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Note

The set of super-stable marriages forms a distributive lattice

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Abstract

Relaxing the total orders of the preference lists of an instance of the stable marriage problem to arbitrary posets, we show after adjusting the notion of stability to the new problem that the set of stable marriages still forms a distributive lattice.

Keywords: Stable marriage problem; Poset, Lattice

The stable marriage problem was introduced by Gale and Shapley [1]. An instance (P) consists of n men and n women each of whom has a totally ordered preference list of the opposite sex. A complete matching M in the bipartite men–women graph is called a *marriage*. A marriage is *unstable* if there is a man–woman pair not in M where both prefer each other to their current partners in M. A marriage is defined to be *stable* if it is not unstable.

It is well known (see e.g. [5]) that the set of stable marriages forms a distributive lattice (see Definition 4) under the following partial ordering:

The matching M is greater or equal to M' ($M \geqslant M'$) if and only if every man is at least as satisfied with his partner in M as he is in M'.

In [4, 5] the question is raised what would happen if one skips the requirement that the preference lists should be totally ordered. We show, adjusting the notion of stability in a natural way, that the set of stable marriages remains being a distributive lattice if we replace the linear preference lists of the men and women by arbitrary posets. Reversing the ordering of the lattice in this generalized problem (GP) gives the distributive lattice ordered from a female point of view.

Now it will be necessary to give the definition of stability for the generalized problem (GP) where the men rank the women and the women rank the men in

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arbitrary posets. We will always denote men with upper-case and women with lower-case letters for sake of simplicity. Of course, the problem would be the same if we interchanged the roles of men and women.

Definition 1. A marriage M (always denoted as $M = (Aa, Bb, \dots, Zz)$ of n ladies $(a, b, \dots, z)$ and n gentlemen $(A, B, \dots, Z)$ where A marries a, B marries $b, \dots$) is called *unstable* if there are two pairs Xx and Yy, where
(a) X prefers y to x and y prefers X to Y, or
(b) X prefers y to x and X and Y are incomparable in y's preference poset, or
(c) y and x are incomparable for X and y prefers X to Y, or
(d) y and x are incomparable for X and Y and X are incomparable in y's poset.
M is called *stable* if it is not unstable.

Remark 1. This generalized notion of stability is called *super-stability* in Irving and Gusfield [3, 4].

Definition 2. Let $(P, \leqslant)$, $P = \{x_1, \dots, x_n\}$ be a poset. Then $L = (x_{i_1}, \dots, x_{i_n})$, $i_j \in \{1, \dots, n\}$, $i_j \neq i_k$ for $j \neq k$ is called a *linear extension* of P if

$$j < k \ \Rightarrow \ x_{i_k} \nless x_{i_j} \quad \forall j, k \in \{1, \dots, n\}.$$

Definition 3. Let I be an instance of (GP) with n men and n women. Let $\boldsymbol{L} := \{\mathscr{L} = (L_1, \dots, L_n, L'_1, \dots, L'_n) \mid L_i$ is a linear extension of the ith man's poset of preferences and L'_j is a linear extension of the jth woman's poset of preferences$\}$.

Let $P_{\mathscr{L}}$ be the instance of (P) corresponding to $\mathscr{L} \in \boldsymbol{L}$.[1] We denote by $D_{\mathscr{L}}$ the set of stable marriages corresponding to the instance $P_{\mathscr{L}}$ of (P).

In the sense of Definition 3 the following theorem holds.

Theorem. (a) *Every stable marriage in* (GP) *remains stable if we replace any of the posets by any of its linear extensions. An instance* (GP) *leads thus to many instances of* (P), *one for each possible replacement.*[2]

(b) *Every marriage which is stable for every replacement of the posets by linear extensions is a stable marriage in* (GP). *In other words: Replace the posets by linear extensions as in* (a). *Find the set $D_{\mathscr{L}}$ of stable marriages for every $P_{\mathscr{L}}$ obtained in this way. The intersection $\mathscr{S}$ of all the sets $D_{\mathscr{L}}$, $\mathscr{L} \in \boldsymbol{L}$, will contain only marriages which are stable in* (GP) *too.*

Proof. (a) Assume $M = (Aa, Bb, \dots)$ is stable in (GP) and unstable for a linear extension of the lists. Let the instability occur for A, b.[3] This means that in A's list b is

[1] As every finite poset has only finitely many linear extensions it holds that $|\boldsymbol{L}| < \infty$.
[2] For sake of convenience we will call this process the *replacement of the posets by their linear extensions.*
[3] The pair Ab is *blocking* M in the terminology of [3].

ranked higher than a and b ranks A higher than B. But for a linear extension of b's list where A is ranked higher than B (in the extension) A is greater than B or A and B are incomparable for b in the original poset. For b is ranked higher than a in A's extension a is ranked higher than b or a and b are incomparable in A's original poset. In all the cases M must be unstable in (GP), a contradiction.

(b) For every unstable marriage M in (GP) one easily finds an $\mathscr{L} \in \boldsymbol{L}$ such that M is unstable in the corresponding instance of (P). Suppose $M = (Aa, Bb, \ldots)$ is unstable in (GP) and Ab is a blocking pair. We consider exemplary the case (d) of Definition 1, i.e., a and b are incomparable in A's list, A and B are incomparable in b's list. Take a linear extension L of A's list in which A prefers b to a. Take a linear extension K of b's list in which b prefers A to B. Replace all posets by linear extensions and especially A's poset by L and b's poset by the linear extension K. Then in every marriage of this instance of (P) containing the pairs Aa and Bb the pair Ab obviously is a blocking one. But then M is not contained in $\mathscr{S}$, a contradiction. □

Definition 4. A poset[4] $(D, \leqslant)$ is called a *lattice* if for every two elements $a, b \in D$ there exist
(a) $a \vee b := \inf\{x \in D \mid x \geqslant a \text{ and } x \geqslant b\} \in D$ and
(b) $a \wedge b := \sup\{x \in D \mid x \leqslant a \text{ and } x \leqslant b\} \in D$.
We will call a lattice D *distributive* if for all $a, b, c \in D$ it holds

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \text{ and } a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Remark 2. If the preference lists are totally ordered sets we have the original stable marriage problem (P), where the stable marriages are known to form a distributive lattice (see [5]).

Summarizing we have the following corollary.

Corollary. *The set $\mathscr{S}$ of stable marriages of* (GP) *is either empty or it forms a distributive lattice.*

Proof. With the foregoing theorem, the set $\mathscr{S}$ of stable marriages of (GP) is the intersection of all the distributive lattices $D_{\mathscr{L}}$ of stable marriages of the problem instances $P_{\mathscr{L}}$ of (P) corresponding to all $\mathscr{L} \in \boldsymbol{L}$. As an intersection of distributive lattices is either a distributive lattice or empty the corollary is proved.[5] □

In [1] it was shown that in the original problem (P) the *male maximal* stable marriage (i.e. the greatest element in the distributive lattice of stable marriages) was

[4] This is the usual abbreviation for *partially ordered set*.
[5] The possibility of $\mathscr{S}$ being empty is seen already with trivial examples of two men and two women.

the *female minimal* stable marriage (i.e., the lowest element in the lattice of stable marriages under the female ordering). By reversing the ordering of the set of stable marriages we get the distributive lattice ordered from the ladies' point of view. This also holds in (GP).

Lemma. *If $\mathscr{S}$ is the set of stable marriages of an instance* (GP) *it holds for* $M, M' \in \mathscr{S}$

$$M \leqslant M' \Leftrightarrow M' \preceq M$$

if $\preceq$ is the female partial ordering of $\mathscr{S}$ (i.e., $M' \preceq M$ if every women is at least as satisfied with her partner in M as she is in M').

Proof. "$\Rightarrow$" Let

$$(Aa, Xx, Yy, Zz, \ldots) = M \leqslant M' = (Aa', Xx', Yy', Zz', \ldots)$$

and $M \neq M'$ (otherwise the lemma holds trivially).

Assume now (a) $M \prec M'$[6] or (b) M and M' are incomparable for $\prec$.

(a) In M we have a man A who prefers a' to a, $a \neq a'$ and a man X ($\neq A$) who prefers $x' = a$ to x or finds x' and x incomparable. Because of the fact that $M \prec M'$ it holds that a prefers X to A or X and A are incomparable in a's poset. But then M would be unstable.

(b) There is a y ($\neq z$) with y ranks Z higher than Y or thinks of Z and Y as incomparable. At the same time z' ($= y$) is the partner of Z in M'. But then M must be unstable, a contradiction.

"$\Leftarrow$" Can be proved analogously. □

Remark 3. The set of stable marriages equipped with the female ordering is the distributive lattice obtained by reversing the male ordering.

It is well known (cf. [2]) that in the original problem (P) the so-called *Pareto optimality* holds:

There is no matching (stable or not) in which every man has a partner strictly better than in the male optimal stable matching.

The subsequent example shows that this is no longer the case for (GP).

Example. We consider an instance of (GP) with three men (A, B, C) and three women (a, b, c) with the following preference lists:

A cannot compare a and b who are both ranked higher than c.

B cannot compare b and c who are both ranked higher than a.

C cannot compare c and a who are both ranked higher than b.

a cannot compare C and A who are both ranked lower than B.

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[6] $M \prec M' :\Leftrightarrow M \preceq M'$ and $M \neq M'$.

b cannot compare A and B who are both ranked lower than C.

c cannot compare B and C who are both ranked lower than A.

One easily checks that there is only one stable marriage $M = (Ac, Ba, Cb)$ whereas the (unstable) marriage $\tilde{M} = (Aa, Bb, Cc)$ is strictly better for every man.

Now we could ask whether there is a polynomial-time algorithm which finds a stable marriage for an instance of the problem (GP) or states its nonexistence. Such an algorithm exists. It extends the algorithm given in [3] for finding a super-stable matching in the problem instance (WP) where both sexes have special posets, namely so-called *weak orders*, as preference lists:[7]

Every man makes a proposal to the woman (women) in the highest position[8] in his list.
Each women cancels all the proposers ranked lower than her highest proposer(s) and is crossed out in the lists of the cancelled men.
If a women still has more than one proposer she delets all men on her list but those who are ranked higher than all her proposers. The cancelled men cross out this woman in their list.
STOP if there is an empty list and output that there exits no stable marriage.
STOP if every woman has exactly one proposer. Give every woman her proposer and celebrate a stable marriage.
START the ceremony again.

Each of the n men in every "round" of the algorithm proposes to at most n women. Due to the fact that in every round the algorithm stops or at least one women is cancelled from a man's list the complexity of the algorithm is bounded by $O(n^4)$.

The correctness of the algorithm is a consequence of the following facts. Their proofs go along the same lines as the proof of the correctness of the original Gale–Shapley algorithm in [1]. Therefore these proofs will only be sketched.

(a) If the algorithm stops with the output of a marriage this marriage will be stable.
(b) If a woman, a say, cancels a man, A say, during the algorithm the pair Aa is *unstable* (i.e., no stable marriage with the pair Aa exists).[9]
(c) If there is no stable matching the algorithm will stop with this answer.
(d) The algorithm finds the greatest element of the distributive lattice of stable marriages if this lattice is not empty.

Consider (a). Suppose every woman has exactly one proposer. Then these will be pairwise distinct because if there was a man not proposing to any woman his preference list was empty and the algorithm had already terminated.

[7] Ties are allowed in the preference lists.
[8] The maximal elements in his poset of not yet cancelled women.
[9] A pair is called *stable* if it is not unstable, i.e., there exists a stable marriage containing this pair.

Let the resulting marriage $M = (Aa, Bb, \ldots)$ be unstable and let Ab be a blocking pair. We consider for example the case (b) of Definition 1 where A prefers b to a and b cannot compare A and B. As A once must have proposed to b and later b was cancelled from his list, b must have had a proposal from a gentleman incomparable to A or someone she prefers to A. This is a contradiction because in either case B had been deleted from b's list and the algorithm never had assigned B to b.

Consider (b). By induction. Assume that up to a given point no proposer was cancelled from a ladies' list who was *possible* for him (i.e., if this particular pair was stable). Suppose now a proposer is deleted from a possible partner's list. Then we are lead to a contradiction in a similar way as in (a).
The remaining points are immediate consequences of (a) and (b).

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