LETTER TO THE EDITOR

Triviality of the prolongation algebra of the Kuramoto-Sivashinsky equation

H Nijs and R Martini

Department of Applied Mathematics, Twente University of Technology, PO Box 217, 7500 AE Enschede, The Netherlands

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Abstract. We apply the well known Wahlquist-Estabrook prolongation technique to the Kuramoto-Sivashinsky equation. The prolongation algebra turns out to be trivial in the sense that it is commutative. This supports non-integrability of the equation.

The Kuramoto-Sivashinsky equation is a non-linear partial differential equation frequently encountered in the study of continuous media. It describes, for instance, the fluctuation of a flame front or an oscillating chemical reaction in a homogeneous medium. A review may be found in [1]. In conservative form it is written as

$$u_t + uu_x + \mu u_{xx} + \nu u_{xxxx} = 0. \quad (1)$$

This equation is invariant under the Galilean transformation

$$(u, x, t) \rightarrow (u + c, x - ct, t). \quad (2)$$

In [1] the well known Painlevé analysis is applied to the Kuramoto-Sivashinsky equation (1), indicating non-integrability of this equation. This is not surprising because it is known that the Kuramoto-Sivashinsky equation exhibits chaotic behaviour. In this letter we prove, following the well known prolongation technique of Wahlquist and Estabrook, that the prolongation algebra for the Kuramoto-Sivashinsky equation is trivial in the sense that it is commutative. This result is also a strong indication of the non-integrability of the equation. Although there is no rigorous proof that commutativity of the prolongation algebra implies non-integrability, it is a general conjecture that this is indeed the case. This conjecture is supported by many known examples, e.g. the Korteweg-de Vries-Burgers equation [2].

In order to write equation (1) as a system of differential forms we introduce $p = u_x$, $q = u_{xx}$, $r = u_{xxx}$. Equation (1) can now be written as

$$u_t + up + \mu q + \nu r_x = 0. \quad (3)$$

In order not to discuss a degenerate case we assume $\nu \neq 0$. When we adopt in the six-dimensional space of dependent and independent variables $\{x, t, u, p, q, r\}$ the basis forms $\{dx, dt, du, dp, dq, dr\}$, we can represent the equation (3) by the following set of 2-forms:

$$\begin{aligned} \alpha_1 &= du \wedge dt - p \, dx \wedge dt \\ \alpha_2 &= dp \wedge dt - q \, dx \wedge dt \\ \alpha_3 &= dq \wedge dt - r \, dx \wedge dt \\ \alpha_4 &= \nu \, dr \wedge dt - (\mu q + up) \, dx \wedge dt - du \wedge dx. \end{aligned} \quad (4)$$

Let $S = \{(x, t, u(x, t), p(x, t), q(x, t), r(x, t))\}$ be a regular two-dimensional manifold in six-dimensional space. Any S that is a solution of (3) will annul the set of forms (4) and vice versa [3]. The ideal of 2-forms (4) is closed (i.e. $\sum_{i=1}^4 \eta_i \wedge \alpha_i$, $i = 1, \dots, 4$, η_i being a set of 1-forms) which means that equation (3) satisfies the integrability conditions. To obtain a prolongation structure we introduce the prolongation variable y (the number of prolongation variables will not influence the process and will therefore be taken equal to 1). We also introduce the 1-form $\omega = dy + F dx + G dt$. Following [3] we take $F = F(u, p, q, r, y)$ and $G = G(u, p, q, r, y)$. The extended ideal $I(\alpha_1, \dots, \alpha_4, \omega)$ should be closed now, which means in particular that $d\omega = \sum_{i=1}^4 f_i \alpha_i + n \wedge \omega$ (f_i are 0-forms and n is a 1-form). We define $[G, F] = GF_y - G_y F$ and obtain the following equations for F and G :

$$F_p = 0 \quad F_q = 0 \quad F_r = 0 \quad (5)$$

$$\nu F_u + G_r = 0 \quad (6)$$

$$[G, F] + pG_u + qG_p + rG_q - G_r(\mu q + up)/\nu = 0. \quad (7)$$

Equations (5) give $F = F(u, y)$ and integrating (6) renders

$$G = -\nu r F_u + G^1(u, p, q, y).$$

Substituting this in equation (7) we derive an equation linear in r . After separating powers of r and integration with respect to q this yields

$$G^1 = \nu q(pF_{uu} + [F_u, F]) + G^2(u, p, y)$$

which gives, after substituting this in the remaining part, an equation quadratic in q . Collecting powers gives the following three equations:

$$F_{uu} = 0 \quad (8)$$

$$2\nu p[F_{uu}, f] + \nu[[F_u, F], F] + \nu p^2 F_{uuu} + \mu F_u + G_p^2 = 0 \quad (9)$$

$$[G^2, F] + pG_u^2 + upF_u = 0. \quad (10)$$

From equation (8) we find $F = X_1 + uX_2$. Let us introduce the shorthand notation $[X_i, X_j] = [i, j]$ and $\mathfrak{J}(i, j, k)$ for 'Jacobi identity applied for X_i, X_j and X_k '. Further we define

$$[1, 2] = X_4 \quad [1, 4] = X_5 \quad [1, 5] = X_6 \quad [1, 6] = X_7. \quad (11)$$

The result for F together with relations (2.9) gives us, after integrating equation (9) with respect to p ,

$$G^2 = -p(\mu X_2 + \nu X_5 + \nu u[2, 4]) + G^3(u, y).$$

Together with (10) this gives an equation quadratic in p and after collecting powers we find

$$[2, 4] = 0 \quad (12)$$

$$-[(\mu X_2 + \nu X_5 + \nu u[2, 4]), (X_1 + uX_2)] + uX_2 + G_u^3 = 0 \quad (13)$$

$$[G^3, (X_1 + uX_2)] = 0. \quad (14)$$

$\mathfrak{J}(1, 2, 4)$ together with (12) gives $[2, 5] = 0$. With this we find after integrating (13)

$$G^3 = -\frac{1}{2}u^2 X_2 - u(\mu X_4 + \nu X_6) + X_3.$$

Together with (14) this leads to a quadratic equation in u which splits up into

$$[1, 3] = 0 \quad [2, 3] = \mu X_5 + \nu X_7 \quad [2, 6] = -X_4/2\nu \quad [2, 4] = 0 \quad (15)$$

as we already found. Relations (15) together with the relations for $X_1, \dots, X_4$ (11) define the prolongation algebra.

Theorem. The prolongation algebra of the Kuramoto–Sivashinsky equation determined by (11) and (15) is commutative.

Proof. By use of the Jacobi identity we shall prove $J_1(2, 5, 6) = 3X_4/4\nu^2$. We already found by use of $J_1(1, 2, 4)$ that $[2, 5] = 0$. This relation together with $J_1(1, 2, 5)$ leads to

$$[4, 5] = X_4/2\nu. \quad (16)$$

From $J_1(1, 2, 3)$ we find a relation between two commutators

$$[3, 4] = -\mu X_6 - \nu[1, 7].$$

Using (16), $J_1(1, 4, 5)$ gives $[4, 6] = X_5/2\nu$. From $J_1(1, 2, 6)$ we find $[2, 7] = -X_5/\nu$. Now $J_1(1, 2, 7)$ gives

$$[2, [1, 7]] + [4, 7] + X_6/\nu = 0 \quad (17)$$

and from $J_1(2, 3, 4)$ we have

$$-\nu[2, [1, 7]] + \nu[4, 7] + \mu X_4/\nu = 0. \quad (18)$$

When we take an appropriate linear combination of (17) and (18) we can eliminate $[2, [1, 7]]$ and therefore we have

$$[4, 7] = -\mu X_4/2\nu^2 - X_6/2\nu.$$

This result we use to calculate $J_1(1, 4, 6)$ which gives us

$$[5, 6] = \mu X_4/2\nu^2 + X_6/\nu.$$

This result we use to show that

$$J_1(2, 4, 6) = 3X_4/4\nu^2$$

To fulfil this last Jacobi identity X_4 has to be zero. It follows easily that $X_5 = 0$, $X_6 = 0$, $X_7 = 0$ and therefore $[1, 2] = 0$ and $[2, 3] = 0$. We already had $[1, 3] = 0$, which proves the theorem.

Remark. Adding an additional term of the form δu_{xxx} to the original equation (1) yields the equation

$$u_t + uu_x + \mu u_{xx} + \delta u_{xxx} + \nu u_{xxxx} = 0. \quad (19)$$

In the same way as previously, we can show that the prolongation algebra commutes in this case also. So for equation (19) also no non-trivial prolongation algebra exists.

In conclusion, we have shown for $\nu \neq 0$ that (11) and (15) define the prolongation algebra for the Kuramoto–Sivashinsky equation. This implies for the generators X_1 , X_2 and X_3 that they commute. Therefore no non-trivial prolongation algebra exists for the Kuramoto–Sivashinsky equation.

References

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