ E\tilde{U} = x,\ (\forall\omega,\omega')\ X_\omega(\tilde{U}(\omega,\omega')) \geqslant \mu\right\}, \\ \overline{C}_\mu &= \left\{x\in R \mid (\exists \tilde{U}\in\tilde{\mathcal{X}})\ E\tilde{U} = x,\ (\forall\omega,\omega')\ \overline{X}_\omega(\tilde{U}(\omega,\omega')) \geqslant \mu\right\}. \end{aligned} \tag{5.10}$$

We observe that $C_0 = \overline{C}_0 = R$. The membership functions $(E\mathbf{X})$ and $(E\overline{\mathbf{X}})$ may respectively be obtained from the families of sets C_μ, $\mu\in[0,1]$, and $\overline{C}_\mu$, $\mu\in[0,1]$, by

$$\begin{aligned} (E\mathbf{X})(x) &= \sup\{\mu\in[0,1] \mid x\in C_\mu\}, \\ (E\overline{\mathbf{X}})(x) &= \sup\{\mu\in[0,1] \mid x\in \overline{C}_\mu\}. \end{aligned} \tag{5.11}$$

We shall prove $(\forall\mu\in[0,1])\ C_\mu=\overline{C}_\mu$, from which the proof of the theorem follows. Let for each $\mu\in[0,1]$

$$a_\mu=\inf(C_\mu),\quad b_\mu=\sup(C_\mu),\quad \bar{a}_\mu=\inf\left(\overline{C}_\mu\right),\quad \bar{b}_\mu=\sup\left(\overline{C}_\mu\right). \tag{5.12}$$

It is noted that $a_0=\bar{a}_0=-\infty$, $b_0=\bar{b}_0=\infty$. Next define for each $\mu\in(0,1]$ and each $\omega\in\Omega$

$$\begin{aligned} U_\mu^*(\omega)&=\inf\{x\in R\,|\,X_\omega(x)\geqslant\mu\}, & U_\mu^{**}(\omega)&=\sup\{x\in R\,|\,X_\omega(x)\geqslant\mu\},\\ \overline{U}_\mu^*(\omega)&=\inf\left\{x\in R\,|\,\overline{X}_\omega(x)\geqslant\mu\right\}, & \overline{U}_\mu^{**}(\omega)&=\sup\left\{x\in R\,|\,\overline{X}_\omega(x)\geqslant\mu\right\}. \end{aligned} \tag{5.13}$$

By the definition of $\overline{X}$ we have $U_\mu^*=\overline{U}_\mu^*$ and $U_\mu^{**}=\overline{U}_\mu^{**}$ and hence for each $\mu\in(0,1]$

$$a_\mu=EU_\mu^*=E\overline{U}_\mu^*=\bar{a}_\mu,\qquad b_\mu=EU_\mu^{**}=E\overline{U}_\mu^{**}=\bar{b}_\mu. \tag{5.14}$$

Furthermore, by the assumption (4.2) we have $a_\mu\in C_\mu$ as well as $b_\mu\in C_\mu$. We prove that $C_\mu=[a_\mu,b_\mu]$. Let $0<\sigma<1$, and $A\in\mathcal{F}'$ be a subset of Ω' such that $\mathcal{P}'(A)=\sigma$. Define the random variable $\hat{U}\in\tilde{\mathfrak{X}}$ such that

$$\hat{U}(\omega,\omega')=\begin{cases} U_\mu^*(\omega) & \text{for}\quad \omega'\in A,\\ U_\mu^{**}(\omega) & \text{for}\quad \omega'\in A^c. \end{cases} \tag{5.15}$$

Then $E\hat{U}=\sigma EU_\mu^*+(1-\sigma)EU_\mu^{**}=\sigma a_\mu+(1-\sigma)b_\mu\in[a_\mu,b_\mu]$, while

$$(\forall\omega,\omega')\quad X_\omega(\hat{U}(\omega,\omega'))\geqslant\min\left[X_\omega(U_\mu^*(\omega)),X_\omega(U_\mu^{**}(\omega))\right]\geqslant\mu. \tag{5.16}$$

It follows that $\sigma a_\mu+(1-\sigma)b_\mu\in C_\mu$, and hence that $C_\mu=[a_\mu,b_\mu]$ for each $\mu\in[0,1]$. Similarly we may prove that $\overline{C}_\mu=[\bar{a}_\mu,\bar{b}_\mu]$. Since $\bar{a}_\mu=a_\mu$ *and* $\bar{b}_\mu=b_\mu$, the proof that $(\forall\mu\in[0,1])\ C_\mu=\overline{C}_\mu$ is complete and the theorem is proved. ■

The result expressed by the preceding theorem may be viewed as follows. The map $E:\tilde{\mathfrak{X}}\to R$ may be decomposed into two maps as follows. Let $\tilde{U}\in\tilde{\mathfrak{X}}$. Then by writing

$$E\tilde{U}=\int d\mathcal{P}(\omega)\int\tilde{U}(\omega,\omega')\,d\mathcal{P}'(\omega'), \tag{5.17}$$

we see that $E\tilde{U}$ is the composition of the following two maps:

(i) $\tilde{\mathfrak{X}} \to \mathfrak{X}$ specified by $\tilde{U} \mapsto \bar{U}$ with $\bar{U}(\omega) = \int \tilde{U}(\omega,\omega')\,d\mathcal{P}'(\omega')$,
(ii) $\mathfrak{X} \to R$ specified by $\bar{U} \mapsto E\bar{U}$.

The image of $\mathbf{X}$ under the first map is $\bar{\mathbf{X}}$; the image of $\bar{\mathbf{X}}$ under the second map is $E\bar{\mathbf{X}}$. By Theorem 3.1, the image $E\mathbf{X}$ of $\mathbf{X}$ under the composite map (5.17) is $E\bar{\mathbf{X}}$, which is exactly the content of Theorem 5.2.

The next theorem follows as a corollary of the proof of Theorem 5.2.

THEOREM 5.3. *$E\bar{\mathbf{X}}$ is unimodal.*

Proof. It follows from (5.11) and from the fact that C_μ decreases monotonically with increasing μ that $\{x \in R | (E\mathbf{X})(x) > \mu\} = \{x \in R | \sup\{\mu' \in [0,1] | x \in [a_\mu, b_\mu]\} > \mu\} = [a_\mu, b_\mu]$. Evidently this set is convex and hence $E\mathbf{X}$ is unimodal. ■

Later we shall encounter expectations of products of fuzzy random variables. If $\mathbf{X} = (\tilde{\mathfrak{X}}, X)$ and $\mathbf{Y} = (\tilde{\mathfrak{Y}}, Y)$ are fuzzy random variables, we define $E\mathbf{XY}$ as the fuzzy number that is the image of the product fuzzy set $\mathbf{X}\times\mathbf{Y} = (\tilde{\mathfrak{X}} \times \tilde{\mathfrak{Y}}, X \times Y)$ in R under the mapping $(\tilde{U}, \tilde{V}) \mapsto E\tilde{U}\tilde{V}$. This means that the fuzzy number $E\mathbf{XY} = (R, (E\mathbf{XY}))$ has membership function

$$(E\mathbf{XY})(z) = \sup_{\tilde{U} \in \tilde{\mathfrak{X}},\, \tilde{V} \in \tilde{\mathfrak{Y}} : E\tilde{U}\tilde{V} = z} \; \inf_{\substack{\omega \in \Omega \\ \omega' \in \Omega'}} \min\left[X_\omega(\tilde{U}(\omega,\omega')), Y_\omega(\tilde{V}(\omega,\omega'))\right], \tag{5.18}$$

for all $z \in R$, since the pair $(\tilde{U}, \tilde{V}) \in \tilde{\mathfrak{X}} \times \tilde{\mathfrak{Y}}$ has degree of membership

$$\min\left[\inf_{\substack{\omega \in \Omega \\ \omega' \in \Omega'}} X_\omega(\tilde{U}(\omega,\omega')),\ \inf_{\substack{\omega \in \Omega \\ \omega' \in \Omega'}} Y_\omega(\tilde{V}(\omega,\omega'))\right] = \inf_{\substack{\omega \in \Omega \\ \omega' \in \Omega'}} \min\left[X_\omega(\tilde{U}(\omega,\omega')), Y_\omega(\tilde{V}(\omega,\omega'))\right] \tag{5.19}$$

in the fuzzy set $\mathbf{X}\times\mathbf{Y}$.

A fuzzy random variable $\mathbf{X}$ is called *nonnegative* if $X_\omega(x) = 0$ for $x < 0$ and all $\omega \in \Omega$. For $\mathbf{X} = (\tilde{\mathfrak{X}}, X)$ and $\mathbf{Y} = (\tilde{\mathfrak{Y}}, Y)$ given fuzzy random variables with "unimodalized" versions $\bar{\mathbf{X}}$ and $\bar{\mathbf{Y}}$, respectively [see (5.9)], the following result may be established.

THEOREM 5.4. *If* **X** *and* **Y** *are nonnegative, then*

$$(E\mathbf{XY})(z)=\sup_{U\in\mathfrak{X},\,V\in\mathfrak{Y}:EUV=z}\ \inf_{\omega\in\Omega}\min\left[\bar{X}_\omega(U(\omega)),\bar{Y}_\omega(V(\omega))\right],\qquad z\in R. \tag{5.20}$$

Proof. For each $\mu\in[0,1]$ we define the sets

$$\begin{aligned}C_\mu=\Big\{z\in R|(\exists\tilde{U}\in\tilde{\mathfrak{X}},\ \tilde{V}\in\tilde{\mathfrak{Y}})\ E\tilde{U}\tilde{V}=z,\\ (\forall\omega,\omega')\ X_\omega(\tilde{U}(\omega,\omega'))\geqslant\mu,\ Y_\omega(\tilde{V}(\omega,\omega'))\geqslant\mu\Big\},\\ \bar{C}_\mu=\Big\{z\in R|(\exists U\in\mathfrak{X},\ V\in\mathfrak{Y})\ EUV=z,\\ (\forall\omega)\ \bar{X}_\omega(U(\omega))\geqslant\mu,\ \bar{Y}_\omega(V(\omega))\geqslant\mu\Big\}.\end{aligned} \tag{5.21}$$

We observe that $C_0=\bar{C}_0=R$. The families of sets $C_.$ and $\bar{C}_.$ are respectively related to $E\mathbf{XY}$ and the right-hand side of (5.20) by

$$(E\mathbf{XY})(z)=\sup\{\mu\in[0,1]|z\in C_\mu\},$$

$$\sup_{U\in\mathfrak{X},V\in\mathfrak{Y}:EUV=z}\ \inf_{\omega\in\Omega}\min\left[X_\omega(U(\omega)),Y_\omega(V(\omega))\right]=\sup\{\mu\in[0,1]|z\in\bar{C}_\mu\}. \tag{5.22}$$

Let, for each $\mu\in[0,1]$,

$$a_\mu=\inf(C_\mu),\quad b_\mu=\sup(C_\mu),\quad \bar{a}_\mu=\inf(\bar{C}_\mu),\quad \bar{b}_\mu=\sup(\bar{C}_\mu). \tag{5.23}$$

Evidently $a_0=\bar{a}_0=-\infty$, $b_0=\bar{b}_0=\infty$. Defining for each $\mu\in(0,1]$ and each $\omega\in\Omega$

$$\begin{aligned}U_\mu^*(\omega)&=\inf\{x\in R|X_\omega(x)\geqslant\mu\}, & U_\mu^{**}(\omega)&=\sup\{x\in R|X_\omega(x)\geqslant\mu\},\\ V_\mu^*(\omega)&=\inf\{y\in R|Y_\omega(y)\geqslant\mu\}, & V_\mu^{**}(\omega)&=\sup\{y\in R|Y_\omega(y)\geqslant\mu\},\end{aligned} \tag{5.24}$$

and similarly $\bar{U}_\mu^*$, $\bar{U}_\mu^{**}$, $\bar{V}_\mu^*$, *and* $\bar{V}_\mu^{**}$ by replacing respectively X with $\bar{X}$ and Y with $\bar{Y}$, we have $U_\mu^*=\bar{U}_\mu^*$, $U_\mu^{**}=\bar{U}_\mu^{**}$, $V_\mu^*=\bar{V}_\mu^*$, $V_\mu^{**}=\bar{V}_\mu^{**}$, and hence for each

$\mu\in(0,1]$, $a_\mu=EU_\mu^* V_\mu^*=E\bar{U}_\mu^*\bar{V}_\mu^*=\bar{a}_\mu$, $b_\mu=EU_\mu^{**}V_\mu^{**}=E\bar{U}_\mu^{**}\bar{V}_\mu^{**}=\bar{b}_\mu$. Furthermore, by the assumption (4.2) we have $a_\mu\in C_\mu$ as well as $b_\mu\in C_\mu$. We prove that $C_\mu=[a_\mu,b_\mu]$. Let $c\in[a_\mu,b_\mu]$, and let $A\in\mathcal{F}'$ and $B\in\mathcal{F}'$ be subsets of Ω' such that $\mathcal{P}'(A)=\lambda$, $\mathcal{P}'(B)=\sigma$, $\mathcal{P}'(A\cap B)=\lambda\sigma$ with $\lambda\in[0,1]$ and $\sigma\in[0,1]$. We define the random variables $\hat{U}\in\tilde{\mathfrak{X}}$ and $\hat{V}\in\tilde{\mathfrak{Y}}$ as

$$\hat{U}(\omega,\omega')=\begin{cases}U_\mu^*(\omega) & \text{if} \quad \omega'\in A,\\ U_\mu^{**}(\omega) & \text{if} \quad \omega'\in A^c,\end{cases}$$
$$\hat{V}(\omega,\omega')=\begin{cases}V_\mu^*(\omega) & \text{if} \quad \omega'\in B,\\ V_\mu^{**}(\omega) & \text{if} \quad \omega\in B^c.\end{cases} \tag{5.25}$$

Then by the nonnegativity of U_μ^*, U_μ^{**}, V_μ^*, and V_μ^{**} we can always choose λ and σ such that

$$E\hat{U}\hat{V}=\lambda\sigma EU_\mu^* V_\mu^*+\lambda(1-\sigma)EU_\mu^* V_\mu^{**}+(1-\lambda)\sigma EU_\mu^{**}V_\mu^*+$$
$$(1-\lambda)(1-\sigma)EU_\mu^{**}V_\mu^{**}=c. \tag{5.26}$$

Furthermore

$$(\forall\omega,\omega')\quad X_\omega(\hat{U}(\omega,\omega'))\geqslant\min\left[X_\omega(U_\mu^*(\omega)),X_\omega(U_\mu^{**}(\omega))\right]\geqslant\mu,$$
$$Y_\omega(\hat{V}(\omega,\omega'))\geqslant\min\left[Y_\omega(V_\mu^*(\omega)),Y_\omega(V_\mu^{**}(\omega))\right]\geqslant\mu. \tag{5.27}$$

It follows that $c\in C_\mu$, and hence that $(\forall\mu\in[0,1])$ $C_\mu=[a_\mu,b_\mu]$. We next prove that $(\forall\mu\in[0,1])$ $\bar{C}_\mu=[\bar{a}_\mu,\bar{b}_\mu]=[a_\mu,b_\mu]=C_\mu$, which by (5.22) proves the theorem. Let $c\in[\bar{a}_\mu,\bar{b}_\mu]$, and choose $\lambda\in[0,1]$ and $\sigma\in[0,1]$ such that $E\hat{U}\hat{V}=c$, where $\hat{U}\in\mathfrak{X}$, $\hat{V}\in\mathfrak{Y}$ such that $\hat{U}=\lambda\bar{U}_\mu^*+(1-\lambda)\bar{U}_\mu^{**}$, $\hat{V}=\sigma\bar{V}_\mu^*+(1-\sigma)\bar{V}_\mu^{**}$. By the unimodality of $\bar{X}$ and $\bar{Y}$ we have

$$(\forall\omega)\quad \bar{X}_\omega(\hat{U}(\omega))\geqslant\min\left[\bar{X}_\omega\left(\bar{U}_\mu^*(\omega)\right),\bar{X}_\omega\left(\bar{U}_\mu^{**}(\omega)\right)\right]\geqslant\mu,$$
$$\bar{Y}_\omega(\hat{V}(\omega))\geqslant\min\left[\bar{Y}_\omega\left(\bar{V}_\mu^*(\omega)\right),\bar{Y}_\omega\left(\bar{V}_\mu^{**}(\omega)\right)\right]\geqslant\mu, \tag{5.28}$$

which shows that $c\in\bar{C}_\mu$. This concludes the proof that $(\forall\mu\in[0,1])$ $\bar{C}_\mu=[\bar{a}_\mu,b_\mu]$ and at the same time the proof of the theorem. ■

6. INDEPENDENT FUZZY RANDOM VARIABLES

The fuzzy random variables $\mathbf{X}_i=(R,X^i)$, $i=1,2,\ldots,n$, are said to be independent if the sigma algebras $\sigma(X^i)$, $i=1,2,\ldots,n$, of subsets of Ω successively generated by X^i, $i=1,2,\ldots,n$, are independent. At this point the following property of independent fuzzy random variables can be established. Other properties will follow in subsequent sections.

THEOREM 6.1. *If the two nonnegative fuzzy random variables* $\mathbf{X}$ *and* $\mathbf{Y}$ *are independent, then* $E\mathbf{XY}=E\mathbf{X}E\mathbf{Y}$.

Proof. By Theorem 5.4, we may assume without loss of generality that $\mathbf{X}$ and $\mathbf{Y}$ are unimodal. By the independence of $\mathbf{X}$ and $\mathbf{Y}$, any random variables $U\in\mathfrak{X}$ and $V\in\mathfrak{Y}$ are independent. It follows from Theorem 5.4 that

$$(E\mathbf{XY})(z)=\sup_{U\in\mathfrak{X},\,V\in\mathfrak{Y}:EUEV=z}\ \inf_{\omega\in\Omega}\min\left[X_\omega(U(\omega)),Y_\omega(V(\omega))\right]$$

$$=\sup_{\substack{U\in\mathfrak{X},\,V\in\mathfrak{Y},\,u,v\in R:\\ EU=u,\,EV=v,\,uv=z}}\ \inf_{\omega\in\Omega}\min\left[X_\omega(U(\omega)),Y_\omega(V(\omega))\right]$$

$$=\sup_{u,v\in R:uv=z}\ \sup_{U\in\mathfrak{X},V\in\mathfrak{Y}:EU=u,\,EV=v}\ \inf_{\omega\in\Omega}\min\left[X_\omega(U(\omega)),Y_\omega(V(\omega))\right]$$

$$=\sup_{u,v\in R:uv=z}\min\left[\sup_{U\in\mathfrak{X}:EU=u}\ \inf_{\omega\in\Omega}X_\omega(U(\omega)),\ \sup_{V\in\mathfrak{Y}:EV=v}\ \inf_{\omega\in\Omega}Y_\omega(V(\omega))\right]$$

$$=\sup_{u,v\in R:uv=z}\min\left[(E\mathbf{X})(u),(E\mathbf{Y})(v)\right]$$

$$=(E\mathbf{X}E\mathbf{Y})(z), \tag{6.1}$$

which concludes the proof. ■

7. FUZZY PROBABILITY AND FUZZY EVENTS

Let $\mathbf{X}$ be a fuzzy random variable, and A a Borel set in R. Then the (fuzzy) probability $\Pr(\mathbf{X}\in A)$ is defined as the image of $\mathbf{X}=(\tilde{\mathfrak{X}},X)$ in R under the map specified by $\tilde{U}\mapsto\tilde{\mathcal{P}}(\tilde{U}\in A)$. Hence we have $\Pr(\mathbf{X}\in A)=\left(R,(\Pr(\mathbf{X}\in A))\right)$, with

$$(\Pr(\mathbf{X}\in A))(p)=\sup_{\tilde{U}\in\tilde{\mathfrak{X}}:\tilde{\mathcal{P}}(\tilde{U}\in A)=p}\ \inf_{\substack{\omega\in\Omega\\ \omega'\in\Omega'}}X_\omega(\tilde{U}(\omega,\omega')). \tag{7.1}$$

Since $0\leqslant\tilde{\mathcal{P}}(\tilde{U}\in A)\leqslant 1$, $(\Pr(\mathbf{X}\in A))(p)=0$ for $p\notin[0,1]$. It will be helpful to consider $\Pr(\mathbf{X} \in A)$ as the image of $\mathbf{X}$ under a composition of two maps. To this end, we write

$$\tilde{\mathcal{P}}(\tilde{U} \in A) = \int d\mathcal{P}(\omega) \int_{\omega' : \tilde{U}(\omega, \omega') \in A} d\mathcal{P}'(\omega'), \tag{7.2}$$

and specify the first map: $\tilde{\mathcal{X}} \to \mathcal{X}$ by

$$\tilde{U} \mapsto \int_{\omega' : \tilde{U}(\cdot, \omega') \in A} d\mathcal{P}'(\omega'). \tag{7.3}$$

The second map: $\mathcal{X} \to R$ is specified by

$$U \mapsto EU. \tag{7.4}$$

THEOREM 7.1. *The image of* $\mathbf{X} = (\tilde{\mathcal{X}}, X)$ *in* $\mathcal{X}$ *under the map*: $\tilde{\mathcal{X}} \to \mathcal{X}$ *given by*

$$\tilde{U} \mapsto \int_{\omega' : \tilde{U}(\cdot, \omega') \in A} d\mathcal{P}'(\omega') \tag{7.5}$$

is $\mathbf{I}^{\mathbf{X} \in A} = (\mathcal{X}, I^{\mathbf{X} \in A})$, *where*

$$I_\omega^{\mathbf{X} \in A}(\pi) = \sup_{U' \in \mathcal{X}' : \mathcal{P}'(U' \in A) = \pi} \ \inf_{\omega' \in \Omega'} X_\omega(U'(\omega')), \qquad \pi \in R, \tag{7.6}$$

with $\mathcal{X}'$ *the set of all random variables defined on* $(\Omega', \mathcal{F}', \mathcal{P}')$.

Proof. Let $V \in \mathcal{X}$. Then the degree of membership of V in the image of $\mathbf{X}$ under the map (7.5) is

$$\begin{aligned} &\sup_{\tilde{U} \in \tilde{\mathcal{X}} : (\forall \omega)\ \mathcal{P}'(\tilde{U}(\omega, \cdot) \in A) = V(\omega)} \ \inf_{\substack{\omega \in \Omega \\ \omega' \in \Omega'}} X_\omega(\tilde{U}(\omega, \omega')) \\ &= \inf_{\omega \in \Omega} \ \sup_{U' \in \mathcal{X}' : \mathcal{P}'(U' \in A) = V(\omega)} \ \inf_{\omega' \in \Omega} X_\omega(U'(\omega')), \end{aligned} \tag{7.7}$$

which proves the theorem. ■

$\mathbf{I}^{\mathbf{X} \in A}$ is called the *indicator function* of the fuzzy event $\mathbf{X} \in A$ for reasons to be explained later. For fixed $\omega \in \Omega$ and $\pi \in [0, 1]$ the number $I_\omega^{\mathbf{X} \in A}(\pi)$ indicates the acceptability that a fraction π of the point ω belongs to the fuzzy event $\mathbf{X} \in A$. Clearly $I_\omega^{\mathbf{X} \in A}(\pi) = 0$ for π not in $[0, 1]$. In general, a fuzzy random variable $\mathbf{I}$ defined on $\mathfrak{X}$ (rather than $\tilde{\mathfrak{X}}$) with a unimodal membership function I such that $I_\omega(\pi)=0$ for $\pi \notin [0,1]$ will be said to be the indicator function of a fuzzy event.

THEOREM 7.2. *Define*

$$r'_A(\omega)=\sup_{x\in A} X_\omega(x), \qquad r''_A(\omega)=\sup_{x\in A^c} X_\omega(x). \tag{7.8}$$

Then the indicator function $\mathbf{I}^{\mathbf{X}\in A}$ *may be specified by*

$$I_\omega^{\mathbf{X}\in A}(\pi)=\begin{cases} r''_A(\omega) & \text{if } \pi=0, \\ \min[r'_A(\omega), r''_A(\omega)] & \text{if } 0<\pi<1, \\ r'_A(\omega) & \text{if } \pi=1, \end{cases} \tag{7.9}$$

and hence is unimodal.

Proof. The result is evident if we write

$$I_\omega^{\mathbf{X}\in A}(\pi)=\sup_{U'\in\mathfrak{X}':\mathcal{P}'(U'\in A)=\pi} \min\Big[\inf_{\omega':U'(\omega')\in A} X_\omega(U'(\omega')), \inf_{\omega':U'(\omega')\in A^c} X_\omega(U'(\omega'))\Big], \tag{7.10}$$

and consider the cases $\pi=0$, $0<\pi<1$, and $\pi=1$ separately. ∎

We are now in a position to explain why $\mathbf{I}^{\mathbf{X}\in A}$ is called an indicator function. Suppose that $\mathbf{X}=(\tilde{\mathfrak{X}},X)$ characterizes an ordinary, nonfuzzy random variable, i.e., there exists a random variable $Z\in\mathfrak{X}$, defined on $(\Omega,\mathcal{F},\mathcal{P})$, such that

$$X_\omega(x)=\begin{cases} 1 & \text{if } x=Z(\omega), \\ 0 & \text{if } x\neq Z(\omega). \end{cases} \tag{7.11}$$

Then using Theorem 7.2 it is easily found that

$$\begin{aligned} I_\omega^{\mathbf{X}\in A}(\pi)&=\begin{Bmatrix} 1 & \text{if } \pi=1 \\ 0 & \text{otherwise} \end{Bmatrix} \quad \text{if } Z(\omega)\in A, \\ I_\omega^{\mathbf{X}\in A}(\pi)&=\begin{Bmatrix} 1 & \text{if } \pi=0 \\ 0 & \text{otherwise} \end{Bmatrix} \quad \text{if } Z(\omega)\notin A. \end{aligned} \tag{7.12}$$

In this case $\mathbf{I}^{\mathbf{X}\in A}$ characterizes an ordinary indicator function.

THEOREM 7.3. $\Pr(\mathbf{X}\in A)=E\mathbf{I}^{\mathbf{X}\in A}$.

Proof. Since $\tilde{\mathcal{P}}(\tilde{U}\in A)$ is the composition of the map $:\tilde{\mathfrak{X}}\to\mathfrak{X}$ defined by (7.3) and the map $\mathfrak{X}\to R$ defined by (7.4), the proof of Theorem 7.3 follows from Theorem 3.1. ∎

Suppose that $\mathbf{X}=(\tilde{\mathfrak{X}},X)$ and $\mathbf{Y}=(\tilde{\mathfrak{Y}},Y)$ are two fuzzy random variables, and let A and B be Borel sets in R. Then we define the joint probability $\Pr(\mathbf{X}\in A,\ \mathbf{Y}\in B)$ as the fuzzy number with membership function

$$(\Pr(\mathbf{X}\in A,\mathbf{Y}\in B))(p)$$
$$=\sup_{\tilde{U}\in\tilde{\mathfrak{X}},\tilde{V}\in\tilde{\mathfrak{Y}}:\tilde{\mathcal{P}}(\tilde{U}\in A,\tilde{V}\in B)=p}\ \inf_{\substack{\omega\in\Omega\\ \omega'\in\Omega'}}\min\left[X_\omega(\tilde{U}(\omega,\omega'),Y_\omega(\tilde{V}(\omega,\omega'))\right],\tag{7.13}$$

for $p\in[0,1]$. Evidently, $\Pr(\mathbf{X}\in A,\ Y\in B)$ is defined as the image of $\mathbf{X}\times\mathbf{Y}$ in R under the map $(\tilde{U},\tilde{V})\mapsto\tilde{\mathcal{P}}(\tilde{U}\in A,\ \tilde{V}\in B)$. Let us rewrite

$$\tilde{\mathcal{P}}(\tilde{U}\in A,\ \tilde{V}\in B)=\int d\mathcal{P}(\omega)\int_{\omega':\tilde{U}\in A,\ \tilde{V}\in B}d\mathcal{P}'(\omega').\tag{7.14}$$

This shows that the map $(\tilde{U},\tilde{V})\mapsto\tilde{\mathcal{P}}(\tilde{U}\in A,\ \tilde{V}\in B)$ is the composition of the two following maps. The first map maps $\tilde{\mathfrak{X}}\times\tilde{\mathfrak{Y}}$ into $\mathfrak{Z}$, which is the class of random variables defined on $(\Omega,\mathfrak{F},\mathcal{P})$ that are $\sigma(X,Y)$-measurable, and is given by

$$(\tilde{U},\tilde{V})\mapsto\int_{\omega':\tilde{U}\in A,\ \tilde{V}\in B}d\mathcal{P}'(\omega').\tag{7.15}$$

Here $\sigma(X,Y)\subset\mathfrak{F}$ is the smallest sigma algebra containing both $\sigma(X)$ and $\sigma(Y)$. The second map maps $\mathfrak{Z}$ into R, and is specified by $Z\mapsto EZ$.

The image of the product fuzzy set $\mathbf{X}\times\mathbf{Y}$ in $\mathfrak{Z}$ under the first map is the indicator function $\mathbf{I}^{\mathbf{X}\in A,\,\mathbf{Y}\in B}=(\mathfrak{Z},I^{\mathbf{X}\in A,\,\mathbf{Y}\in B})$, where

$$I_\omega^{\mathbf{X}\in A,\,\mathbf{Y}\in B}(\pi)$$
$$=\sup_{U',V'\in\mathfrak{X}':\mathcal{P}'(U'\in A,\,V'\in B)=\pi}\ \inf_{\omega'\in\Omega'}\min\left[X_\omega(U'(\omega')),Y_\omega(V'(\omega'))\right].\tag{7.16}$$

Since $\Pr(\mathbf{X}\in A,\ \mathbf{Y}\in B)$ is the image of $\mathbf{I}^{\mathbf{X}\in A,\,\mathbf{Y}\in B}$ under the map $\mathfrak{Z}\to R$ defined by $Z \mapsto EZ$, we have

$$\Pr(\mathbf{X} \in A, \mathbf{Y} \in B) = E\mathbf{I}^{\mathbf{X} \in A, \mathbf{Y} \in B}. \tag{7.17}$$

THEOREM 7.4. *Define $r'_A(\omega)$ and $r''_A(\omega)$ as in Theorem 6.1, and let*

$$s'_B(\omega) = \sup_{y \in B} Y_\omega(y), \qquad s''_B(\omega) = \sup_{y \in B^c} Y_\omega(y). \tag{7.18}$$

Then

$$I_\omega^{\mathbf{X} \in A, \mathbf{Y} \in B}(\pi) = \begin{cases} \max\left[r''_A(\omega), s''_B(\omega) \right] & \text{if} \quad \pi = 0, \\ \min\left[r'_A(\omega), s'_B(\omega), \max\left[r''_A(\omega), s''_B(\omega) \right] \right] & \text{if} \quad 0 < \pi < 1, \\ \min\left[r'_A(\omega), s'_B(\omega) \right] & \text{if} \quad \pi = 1. \end{cases} \tag{7.19}$$

Proof. The proof is similar to that of Theorem 7.2. ■

THEOREM 7.5. $E\mathbf{I}^{\mathbf{X} \in A, \mathbf{Y} \in B} = E\mathbf{I}^{\mathbf{X} \in A}\mathbf{I}^{\mathbf{Y} \in B}$.

Proof. We have respectively

$$\begin{aligned} (E\mathbf{I}^{\mathbf{X} \in A, \mathbf{Y} \in B})(z) &= \sup_{z \in \mathfrak{Z} : EZ = z} \; \inf_{\omega \in \Omega} I_\omega^{\mathbf{X} \in A, \mathbf{Y} \in B}(Z(\omega)), \\ (E\mathbf{I}^{\mathbf{X} \in A}\mathbf{I}^{\mathbf{Y} \in B})(z) &= \sup_{U \in \mathfrak{X}, V \in \mathfrak{Y} : EUV = z} \; \inf_{\omega \in \Omega} \min\left[I_\omega^{\mathbf{X} \in A}(U(\omega)), I_\omega^{\mathbf{Y} \in B}(V(\omega)) \right]. \end{aligned} \tag{7.20}$$

Therefore, consider the sets, defined for each $\mu \in [0, 1]$,

$$\begin{aligned} C_\mu &= \left\{ z \in R \,|\, (\exists Z \in \mathfrak{Z})\; EZ = z,\; (\forall \omega)\; I_\omega^{\mathbf{X} \in A, \mathbf{Y} \in B}(Z(\omega)) \geq \mu \right\}, \\ D_\mu &= \left\{ z \in R \,|\, (\exists U \in \mathfrak{X},\; V \in \mathfrak{Y})\; EUV = z, \right. \\ &\qquad \left. (\forall \omega)\; I_\omega^{\mathbf{X} \in A}(U(\omega)) \geq \mu,\; I_\omega^{\mathbf{Y} \in B}(V(\omega)) \geq \mu \right\}. \end{aligned} \tag{7.21}$$

The theorem has been proved if we can show that $(\forall \mu \in [0, 1])\; C_\mu = D_\mu$. Define for each $\mu \in (0, 1]$ the random variables $Z_\mu^*, Z_\mu^{**} \in \mathfrak{Z}$, $U_\mu^*, U_\mu^{**} \in \mathfrak{X}$, and V_μ^*, V_μ^{**}

$\in \mathcal{Y}$ as follows:

$$
\begin{aligned}
U_\mu^*(\omega) &= \inf\left\{x \in R \mid I_\omega^{X \in A}(x) \geqslant \mu\right\},\\
U_\mu^{**}(\omega) &= \sup\left\{x \in R \mid I_\omega^{X \in A}(x) \geqslant \mu\right\},\\
V_\mu^*(\omega) &= \inf\left\{y \in R \mid I_\omega^{Y \in B}(y) \geqslant \mu\right\},\\
V_\mu^{**}(\omega) &= \sup\left\{y \in R \mid I_\omega^{Y \in B}(y) \geqslant \mu\right\},\\
Z_\mu^*(\omega) &= \inf\left\{z \in R \mid I_\omega^{X \in A, Y \in B}(z) \geqslant \mu\right\},\\
Z_\mu^{**}(\omega) &= \sup\left\{z \in R \mid I_\omega^{X \in A, Y \in B}(z) \geqslant \mu\right\}.
\end{aligned}
\tag{7.22}
$$

It follows from Theorems 7.2 and 7.4 that

$$
\begin{aligned}
U_\mu^*(\omega) &= \begin{cases} 0 & \text{if } r_A''(\omega) \geqslant \mu, \\ 1 & \text{if } r_A''(\omega) < \mu, \end{cases}\\
V_\mu^*(\omega) &= \begin{cases} 0 & \text{if } s_B''(\omega) \geqslant \mu, \\ 1 & \text{if } s_B''(\omega) < \mu, \end{cases}\\
Z_\mu^*(\omega) &= \begin{cases} 0 & \text{if } \max\left[r_A''(\omega), s_B''(\omega)\right] \geqslant \mu, \\ 1 & \text{if } \max\left[r_A''(\omega), s_B''(\omega)\right] < \mu, \end{cases}\\
U_\mu^{**}(\omega) &= \begin{cases} 0 & \text{if } r_A'(\omega) \geqslant \mu, \\ 1 & \text{if } r_A'(\omega) \geqslant \mu, \end{cases}\\
V_\mu^{**}(\omega) &= \begin{cases} 0 & \text{if } s_B'(\omega) < \mu, \\ 1 & \text{if } s_B'(\omega) \geqslant \mu, \end{cases}\\
Z_\mu^{**}(\omega) &= \begin{cases} 0 & \text{if } \min\left[r_A'(\omega), s_B'(\omega)\right] < \mu, \\ 1 & \text{if } \min\left[r_A'(\omega), s_B'(\omega)\right] \geqslant \mu. \end{cases}
\end{aligned}
\tag{7.23}
$$

[Note that we need here the assumed normality of all random variables, which implies either $r_A'(\omega) = 1$ or $r_A''(\omega) = 1$, as well as either $s_B'(\omega) = 1$ or $s_B''(\omega) = 1$.] It is easily verified from these formulas that $(\forall \mu \in (0,1])$, $Z_\mu^* = U_\mu^* V_\mu^*$ and $Z_\mu^{**} =$

$U_\mu^{**} V_\mu^{**}$. This proves that for each $\mu \in (0,1]$

$$\begin{aligned} \inf(C_\mu) &= EZ_\mu^* = EU_\mu^* V_\mu^* = \inf(D_\mu), \\ \sup(C_\mu) &= EZ_\mu^{**} = EU_\mu^{**} V_\mu^{**} = \sup(D_\mu). \end{aligned} \tag{7.24}$$

Since $(\forall \mu \in (0,1])$ both C_μ and D_μ are convex, $\inf(C_\mu) \in C_\mu$, $\sup(C_\mu) \in C$, inf $(D_\mu) \in D_\mu$, $\sup(D_\mu) \in D_\mu$, and finally $C_0 = D_0 = R$, we have proved $(\forall \mu \in [0,1])$ $C_\mu = D_\mu$, which concludes the demonstration of the theorem. ■

The next result follows immediately from the previous theorem.

Theorem 7.6. *If* **X** *and* **Y** *are independent fuzzy random variables, and A and B Borel sets in R, then* $\Pr(\mathbf{X} \in A, \mathbf{Y} \in B) = \Pr(\mathbf{X} \in A)\Pr(\mathbf{Y} \in B)$.

Proof. Since **X** and **Y** are independent, also $\mathbf{I}^{\mathbf{X} \in A}$ and $\mathbf{I}^{\mathbf{Y} \in B}$ are independent. It follows from the previous theorem and Theorem 6.1 that

$$\Pr(\mathbf{X} \in A, \mathbf{Y} \in B) = E\mathbf{I}^{\mathbf{X} \in A}\mathbf{I}^{\mathbf{Y} \in B} = E\mathbf{I}^{\mathbf{X} \in A} E\mathbf{I}^{\mathbf{Y} \in B} = \Pr(\mathbf{X} \in A)\Pr(\mathbf{Y} \in B). \tag{7.25}$$

■

8. FUZZY CONDITIONAL EXPECTATION

In this section we shall discuss conditional expectations of fuzzy random variables. Let $\mathbf{X} = (\tilde{\mathfrak{X}}, X)$ be a fuzzy random variable defined on $(\Omega, \mathfrak{F}, \mathfrak{P})$, and suppose that $\mathfrak{A}$ is a sub-sigma-algebra of $\mathfrak{F}$. Let $\tilde{\mathfrak{A}}$ denote the sub-sigma-algebra of $\tilde{\mathfrak{F}}$, generated by all cylinder sets of the form $A \times \Omega'$ with $A \in \mathfrak{A}$. Let $\tilde{U} \in \tilde{\mathfrak{X}}$, and consider the conditional expectation $E^{\tilde{\mathfrak{A}}}\tilde{U}$. We claim that

$$E^{\tilde{\mathfrak{A}}}\tilde{U} = E^{\mathfrak{A}}\overline{U} \qquad \text{a.e. } (\tilde{\mathfrak{P}}), \tag{8.1}$$

where

$$\overline{U}(\omega) = \int \tilde{U}(\omega, \omega')\, d\mathfrak{P}'(\omega'). \tag{8.2}$$

The conditional expectation $E^{\tilde{\mathfrak{A}}}\tilde{U}$ is defined by the requirements that (a) it is an $\tilde{\mathfrak{A}}$-measurable random variable, and (b) $EI_{\tilde{A}} E^{\tilde{\mathfrak{A}}}\tilde{U} = EI_{\tilde{A}}\tilde{U}$ for any set $\tilde{A} \in \tilde{\mathfrak{A}}$. Here $I_{\tilde{A}}$ is the indicator function of $\tilde{A}$. It is easy to see that $E^{\mathfrak{A}}\overline{U}$ is $\tilde{\mathfrak{A}}$-measurable. To prove (b), we write, letting $\tilde{A} = A \times \Omega'$,

$$\begin{aligned} EI_{\tilde{A}} E^{\mathfrak{A}}\overline{U} &= EI_{A \times \Omega'} E^{\mathfrak{A}}\overline{U} = EI_A E^{\mathfrak{A}}\overline{U} = EI_A \overline{U} \\ &= \int I_A(\omega)\, d\mathfrak{P}(\omega) \int \tilde{U}(\omega, \omega')\, d\mathfrak{P}'(\omega') = EI_{\tilde{A}}\tilde{U} \end{aligned} \tag{8.3}$$

for any $\tilde{A}\in\tilde{\mathscr{Q}}$. This proves that $E^{\tilde{\mathscr{Q}}}\tilde{U}=E^{\mathscr{Q}}\bar{U}$ a.e. ($\tilde{\mathscr{P}}$). We consider $E^{\tilde{\mathscr{Q}}}$ as a function mapping $\tilde{\mathscr{X}}$ into $\mathscr{Z}$, which here is the class of all $\mathscr{Q}$-measurable random variables defined on $(\Omega,\mathscr{F},\mathscr{P})$. The result just obtained shows that $E^{\tilde{\mathscr{Q}}}$ is the composition of two maps. The first map : $\tilde{\mathscr{X}}\to\mathscr{X}$ is characterized by $\tilde{U}\mapsto\bar{U}$, while the second map : $\mathscr{X}\to\mathscr{Z}$ is the map $E^{\mathscr{Q}}$.

We define the conditional expectation of $\mathbf{X}=(\tilde{\mathscr{X}},X)$ given $\mathscr{Q}$ as the image of $\mathbf{X}$ in $\mathscr{Z}$ under the map $E^{\tilde{\mathscr{Q}}}$. We shall prove that $E^{\mathscr{Q}}\mathbf{X}$, thus defined, is a fuzzy random variable $E^{\mathscr{Q}}\mathbf{X}=\big(\mathscr{Z},(E^{\mathscr{Q}}\mathbf{X})\big)$. Since, as in the case of ordinary random variables, conditional expectations can only be defined almost everywhere, the definition of a fuzzy random variable has to be slightly modified at this point. Henceforth, the degree of membership of a random variable $Z\in\mathscr{Z}$ in the fuzzy set $\big(\mathscr{Z},(E^{\mathscr{Q}}\mathbf{X})\big)$ will be given by

$$\operatorname*{essinf}_{\omega\in\Omega}(E^{\mathscr{Q}}\mathbf{X})\big(Z(\omega)\big). \tag{8.4}$$

THEOREM 8.1. *The conditional expectation of* $\mathbf{X}=(\tilde{\mathscr{X}},X)$ *given* $\mathscr{Q}$ *is a fuzzy random variable* $E^{\mathscr{Q}}\mathbf{X}=\big(\mathscr{Z},(E^{\mathscr{Q}}\mathbf{X})\big)$ *with*

$$(E^{\mathscr{Q}}\mathbf{X})_{\omega}(x)=\sup\big\{\mu\in[0,1]\,|\,x\in D_{\mu}(\omega)\big\}. \tag{8.5}$$

The set $D_{\mu}(\omega)\subset R$ *is specified as follows. Let* V_{μ}^{*}, $\mu\in(0,1]$, *be a separable version of the process* $E^{\mathscr{Q}}U_{\mu}^{*}$, $\mu\in(0,1]$, *and let* V_{μ}^{**}, $\mu\in(0,1]$, *be a separable version of the process* $E^{\mathscr{Q}}U_{\mu}^{**}$, $\mu\in(0,1]$, *where*

$$U_{\mu}^{*}(\omega)=\inf\big\{x\in R\,|\,\bar{X}_{\omega}(x)\geqslant\mu\big\},\qquad U_{\mu}^{**}(\omega)=\sup\big\{x\in R\,|\,\bar{X}_{\omega}(x)\geqslant\mu\big\}, \tag{8.6}$$

with $\bar{X}$ *the unimodalized version of* X. *Then* $D_{\mu}(\omega)=[V_{\mu}(\omega),\,V_{\mu}^{**}(\omega)]$ *for* $\mu\in(0,1]$, *while* $D_0(\omega)=R$.

Proof. $E^{\tilde{\mathscr{Q}}}$ is the composition of the map: $\tilde{\mathscr{X}}\to\mathscr{X}$ defined by $\tilde{U}\mapsto\bar{U}$ and the map $E^{\mathscr{Q}}$. The image of $\mathbf{X}=(\tilde{\mathscr{X}},X)$ under the first map is $\bar{\mathbf{X}}=(\mathscr{X},\bar{X})$. Consider the image of $(\mathscr{X},\bar{X})$ in $\mathscr{Z}$ under the map $E^{\mathscr{Q}}$. Let $V\in\mathscr{Z}$. The degree of membership of V in the image of $(\mathscr{X},\bar{X})$ under $E^{\mathscr{Q}}$ is given by

$$\sup_{U\in\mathscr{X}:E^{\mathscr{Q}}U=V}\ \inf_{\omega\in\Omega}\bar{X}_{\omega}\big(U(\omega)\big). \tag{8.7}$$

Define for $\mu\in[0,1]$ the set $C_{\mu}\subset\mathscr{Z}$ by

$$C_{\mu}=\big\{V\in\mathscr{Z}\,|\,(\forall U\in\mathscr{X})\ E^{\mathscr{Q}}U=V,\ (\forall\omega)\ \bar{X}_{\omega}\big(U(\omega)\big)\geqslant\mu\big\}. \tag{8.8}$$

It follows from the unimodality of $\bar{X}$ that C_μ is convex. Furthermore, $C_0 = \mathfrak{Z}$ and C_μ decreases with increasing μ. We may write

$$\sup_{U \in \mathfrak{X} : E^{\mathfrak{a}} U = V} \inf_{\omega \in \Omega} \bar{X}_\omega(U(\omega)) = \sup\{\mu \mid V \in C_\mu\}. \tag{8.9}$$

By the monotonicity of the conditional expectation and the convexity of C_μ it follows that

$$C_\mu = \{V \in \mathfrak{Z} \mid V(\omega) \in D_\mu(\omega) \text{ a.e.}\}, \tag{8.10}$$

with $D_\mu(\omega)$ defined as above. Now by separability $V_\mu^*(\omega)$ is nonincreasing and $V_\mu^{**}(\omega)$ nondecreasing except on a null set contained in Ω, which means that $D_\mu(\omega)$ is nonincreasing except on a null set. Hence

$$\begin{aligned}\sup_{U \in \mathfrak{X} : E^{\mathfrak{a}} U = V} \inf_{\omega \in \Omega} \bar{X}_\omega(U(\omega)) &= \sup\{\mu \mid V \in C_\mu\} \\ &= \sup\{\mu \mid V(\omega) \in D_\mu(\omega) \text{ a.e.}\} \\ &= \operatorname*{essinf}_{\omega \in \Omega} \sup\{\mu \mid V(\omega) \in D_\mu(\omega)\} \\ &= \operatorname*{essinf}_{\omega \in \Omega} (E^{\mathfrak{a}} \mathbf{X})_\omega(V(\omega)),\end{aligned} \tag{8.11}$$

with $(E^{\mathfrak{a}}\mathbf{X})$ as defined above. This proves the theorem. ■

The following fact is immediate.

THEOREM 8.2. *If $\mathfrak{a}$ and $\sigma(X)$ are independent, then $E^{\mathfrak{a}}\mathbf{X} = E\mathbf{X}$.*

Proof. Since U_μ^* and U_μ^{**} are $\sigma(X)$-measurable, $V_\mu^* = EU_\mu^*$ and $V_\mu^{**} = EU_\mu^{**}$, which proves the theorem. ■

With these results in hand we can consider various questions involving conditional expectations. Let us for example study the conditional expectation $E(\mathbf{X} \mid \mathbf{Y} \in A)$, with $\mathbf{X}$ and $\mathbf{Y}$ fuzzy random variables, and $A \subset R$ a Borel set. We define $E(\mathbf{X} \mid \mathbf{Y} \in A)$ as the image of the product fuzzy set $\mathbf{X} \times \mathbf{Y}$ in R under the map: $\tilde{\mathfrak{X}} \times \tilde{\mathfrak{Y}} \to R$ defined by $(\tilde{U}, \tilde{V}) \mapsto E(\tilde{U} \mid \tilde{V} \in B)$.

THEOREM 8.3. *$E(\mathbf{X} \mid \mathbf{Y} \in B)$ is the fuzzy number defined by*

$$\begin{aligned}&(E(\mathbf{X} \mid \mathbf{Y} \in B))(z) \\ &= \sup_{\substack{U \in \mathfrak{X},\, \Pi \in \mathfrak{Y} \\ E(U\Pi)/E(\Pi) = z}} \inf_{\omega \in \Omega} \min\left[\bar{X}_\omega(U(\omega)), I_\omega^{\mathbf{Y} \in B}(\Pi(\omega))\right], \qquad z \in R.\end{aligned} \tag{8.12}$$

Proof. We write

$$E(\tilde{U}|\tilde{V}\in B)=\frac{\iint\limits_{\tilde{V}\in B}\tilde{U}(\omega,\omega')\,d\mathcal{P}(\omega)\,d\mathcal{P}'(\omega')}{\iint\limits_{\tilde{V}\in B}d\mathcal{P}(\omega)\,d\mathcal{P}'(\omega')}$$

$$=\frac{\int d\mathcal{P}(\omega)\dfrac{\int_{\tilde{V}\in B}\tilde{U}(\omega,\omega')\,d\mathcal{P}'(\omega')}{\int_{\tilde{V}\in B}d\mathcal{P}'(\omega')}\int_{\tilde{V}\in B}d\mathcal{P}'(\omega')}{\int d\mathcal{P}(\omega)\int_{\tilde{V}\in B}d\mathcal{P}'(\omega')}$$

$$=\frac{\int\check{U}(\omega)\Pi(\omega)\,d\mathcal{P}(\omega)}{\int\Pi(\omega)\,d\mathcal{P}(\omega)},\tag{8.13}$$

where

$$\Pi(\omega)=\int_{\tilde{V}\in B}d\mathcal{P}'(\omega'),\tag{8.14a}$$

$$\check{U}(\omega)=\begin{cases}\dfrac{1}{\Pi(\omega)}\displaystyle\int_{\tilde{V}\in B}\tilde{U}(\omega,\omega')\,d\mathcal{P}'(\omega') & \text{if}\quad \Pi(\omega)\neq 0,\\ \text{arbitrary but finite} & \text{if}\quad \Pi(\omega)=0.\end{cases}\tag{8.14b}$$

Let us now introduce the probability measure $\hat{\mathcal{P}}$ on Ω defined by

$$d\hat{\mathcal{P}}(\omega)=\frac{\Pi(\omega)\,d\mathcal{P}(\omega)}{\int\Pi(\omega)\,d\mathcal{P}(\omega)}.\tag{8.15}$$

Denoting the expectation with respect to this measure as $\hat{E}$, we may write

$$E(\tilde{U}|\tilde{V}\in B)=\hat{E}\check{U}=\hat{E}\,(\hat{E}^{\sigma(X)}\check{U})=\hat{E}U=\frac{\int U(\omega)\Pi(\omega)\,d\mathcal{P}(\omega)}{\int\Pi(\omega)\,d\mathcal{P}(\omega)},\tag{8.16}$$

where

$$U=\hat{E}^{\sigma(X)}\check{U}.\tag{8.17}$$

The image of $\mathbf{Y}$ in $\mathfrak{Y}$ under the map $:\tilde{\mathfrak{Y}}\to\mathfrak{Y}$ defined by (8.14a) is $\mathbf{I}^{\mathbf{Y}\in B}$. We shall prove that the image of $\mathbf{X}$ under the composite map: $\tilde{\mathfrak{X}}\to\mathfrak{X}$ defined by (8.14b) and (8.17) such that $\tilde{U}\mapsto\check{U}\mapsto U$ is given by $\overline{\mathbf{X}}$ (for each $\tilde{V}$ and each B). Then it follows from (8.16) that $E(\mathbf{X}|\mathbf{Y}\in B)$ is the image in R of $\mathbf{X}\times\mathbf{I}^{\mathbf{Y}\in B}$ under the map: $\mathfrak{X}\times\mathfrak{Y}\to R$ defined by $(U,\Pi)\to EU\Pi/E\Pi$, and that hence (8.12) is correct.

It remains to demonstrate that the image of $\mathbf{X}$ under the map $\tilde{U}\mapsto\check{U}\mapsto U$ is $\overline{\mathbf{X}}$. It is not difficult to see that the image of $\mathbf{X}$ in $\mathfrak{Z}$ [here $\mathfrak{Z}$ is the class of random variables defined on $(\Omega,\mathfrak{F},\mathfrak{P})$ that are $\sigma(X,Y)$-measurable] under the map defined by (8.14b) is given by $\overline{\mathbf{X}}^{\Pi}=(\mathfrak{Z},\overline{X}^{\Pi})$, where

$$\overline{X}^{\Pi}_{\omega}(x)=\begin{cases}\overline{X}_{\omega}(x) & \text{if } \Pi(\omega)>0,\\ 1 & \text{if } \Pi(\omega)=0.\end{cases} \tag{8.18}$$

We determine $\check{E}^{\sigma(X)}\overline{\mathbf{X}}^{\Pi}$ with the aid of Theorem 8.1. It is not difficult to establish that

$$\inf\{x\in R|\overline{X}^{\Pi}_{\omega}(x)\geqslant\mu\}=\begin{cases}U^{*}_{\mu}(\omega) & \text{if } \Pi(\omega)>0,\\ -\infty & \text{if } \Pi(\omega)=0,\end{cases} \tag{8.19}$$

$$\sup\{x\in R|\overline{X}^{\Pi}_{\omega}(x)\geqslant\mu\}=\begin{cases}U^{**}_{\mu}(\omega) & \text{if } \Pi(\omega)>0,\\ +\infty & \text{if } \Pi(\omega)=0,\end{cases} \tag{8.20}$$

with U^{*}_{μ} and U^{**}_{μ} defined as in Theorem 8.1. Evidently $\inf\{x\in R|\overline{X}^{\Pi}_{\omega}(x)\geqslant\mu\}=U^{*}_{\mu}(\omega)$ a.e. with respect to the measure $\hat{\mathfrak{P}}$, and $\sup\{x\in R|\overline{X}^{\Pi}_{\omega}(x)\geqslant\mu\}=U^{**}_{\mu}(\omega)$ a.e. with respect to the measure $\hat{\mathfrak{P}}$. Since both U^{*}_{μ} and U^{**}_{μ} are $\sigma(X)$-measurable, we obtain $V^{*}_{\mu}=U^{*}_{\mu}$ a.e. and $V^{**}_{\mu}=U^{**}_{\mu}$ a.e., both with respect to $\hat{\mathfrak{P}}$, so that

$$(\check{E}^{\sigma(X)}\overline{\mathbf{X}}^{\Pi})_{\omega}=\overline{X}_{\omega}\qquad\text{a.e. }(\hat{\mathfrak{P}}). \tag{8.21}$$

This result is implied if we set

$$(\check{E}^{\sigma(X)}\overline{X}^{\Pi})_{\omega}=\overline{X}_{\omega}, \tag{8.22}$$

which concludes the proof. ∎

COROLLARY of Theorem 8.3. *If* $\mathbf{X}$ *and* $\mathbf{Y}$ *are independent, then* $E(\mathbf{X}|\mathbf{Y}\in A)=E\mathbf{X}$.

The proof is immediate. Given the notion of conditional expectation, conditional fuzzy probabilities are easily defined. For $\mathbf{X}$ and $\mathbf{Y}$ given fuzzy random variables, and A and B given Borel sets contained in R, we define $\Pr(\mathbf{X}\in A|\mathbf{Y}\in B)$ as the image of $\mathbf{X}\times\mathbf{Y}$ in R under the map $(\tilde{U},\tilde{V})\mapsto\tilde{\mathcal{P}}(\tilde{U}\in A|\tilde{V}\in B)$. We hence have

$$\Pr(\mathbf{X}\in A|\mathbf{Y}\in B)(p)$$

$$=\sup_{\substack{\tilde{U}\in\tilde{\mathfrak{X}},\,\tilde{V}\in\tilde{\mathfrak{Y}}:\\ \tilde{\mathcal{P}}(\tilde{U}\in A|\tilde{V}\in B)=p}}\ \inf_{\substack{\omega\in\Omega\\ \omega'\in\Omega'}}\min\left[X_\omega(\tilde{U}(\omega,\omega')),Y_\omega(\tilde{V}(\omega,\omega'))\right]. \tag{8.23}$$

We write

$$\tilde{\mathcal{P}}(\tilde{U}\in A|\tilde{V}\in B)=\frac{\tilde{\mathcal{P}}(\tilde{U}\in A,\,\tilde{V}\in B)}{\mathcal{P}(\tilde{V}\in B)}=\frac{\int d\mathcal{P}(\omega)\int_{\tilde{U}\in A,\,\tilde{V}\in B}d\mathcal{P}'(\omega')}{\int d\mathcal{P}(\omega)\int_{\tilde{V}\in B}d\mathcal{P}'(\omega')}$$

$$=\frac{\int\check{\Psi}(\omega)\Pi(\omega)\,d\mathcal{P}(\omega)}{\int\Pi(\omega)\,d\mathcal{P}(\omega)}$$

$$=\check{E}\check{\Psi}=\check{E}\check{E}^{\sigma(x)}\check{\Psi}=\check{E}\Psi=\frac{\int\Psi(\omega)\Pi(\omega)\,d\mathcal{P}(\omega)}{\int\Pi(\omega)\,d\mathcal{P}(\omega)}, \tag{8.24}$$

where

$$\Pi(\omega)=\int_{\tilde{V}\in B}d\mathcal{P}'(\omega'),$$

$$\check{\Psi}(\omega)=\begin{cases}\dfrac{1}{\Pi(\omega)}\displaystyle\int_{\tilde{U}\in A,\,\tilde{V}\in B}d\mathcal{P}'(\omega') & \text{if}\quad \Pi(\omega)>0,\\ \text{arbitrary but finite} & \text{if}\quad \Pi(\omega)=0,\end{cases} \tag{8.25}$$

$$\Psi=\check{E}^{\sigma(X)}\check{\Psi},$$

and $\check{E}$ has the same meaning as in the proof of Theorem 8.2. The image of $\mathbf{X}$ in $\mathfrak{X}$ under the map $\tilde{U}\mapsto\check{\Psi}\mapsto\Psi$ is $\mathbf{I}^{\mathbf{X}\in A}$, while the image of $\mathbf{Y}$ in $\mathfrak{Y}$ under the map

$\tilde{V} \mapsto \Pi$ is $\mathbf{I}^{\mathbf{Y} \in B}$. It follows from (8.24) that

$$\Pr(\mathbf{X} \in A | \mathbf{Y} \in B)(p) = \sup_{\substack{\Psi \in \mathfrak{X}, \Pi \in \mathfrak{Y}: \\ E(\Psi\Pi)/E\Pi = p}} \inf_{\omega \in \Omega} \min\left[I_\omega^{\mathbf{X} \in A}(\Psi(\omega)), I_\omega^{\mathbf{Y} \in B}(\Pi(\omega)) \right]. \tag{8.26}$$

This immediately shows that if $\mathbf{X}$ and $\mathbf{Y}$ are independent, then $\Pr(\mathbf{X} \in A | \mathbf{Y} \in B) = \Pr(\mathbf{X} \in A)$.

9. CONCLUSIONS

In this paper we have extended the notion of random variables to fuzzy random variables. The extension consists in allowing impreciseness in the values that are assumed by the random variable. The vehicle that is used to carry this impreciseness is fuzzy set theory and the corresponding multivalued logic. Many properties that are well known for ordinary random variables are reencountered in the context of fuzzy random variables. Not all properties carry over, however. For instance, the well-known property $EXY = EXEY$ for independent random variables only applies to nonnegative fuzzy random variables.

Clearly, many properties of fuzzy random variables remain to be investigated. At this point, however, enough theory is available to consider simple applications of fuzzy random variables. In order to handle these applications, there is a need for various algorithms for the computation of expectations of fuzzy random variables and fuzzy probabilities. These algorithms will be developed in a subsequent paper.

It is a pleasure to acknowledge helpful discussions with Dr. A. Bagchi.

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Received December 1977