Smoothing and Likelihood Ratio for Gaussian Boundary Value Processes

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Abstract—A new derivation, which does not need the invertibility assumption of the covariance matrix of the boundary data, is given for the smoothing of Gaussian two-point boundary value processes (TPBVP). The likelihood ratio for TPBV processes is then derived in terms of the system parameters by using the Krein factorization. The likelihood ratio involves the smoother of the process. An alternate expression for the likelihood ratio based on the filtered estimate of the state is also given.

I. INTRODUCTION

ADAMS, Willsky, and Levy [1], [2] studied the smoothing problem for boundary value processes. By boundary value processes they mean random fields (processes) satisfying partial (ordinary) differential equations with certain noisy boundary conditions. The derivation of the smoother is based on the idea of the complementary process, introduced by Weinert and Desai [3], and holds under the assumption that the covariance matrix of the boundary data is invertible. This, in particular, excludes the case of periodic processes considered by Kwakernaak [4]. In his thesis, Adams [5] extended the results to the case of the singular covariance matrix. In this paper an alternate derivation is given for the smoother in the special case of one parameter (or two-point) boundary value processes (TPBVP), where the invertibility assumption mentioned above is also not required. Recently, Riddle and Weinert [6] have proposed a different derivation of the smoothing formulas for boundary value processes with possible singularity of the covariance matrix. Their approach, however, is still based on the idea of complementary models.

A rigorous mathematical treatment of boundary value processes and the related smoothing and identification problems have been given by Bagchi and Aihara [7]. An expression has also been given there for the likelihood ratio. This expression is, however, not very useful in practice, as it is not given in terms of the system parameters. In this paper we obtain a useful form for the likelihood ratio in the special case of TPBV processes. This is achieved by using the Krein factorization, which also gives an interesting interpretation of the associated smoothing problem.

One parameter boundary value processes and their deterministic counterpart—the acausal systems—have been extensively studied by Krener [8], [9]. Krener has been mainly concerned with the realization problem—both in the deterministic and the stochastic contexts. This is bound to have significant implication in the identification problem. We do not, however, get into these issues here. For an example of TPBVP arising in practice, see [2].

II. PROBLEM FORMULATION

Suppose that $\{x_t, t \geq 0\}$ is an n -dimensional two-point boundary value process (TPBVP) satisfying

$$dx_t = A(t)x_t dt + G(t)u(t) + B(t)dw_t \quad (2.1)$$

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and the boundary condition

$$V^0 x_0 + V^T x_T = v \quad (2.2)$$

where $\{w_t, t \geq 0\}$ is a p -dimensional Brownian motion with

$$Ew_t = 0 \quad Ew_t w_s^* = \min(t, s)I \quad (2.3)$$

(with “*” denoting transpose). The input function $\{u(t), t \geq 0\}$ is assumed to be a (known) deterministic l -dimensional vector, while v is an n -dimensional normally distributed random vector, independent of $\{w_t, 0 \leq t \leq T\}$, with

$$Ev = m_v \quad E(v - m_v)(v - m_v)^* = \Sigma_v. \quad (2.4)$$

$A(\cdot)$, $B(\cdot)$, and $G(\cdot)$ are $n \times n$, $n \times p$, and $n \times l$ matrix-valued functions, respectively, which are assumed to be continuous on $[0, T]$, while V^0 and V^T are constant $n \times n$ matrices. Equation (2.2) determines the process $\{x_t, 0 \leq t \leq T\}$ by means of mixed boundary conditions. $\{x_t\}$ is, therefore, no more a Markov process. As explained in [8], $\{x_t\}$ is a “reciprocal” process. We do not discuss this any further.

The measurement process $\{y_t, 0 \leq t \leq T\}$ is given by

$$dy_t = C(t)x_t dt + dw_{0t} \quad (2.5)$$

$$y_0 = 0$$

where y_t has dimension m and $\{w_{0t}, t \geq 0\}$ is an m -dimensional Brownian motion, independent of w and v , with

$$Ew_{0t} w_{0s}^* = \min(t, s)I \quad (2.6)$$

while $C(t)$ is a continuous matrix-valued function. The extension of our results to the situation when $Ew_{0t} w_{0s}^* = \int_0^{t \wedge s} S(\sigma) d\sigma$, with $S(t) > 0$ a.e. t , is obvious.

In this paper we shall study the likelihood ratio formula for discrimination between the Gaussian processes w_{0t} and y_t . This expression for the likelihood ratio is useful both for detecting signal in additive to white noise and for identifying unknown parameters in the system.

We begin with subtracting the mean terms from the processes of interest as they do not play any significant role in determining the likelihood ratio. Let

$$v^c = v - m_v \quad (2.7a)$$

$$x_t^c = x_t - x^u(t) \quad (2.7b)$$

$$y_t^c = y_t - \int_0^t C(s)x^u(s) ds \quad (2.7c)$$

with

$$x^u(t) = Ex_t. \quad (2.7d)$$

Then both x_t^c and y_t^c have zero mean and they satisfy

$$dx_t^c = A(t)x_t^c dt + B(t)dw_t \quad (2.8a)$$

$$V^0 x_0^c + V^T x_T^c = v^c \quad (2.8b)$$

$$dy_t^c = C(t)x_t^c dt + dw_{0t} \quad (2.8c)$$

$$y_0^c = 0. \quad (2.8d)$$

Under the assumption that

$$F \triangleq V^0 + V^T \Phi(T) \quad (2.9)$$

is invertible, we can explicitly solve for x_t^c as

$$x_t^c = \Phi(t)F^{-1} \left\{ v^c + V^0 \int_0^t \Phi(s)^{-1} B(s) dw_s - V^T \Phi(T) \int_t^T \Phi(s)^{-1} B(s) dw_s \right\} \quad (2.10)$$

where $\Phi(t) \triangleq \Phi(t, 0)$, with $\Phi(t, s)$ denoting the state transition matrix for $\dot{x}(t) = A(t)x(t)$. It is also easy to see that $x^u(t)$ satisfies

$$\dot{x}^u(t) = A(t)x^u(t) + G(t)u(t) \quad (2.11a)$$

$$V^0 x^u(0) + V^T x^u(T) = m_v. \quad (2.11b)$$

Let p_y (respectively, p_y^c) denote the measure induced by the process y_t , $0 \leq t \leq T$ (respectively, y_t^c , $0 \leq t \leq T$) on the space $\mathbb{G}^m[0, T]$ of $\mathbb{R}^m$ -valued continuous functions on $[0, T]$ with p_w denoting the Wiener measure thereon. We shall show in the next section that p_y (respectively, p_y^c) is absolutely continuous with respect to p_w . It is easy to see that the Radon-Nikodym derivative

$$LR(y) \triangleq \frac{dp_y}{dp_w}(y) = \frac{dp_y^c}{dp_w}(y^c + Ey) \quad (2.12a)$$

with

$$Ey_t = \int_0^t C(s)x^u(s) ds \quad (2.12b)$$

which is the likelihood ratio for our model (2.1), (2.2), and (2.5).

III. EVALUATION OF THE LIKELIHOOD RATIO

From (2.10) we see that

$$C(t)x_t^c = V(t)v^c + \int_0^t h(t, u) dw_u, \quad 0 \leq t \leq T \quad (3.1)$$

where

$$V(t) = C(t)\Phi(t)F^{-1}, \quad 0 \leq t \leq T \quad (3.2a)$$

and

$$h(t, u) = \begin{cases} C(t)\Phi(t)F^{-1}V^0\Phi(u)^{-1}B(u), & t \geq u \\ -C(t)\Phi(t)F^{-1}V^T\Phi(T)\Phi(u)^{-1}B(u), & t < u. \end{cases} \quad (3.2b)$$

The covariance function of the signal process $\{C(t)x_t^c\}$ is then

$$\begin{aligned} r(t, u) &\triangleq C(t)E(x_t^c x_u^{c*})C(u)^* \\ &= V(t)\Sigma_v V(u)^* + \int_0^T h(t, z)h(u, z)^* dz. \end{aligned} \quad (3.3)$$

Then with $f, g \in L_2^m[0, T]$, and $[\cdot, \cdot]$ denoting inner product,

$$\begin{aligned} E \left(\int_0^T [f(t), dy_t^c] \right) \left(\int_0^T [g(t), dy_t^c] \right) \\ = \int_0^T [f(t), ((I+R)g)(t)] dt \end{aligned} \quad (3.4)$$

where R is the linear operator from $L_2^m[0, T]$ to $L_2^m[0, T]$ with

$$(Rf)(t) = \int_0^T r(t, u)f(u) du. \quad (3.5)$$

One can easily prove that: 1) $-1 \notin \sigma(R)$, where $\sigma(R)$ denotes the spectrum of R ; and 2) $r(t, u)$ is continuous in $[0, T] \times [0, T]$. It follows from an important result due to Shepp [10] that p_y is absolutely continuous with respect to p_w and the likelihood ratio is given by

$$\begin{aligned} LF(y) &= d^{-1/2} \exp. \left(\frac{1}{2} \int_0^T \int_0^T [g(t, s) dy_s^c, dy_t^c] \right. \\ &\quad \left. + \int_0^T [C(t)x^u(t), dy_t^c] + \frac{1}{2} \int_0^T [C(t)x^u(t), C(t)x^u(t)] dt \right) \end{aligned} \quad (3.6)$$

with

$$d = \det.(I + R) \quad (3.7)$$

where “det.” stands for the determinant of the operator $(I + R)$, $g(t, s)$ is the kernel of the integral operator $R(I + R)^{-1}$, and the first term within the exponential appearing in (3.6) is a double Wiener integral.

Before proceeding further, let us first fix some notations. For an integral operator L , we use $(L dy)_t(t)$ to denote $\int_0^T L(t, s) dy_s$, where $L(t, s)$ is the kernel of the operator L . For a square integrable function h , we use $[h, dy]$ to denote $\int_0^T [h(t), dy_t]$, whereas $[h, x]$ stand for $\int_0^T [h(t), x_t] dt$. It is easy to see that

$$E([h_1, f(\cdot)x^c][h_2, dy^c]) = [h_1, Rh_2]$$

and

$$E([h_3, dy^c][h_4, dy^c]) = [h_3, (I + R)h_4].$$

The Gaussian assumption of the processes implies that

$$C(t)\hat{x}_t^c \triangleq E\{C(t)x_t^c | y_s^c, 0 \leq s \leq T\} = (R(I + R)^{-1} dy^c)(t). \quad (3.8)$$

Thus, the likelihood functional may be rewritten as

$$\begin{aligned} LF(y) &= d^{-1/2} \exp. \left(\frac{1}{2} \int_0^T [C(t)\hat{x}_t^c, dy_t^c] \right. \\ &\quad \left. + \int_0^T [C(t)x^u(t), dy_t^c] + \frac{1}{2} \int_0^T [C(t)x^u(t), C(t)x^u(t)] dt \right). \end{aligned} \quad (3.9)$$

IV. THE SMOOTHING EQUATIONS

Adams, Willsky, and Levy developed an algorithm in [2] for smoothing in a TPBVP model as described below

$$d\hat{x}_t^c = A(t)\hat{x}_t^c dt + B(t)B(t)^* \hat{p}_t^c dt \quad (4.1a)$$

$$d\hat{p}_t^c = C(t)^* C(t)\hat{x}_t^c dt - A(t)^* \hat{p}_t^c - C(t)^* dy_t^c, \quad 0 \leq t \leq T \quad (4.1b)$$

with the boundary conditions

$$V^0 \Sigma_v^{-1} V^0 \hat{x}_0^c + V^0 \sum_{v=1}^{n-1} V^T \hat{x}_T^c - \hat{p}_0^c = 0 \quad (4.2a)$$

$$V^T \Sigma_v^{-1} V^0 \hat{x}_0^c + V^T \sum_{v=1}^{n-1} V^T \hat{x}_T^c + \hat{p}_T^c = 0. \quad (4.2b)$$

An important point to note is that the determination of $\hat{x}_t^c$ needs the help of an auxiliary n -dimensional variable $\hat{p}_t^c$ and we have to effectively work with a $2n$ -dimensional system. One drawback of the preceding algorithm is the necessity of assuming the invertibility of the covariance matrix Σ_v . This precludes important special cases, e.g., a system with (partially) deterministic boundary conditions, a special case of which is the periodic system studied in [4]. We describe here an alternative algorithm that does not need the assumption about the invertibility of Σ_v .

Going back to the solution x_t^c given in (2.10), we see that one can write in operator notation

$$x_t^c = S \begin{pmatrix} v^c \\ dw. \end{pmatrix} (t) \quad (4.3)$$

where $S: M \rightarrow L_2^n[0, T]$, with $M = \mathbb{R}^n \times L_2^n[0, T]$, is a linear operator defined by

$$S \begin{pmatrix} e \\ f \end{pmatrix} = \tilde{V}(t)e + \int_0^T \tilde{h}(t, u)f(u) du, \quad e \in \mathbb{R}^n, f \in L_2^n[0, T]$$

$$\tilde{V}(t) = \Phi(t)F^{-1}, \quad \tilde{h}(t, u) = \begin{cases} \Phi(t)F^{-1}V^0\Phi(u)^{-1}B(u), & u \in [0, t] \\ -\Phi(t)F^{-1}V^T\Phi(T)\Phi(u)^{-1}B(u), & u \in (t, T]. \end{cases} \quad (4.4)$$

The inner product in M is obviously defined by

$$\left[\begin{pmatrix} e_1 \\ f_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ f_2 \end{pmatrix} \right]_M = e_1^* e_2 + [f_1, f_2]. \quad (4.5)$$

As explained in the preceding section, the smoother $\hat{x}_t^c \triangleq E\{x_t^c | y_s^c, 0 \leq s \leq T\}$ is given by

$$\hat{x}_t^c = (R_{xy} R_{yy}^{-1} dy^c)(t)$$

where R_{xy} and R_{yy} are covariance operators defined by

$$E([h_1, x^c][h_2, dy^c]) = [h_1, R_{xy}h_2] \quad (4.6a)$$

$$E([h_3, dy^c][h_4, dy^c]) = [h_3, R_{yy}h_4] \quad (4.6b)$$

with $h_1 \in L_2^n[0, T]$ and $h_2, h_3, h_4 \in L_2^n[0, T]$. Now

$$\begin{aligned} E([h_1, x^c][h_2, dy^c]) &= E([h_1, x^c][h_2, Cx^c + dw_0]) = E([h_1, x^c][C^*h_2, x^c]) \\ &= E \left(\left[h_1, S \begin{pmatrix} v^c \\ dw. \end{pmatrix} \right] \left[C^*h_2, S \begin{pmatrix} v^c \\ dw. \end{pmatrix} \right] \right) \\ &= E \left(\left[S^*h_1, \begin{pmatrix} v^c \\ dw. \end{pmatrix} \right]_M \left[S^*C^*(\cdot)h_2, \begin{pmatrix} v^c \\ dw. \end{pmatrix} \right]_M \right) \end{aligned} \quad (4.7)$$

with S^* the adjoint of the operator S , $(Cf)(t) = C(t)f(t)$ and $(C^*g)(t) = C^*(t)g(t)$.

It is easy to see that

$$\begin{aligned} E \left(\left[\begin{pmatrix} e_1 \\ f_1 \end{pmatrix}, \begin{pmatrix} v^c \\ dw. \end{pmatrix} \right]_M \left[\begin{pmatrix} e_2 \\ f_2 \end{pmatrix}, \begin{pmatrix} v^c \\ dw. \end{pmatrix} \right]_M \right) \\ = \left[\begin{pmatrix} e_1 \\ f_1 \end{pmatrix}, \bar{D} \begin{pmatrix} e_2 \\ f_2 \end{pmatrix} \right] \end{aligned} \quad (4.8a)$$

where $\bar{D}: M \rightarrow M$ is the linear operator defined by

$$\bar{D} \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} \Sigma_v & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}. \quad (4.8b)$$

I stands for the identity map. Applying this to (4.7), we get

$$E([h_1, x^c][h_2, dy^c]) = [S^*h_1, \bar{D}S^*C^*h_2]_M \quad (4.9)$$

so that $R_{xy} = S\bar{D}S^*C^*$.

Similarly,

$$\begin{aligned} E([h_3, dy^c][h_4, dy^c]) &= E([h_3, Cx^c + dw_0][h_4, Cx^c + dw_0]) \\ &= E([h_3, Cx^c][h_4, Cx^c] + [h_3, dw_0][h_4, dw_0]) \\ &= [h_3, (I + CS\bar{D}S^*C^*)h_4] \end{aligned} \quad (4.10)$$

implying that

$$R_{yy} = (I + CS\bar{D}S^*C^*)$$

so that

$$\hat{x}_t^c = (S\bar{D}S^*C^*(I + CS\bar{D}S^*C^*)^{-1}dy^c)(t). \quad (4.11)$$

Our next step is to determine a realization for the operator appearing in (4.11). We start with the operator S . A realization of S is, of course, the TPBVP model itself; that is, if $(\frac{e}{f}) \in M$ and $x \in L_2^n[0, T]$ satisfies

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)f(t) \\ V^0x(0) + V^Tx(T) &= e \end{aligned} \quad (4.12)$$

then $x = S(\frac{e}{f})$. Next we consider S^* . For $g \in L_2^n[0, T]$

$$\begin{aligned} \left[S \begin{pmatrix} e \\ f \end{pmatrix}, g \right] &= \int_0^T g(t)^* \left(\tilde{V}(t)e + \int_0^T \tilde{h}(t, u)f(u) du \right) dt \\ &= \left[\begin{pmatrix} e \\ f \end{pmatrix}, \begin{pmatrix} \int_0^T \tilde{V}(t)^*g(t) dt \\ \int_0^T \tilde{h}(t, \cdot)^*g(t) dt \end{pmatrix} \right]_M \end{aligned} \quad (4.13)$$

where the vector component

$$\int_0^T \tilde{V}(t)^*g(t) dt = F^{-1*} \int_0^T \Phi(t)^*g(t) dt \quad (4.14a)$$

and the function component

$$\begin{aligned} \int_0^T \tilde{h}(u, t)^*g(u) du &= B(t)^*\Phi(t)^{-1*} \left(V^0F^{-1*} \int_t^T \Phi(u)^*g(u) du \right. \\ &\quad \left. - \Phi(T)^*V^TF^{-1*} \int_0^t \Phi(u)^*g(u) du \right). \end{aligned} \quad (4.14b)$$

Let us define

$$\begin{aligned} p(t) &= \Phi(t)^{-1*} \left(V^0F^{-1*} \int_t^T \Phi(u)^*g(u) du \right. \\ &\quad \left. - \Phi(T)^*V^TF^{-1*} \int_0^t \Phi(u)^*g(u) du \right). \end{aligned}$$

Simple calculation leads to

$$\dot{p}(t) = -A(t)*p(t) - g(t) \quad (4.15)$$

with

$$p(0) = V^0*F^{-1}* \int_0^T \Phi(u)*g(u) du \quad (4.16a)$$

and

$$p(T) = -V^T*F^{-1}* \int_0^T \Phi(u)*g(u) du. \quad (4.16b)$$

Therefore,

$$p(0) - \Phi(T)*p(T) = \int_0^T \Phi(u)*g(u) du$$

implying that $p(0) = V^0*F^{-1}*[p(0) - \Phi(T)*p(T)]$ and it follows that:

$$\Phi(T)*V^T*F^{-1}*p(0) + V^0*F^{-1}*\Phi(T)*p(T) = 0. \quad (4.17)$$

This gives us the following realization of S^* :

$$S^*g = \begin{pmatrix} e \\ f \end{pmatrix}$$

which is determined by

$$\begin{aligned} \dot{p}(t) &= -A(t)*p(t) - g(t) \\ \Phi(T)*V^T*F^{-1}*p(0) + V^0*F^{-1}*\Phi(T)*p(T) &= 0 \\ f(t) &= B(t)*p(t) \\ e &= F^{-1}*(p(0) - \Phi(T)*p(T)). \end{aligned} \quad (4.18)$$

Let us now consider the equation

$$z = (I + CS\bar{D}S^*C^*)f. \quad (4.19)$$

To this input-output relationship corresponds the system described by Fig. 1.

$$g(t) = C(t)*f(t) \quad (4.20a)$$

$$\dot{p}(t) = -A(t)*p(t) - g(t) \quad (4.20b)$$

$$\Phi(T)*V^T*F^{-1}*p(0) + V^0*F^{-1}*\Phi(T)*p(T) = 0 \quad (4.20c)$$

$$\begin{pmatrix} m_b \\ m(t) \end{pmatrix} = \begin{pmatrix} F^{-1}*(p(0) - \Phi(T)*p(T)) \\ B(t)*p(t) \end{pmatrix}, \quad \begin{pmatrix} l_b \\ l(t) \end{pmatrix} = \begin{pmatrix} \Sigma_v m_b \\ m(t) \end{pmatrix} \quad (4.20d)$$

$$\dot{x}(t) = A(t)x(t) + B(t)l(t) \quad (4.20e)$$

$$V^0x(0) + V^Tx(T) = l_b \quad (4.20f)$$

$$z(t) = C(t)x(t) + f(t) \quad (4.20g)$$

with $t \in [0, T]$. Substituting $f(t) = z(t) - C(t)x(t)$ in (4.20e) and (4.20a) gives

$$\dot{x}(t) = A(t)x(t) + B(t)B(t)*p(t) \quad (4.21a)$$

$$\dot{p}(t) = C(t)*C(t)x(t) - A(t)*p(t) - C(t)*z(t) \quad (4.21b)$$

$$V^0x(0) + V^Tx(T) = \Sigma_v F^{-1}*(p(0) - \Phi(T)*p(T)) \quad (4.22a)$$

$$\Phi(T)*V^T*F^{-1}*p(0) + V^0*F^{-1}*\Phi(T)*p(T) = 0. \quad (4.22b)$$

Fig. 1. Diagram of the system $(I + CS\bar{D}S^*C^*)$.

From Fig. 1 we see that $x \in L_2^n[0, T]$ in this realization satisfies

$$x = S\bar{D}S^*C^*(I + CS\bar{D}S^*C^*)^{-1}z. \quad (4.23)$$

Using (4.23) we see from (4.11) that the smoother is given by the following set of equations:

$$d\hat{x}_t^c = A(t)\hat{x}_t^c dt + B(t)B(t)^*\hat{p}_t^c dt \quad (4.24a)$$

$$d\hat{p}_t^c = C(t)^*C(t)\hat{x}_t^c dt - A(t)^*\hat{p}_t^c dt - C(t)^*dy_t^c \quad (4.24b)$$

with the boundary conditions

$$V^0\hat{x}_0^c + V^T\hat{x}_T^c = \Sigma_v F^{-1}*(\hat{p}_0^c - \Phi(T)^*\hat{p}_T^c) \quad (4.25a)$$

$$\Phi(T)^*V^T*F^{-1}*\hat{p}_0^c + V^0*F^{-1}*\Phi(T)^*\hat{p}_T^c = 0. \quad (4.25b)$$

The equations (4.24a), (4.24b) are identical to those obtained in [2] and the boundary conditions here do not need the assumption that Σ_v is invertible. In case Σ_v is, indeed, invertible, we can rewrite the boundary conditions (4.25a), (4.25b) in the form (4.2a), (4.2b) as obtained in [2]. As mentioned in the Introduction, Adams [5] derived smoothing algorithms in the situation when the boundary noise covariance is singular. In our notation and specializing to the TPBVP case, the smoother boundary conditions obtained by Adams [5, eq. (2.5.37)] were

$$-V \begin{bmatrix} \hat{x}_0^c \\ \hat{x}_T^c \end{bmatrix} + \Sigma_v W \begin{bmatrix} \hat{p}_0^c \\ \hat{p}_T^c \end{bmatrix} = 0 \quad (4.26a)$$

and

$$M \begin{bmatrix} \hat{p}_0^c \\ \hat{p}_T^c \end{bmatrix} = 0 \quad (4.26b)$$

where $V = [V^0 \ V^T]$, N is an $n \times 2n$ matrix such that $\Gamma = \begin{bmatrix} -N \\ N \end{bmatrix}$ is invertible and $N = [W^0 \ W^T]$, $M = [M^0 \ M^T]$ must satisfy

$$\begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} = -V^*W + N^*M. \quad (4.27)$$

It is easy to check that one may take

$$N = [-\Phi(T) \ I] \quad (4.28a)$$

$$W = [F^{-1} \ -F^{-1}*\Phi(T)^*] \quad (4.28b)$$

$$M = [V^T*F^{-1}*\Phi(T)^*^{-1}V^0*F^{-1}*\Phi(T)^*] \quad (4.28c)$$

and, with these choices, the smoother boundary conditions (4.26a), (4.26b) correspond exactly to the boundary conditions (4.25a), (4.25b).

Let us end this section by considering the special case of purely deterministic boundary conditions in which case we have $\Sigma_v = 0$. Then (4.25a), (4.25b) become

$$V^0\hat{x}_0^c + V^T\hat{x}_T^c = 0 \quad (4.29a)$$

$$\Phi(T)^*V^T*F^{-1}*\hat{p}_0^c + V^0*F^{-1}*\Phi(T)^*\hat{p}_T^c = 0. \quad (4.29b)$$

Let us specialize still further to the case of periodic boundary conditions as studied in [4]. In this case we can take $V^0 = I$ and $V^T = -I$. Since $\Phi(T)$ is invertible and commutes with F^{-1} ,

(4.26a), (4.26b) become in this case

$$\hat{x}_0^c = \hat{x}_T^c \quad (4.30a)$$

and

$$\hat{p}_0^c = \hat{p}_T^c. \quad (4.30b)$$

V. KREIN FACTORIZATION AND THE LIKELIHOOD RATIO

In the previous section we have obtained equations needed to evaluate the exponential part of the likelihood ratio formula obtained in (3.9). We shall now obtain an expression for the determinant term involving the system matrices. We do this by using the Krein factorization for linear operators (see [11] for details). Let us consider the operator $I + R$, with the linear operator R defined by (3.5). The operator R is Hilbert-Schmidt, self-adjoint, and nonnegative definite. By using the Krein factorization theorem, there exists a Volterra operator $L: L_2^m[0, T] \rightarrow L_2^m[0, T]$ with kernel $l(t, s)$ such that

$$(I + R)^{-1} = (I - L^*)(I - L). \quad (5.1)$$

The continuity of the kernel $r(t, s)$ of R implies the continuity of the kernel $l(t, s)$ of L in $0 \leq s \leq t \leq T$. Then $(L + L^*)$ has finite trace and from [12] we know that

$$\log(d^{-1/2}) = -\frac{1}{2} \text{Trace}(L + L^*) = -\frac{1}{2} \int_0^T \text{Tr} \, l(t, t) dt. \quad (5.2)$$

Our objective now is to explicitly determine the kernel $l(t, s)$.

Define the operator $G(t): L_2^n[0, t] \rightarrow L_2^n[0, t]$ by

$$G(t) = R(t)(I(t) + R(t))^{-1} \quad (5.3a)$$

where $R(t): L_2^n[0, t] \rightarrow L_2^n[0, t]$ is given by

$$(R(t)f)(s) = \int_0^t r(s, u)f(u) du, \quad 0 \leq s \leq t \quad (5.3b)$$

and $I(t)$ is the identity map on $L_2^n[0, t]$. Let $g(t; s, u)$ be the kernel of the operator $G(t)$. It follows from the proof of Krein's theorem as given in [9, pp. 130-132] that

$$l(t, s) = g(t; t, s). \quad (5.4)$$

We can also give a probabilistic interpretation of this kernel. Consider the model (2.8c), (2.8d) restricted to the interval $[0, t]$

$$dy_s^c = C(s)x_s^c ds + dw_{0s}, \quad 0 \leq s \leq t \quad (5.5a)$$

$$y_0^c = 0. \quad (5.5b)$$

As we have already seen in (3.8),

$$E\{C(s)x_s^c | y_u^c, 0 \leq u \leq t\} = (R(t)(I(t) + R(t))^{-1} dy_s^c)(s) \quad (5.6)$$

which, in our notation, may be rewritten as

$$\begin{aligned} \int_0^t g(t; s, u) dy_u^c &= E\{C(s)x_s^c | y_u^c, 0 \leq u \leq t\} \\ &\triangleq C(s)\hat{x}_{s|t}^c. \end{aligned} \quad (5.7)$$

In particular, using (5.4), we get

$$\begin{aligned} \int_0^t l(t, s) dy_s^c &= E\{C(t)x_t^c | y_s^c, 0 \leq s \leq t\} \\ &\triangleq \hat{x}_t^{cf} \end{aligned} \quad (5.8)$$

the filtered estimate of x_t^c .

To obtain the determinant d in terms of the system parameters, we consider the model (2.8a)-(2.8d) restricted to the subinterval $[0, t]$, for fixed t . The solution x_s^c , $s \in [0, t]$, as obtained in (2.10), may be reexpressed as

$$x_s^c = \Phi(s)(V^0 + V^t\Phi(t))^{-1} \left(v_t + V^0 \int_0^s \Phi(u)^{-1} B(u) dw_u - V^t\Phi(t) \int_s^t \Phi(u)^{-1} B(u) dw_u \right) \quad (5.9)$$

with

$$V^t = V^T\Phi(T)\Phi(t)^{-1} \quad (5.10a)$$

and

$$v_t = v^c - V^T\Phi(T) \int_t^T \Phi(u)^{-1} B(u) dw_u. \quad (5.10b)$$

If we compare this solution to the solution (2.10) obtained for the model (2.8a), (2.8b), we see immediately that the process $\{x_s^c, s \in [0, t]\}$ satisfies

$$dx_s^c = A(s)x_s^c ds + B(s)dw_s, \quad 0 \leq s \leq t \quad (5.11a)$$

$$V^0 x_0^c + V^t x_t^c = v_t. \quad (5.11b)$$

The random vector v_t is normally distributed with expectation zero and covariance matrix

$$\begin{aligned} \Sigma_v(t) &= \Sigma_v + V^T\Phi(T) \\ &\cdot \int_t^T \Phi(u)^{-1} B(u) B(u)^* \Phi(u)^{-1*} du \Phi(T)^* V^T. \end{aligned} \quad (5.12)$$

Furthermore, v_t is independent of $\{w_s, s \in [0, t]\}$ and $\{w_{0s}, s \in [0, t]\}$. The measurement in $[0, t]$ is, of course, as given in (2.8c), (2.8d) restricted to $[0, t]$; namely,

$$dy_s^c = C(s)x_s^c ds + dw_{0s}, \quad 0 \leq s \leq t \quad (5.11c)$$

$$y_0^c = 0. \quad (5.11d)$$

The transition from (2.8a), (2.8b) to (5.11a), (5.11b) can be described as a factorization of the operator $S(t)$ that maps v^c and dw into x_s^c , $0 \leq s \leq t$: $S(t): \mathbb{R}^n \times L_2^2[0, T] \rightarrow L_2^n[0, t]$ defined by

$$S(t) \begin{pmatrix} e \\ f \end{pmatrix} (s) = \Phi(s)F^{-1} \left(e + V^0 \int_0^s \Phi(u)^{-1} B(u)f(u) du - V^T\Phi(T) \int_s^T \Phi(u)^{-1} B(u)f(u) du \right). \quad (5.13)$$

The factorization takes the following form: $S(t) := S_1(t)U(t)$ with

$$S_1(t): \mathbb{R}^n \times L_2^2[0, t] \rightarrow L_2^n[0, t]$$

$$\begin{aligned} S_1(t) \begin{pmatrix} e \\ f \end{pmatrix} (s) &= \Phi(s)(V^0 + V^t\Phi(t))^{-1} \\ &\cdot \left(e + V^0 \int_0^s \Phi(u)^{-1} B(u)f(u) du - V^t\Phi(t) \int_s^t \Phi(u)^{-1} B(u)f(u) du \right) \end{aligned} \quad (5.14a)$$

and

$$U(t) : \mathbb{R}^n \times L_2^p[0, T] \rightarrow \mathbb{R}^n \times L_2^p[0, t]$$

$$U(t) \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} e - V^T \Phi(T) \int_t^T \Phi(u)^{-1} B(u) f(u) du \\ f(\cdot)|_{[0, t]} \end{pmatrix}. \quad (5.14b)$$

The operator $U(t)$ maps v^c and $\{dw_s, s \in [t, T]\}$ into the new boundary vector v_t , while the operator $S_1(t)$ maps v_t and $\{dw_s, s \in [0, t]\}$ into $x_s^c, s \in [0, t]$. Using the procedure used in deriving (4.11), we see that

$$\begin{aligned} x_{s|t}^c &\triangleq E\{x_s^c | y_u^c, 0 \leq u \leq t\} \\ &= (S(t) \bar{D} S(t)^* C^* (I(t) + CS(t) \bar{D} S(t)^* C^*) dy_s^c)(s). \end{aligned} \quad (5.15)$$

We now determine a realization of (5.15) by using the factorization of $S(t)$ as described above. A realization of the operator $S_1(t)$ is the TPBVP model

$$\dot{x}(s) = A(s)x(s) + B(s)f(s), \quad 0 \leq s \leq t \quad (5.16a)$$

$$V^0 x(0) + V^t x(t) = e \quad (5.16b)$$

and

$$x(s) = S_1(t) \begin{pmatrix} e \\ f(\cdot) \end{pmatrix} (s), \quad 0 \leq s \leq t. \quad (5.17)$$

Following the steps (4.13)–(4.18) we see that a realization for

$$S_1^*(t)g = \begin{pmatrix} e \\ f \end{pmatrix}, \quad 0 \leq s \leq t \quad (5.18)$$

is given by

$$\dot{p}(s) = -A(s)^* p(s) - g(s), \quad 0 \leq s \leq t \quad (5.19a)$$

$$\Phi(t)^* V^t F^{-1} p(0) + V^0 F^{-1} \Phi(t)^* p(t) = 0 \quad (5.19b)$$

$$f(s) = B(s)^* p(s), \quad 0 \leq s \leq t \quad (5.19c)$$

$$e = F^{-1} (p(0) - \Phi(t)^* p(t)) \quad (5.19d)$$

(note that $\Phi(t)^* V^t = \Phi(T)^* V^T$).

We also have to determine $U(t)^* : \mathbb{R}^n \times L_2^p[0, t] \rightarrow \mathbb{R}^n \times L_2^p[0, T]$. Let $e_1, e_2 \in \mathbb{R}^n, f_1 \in L_2^p[0, T]$, and $f_2 \in L_2^p[0, t]$. Then

$$\begin{aligned} &\left[U(t) \begin{pmatrix} e_1 \\ f_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ f_2 \end{pmatrix} \right] \\ &= e_2^* \left(e_1 - V^T \Phi(T) \int_t^T \Phi(u)^{-1} B(u) f_1(u) du \right) \\ &\quad + \int_0^t f_2(u)^* f_1(u) du + e_1^* e_2 + \int_0^t f_1(u)^* f_2(u) du \\ &\quad - \int_t^T f_1(u)^* B(u)^* \Phi(u)^{-1} du \Phi(T)^* V^T e_2 \\ &= \left[\begin{pmatrix} e_1 \\ f_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ \begin{cases} f_2(s), & 0 \leq s \leq t \\ -B(s)^* \Phi(s)^{-1} \Phi(T)^* V^T e_2, & t < s \leq T \end{cases} \end{pmatrix} \right] \end{aligned}$$

so that

$$U(t)^* \begin{pmatrix} e_2 \\ f_2 \end{pmatrix} = \begin{pmatrix} e_2 \\ \begin{cases} f_2(s), & 0 \leq s \leq t \\ -B(s)^* \Phi(s)^{-1} \Phi(T)^* V^T e_2, & t < s \leq T \end{cases} \end{pmatrix}.$$

Fig. 2. Diagram of the system $[I(t) + CS_1(t)U(t)\bar{D}U(t)^*S_1(t)^*C^*]$.

The equation

$$z = (I(t) + CS_1(t)U(t)\bar{D}U(t)^*S_1(t)^*C^*)f$$

is thus described by Fig. 2.

From the realization of $S_1(t)^*$, we know that

$$\alpha = F^{-1} (p(0) - \Phi(t)^* p(t)); \quad g(s) = B(s)^* p(s), \quad 0 \leq s \leq t$$

where

$$\dot{p}(s) = -A(s)^* p(s) - C(s)^* f(s)$$

$$\Phi(t)^* V^t F^{-1} p(0) + V^0 F^{-1} \Phi(t)^* p(t) = 0.$$

Similarly, from the realization of $U(t)^*$,

$$\beta = \alpha; \quad h(s) = \begin{cases} g(s), & 0 \leq s \leq t \\ -B(s)^* \Phi(s)^{-1} \Phi(t)^* V^T \alpha, & s < t \leq T. \end{cases}$$

Furthermore, we have

$$\gamma = \Sigma_v \beta; \quad k(s) = h(s), \quad 0 \leq s \leq T$$

$$\delta = \gamma - V^T \Phi(T) \int_t^T \Phi(u)^{-1} B(u) k(u) du; \quad l(s) = k(s), \quad 0 \leq s \leq t$$

$$\dot{x}(s) = A(s)x(s) + B(s)l(s), \quad 0 \leq s \leq t$$

$$V^0 x(0) + V^t x(t) = \delta$$

$$z(s) = C(s)x(s) + f(s), \quad 0 \leq s \leq t. \quad (5.21)$$

Substitution of $f(s) = z(s) - C(s)x(s)$ yields

$$\dot{x}(s) = A(s)x(s) + B(s)B(s)^* p(s) \quad (5.22a)$$

$$\dot{p}(s) = C(s)^* C(s)x(s) - A(s)^* p(s) - C(s)^* z(s) \quad (5.22b)$$

$$V^0 x(0) + V^t x(t) = \Sigma_v(t) F^{-1} (p(0) - \Phi(t)^* p(t)) \quad (5.22c)$$

$$\Phi(T)^* V^T F^{-1} p(0) + V^0 F^{-1} \Phi(t)^* p(t) = 0 \quad (5.22d)$$

where $\Sigma_v(t)$ is the covariance matrix given in (5.12). Furthermore,

$$\begin{aligned} x &= S_1(t)U(t)\bar{D}U(t)^*S_1(t)^*C^*(I(t) \\ &\quad + CS_1(t)U(t)\bar{D}U(t)^*S_1(t)^*C^*)^{-1}z. \end{aligned} \quad (5.23)$$

The estimates $\hat{x}_{s|t}^c$ may now be determined as the solution of the following set of differential equations:

$$d\hat{x}_{s|t}^c = A(s)\hat{x}_{s|t}^c ds + B(s)B(s)^* \hat{p}_{s|t}^c ds \quad (5.24a)$$

$$d\hat{p}_{s|t}^c = C(s)^* C(s)\hat{x}_{s|t}^c ds - A(s)^* \hat{p}_{s|t}^c ds - C(s)^* dy_s^c \quad (5.24b)$$

with the boundary conditions

$$V^0 \hat{x}_{0|t}^c + V^t \hat{x}_{t|t}^c = \Sigma_v(t) F^{-1} (\hat{p}_{0|t}^c - \Phi(t)^* \hat{p}_{t|t}^c) \quad (5.25a)$$

$$\Phi(T)^* V^T F^{-1} \hat{p}_{0|t}^c + V^0 F^{-1} \Phi(t)^* \hat{p}_{t|t}^c = 0. \quad (5.25b)$$

Suppose now that the matrix $\Sigma_v(t)$ is invertible. We can, then, reexpress the boundary conditions (5.25a), (5.25b) as

$$\pi_0(t)\hat{x}_{0|t}^c + \pi_1(t)\hat{x}_{t|t}^c - \hat{p}_{0|t}^c = 0 \quad (5.26a)$$

$$\pi_1(t)^* \hat{x}_{0|t}^c + \pi_2(t)\hat{x}_{t|t}^c + \hat{p}_{t|t}^c = 0 \quad (5.26b)$$

where the $n \times n$ matrices $\pi_0(t)$, $\pi_1(t)$, and $\pi_2(t)$ are given by

$$\pi_0(t) = V^{0*} \Sigma_v(t)^{-1} V^0 \quad (5.27a)$$

$$\pi_1(t) = V^{0*} \Sigma_v(t)^{-1} V^T \quad (5.27b)$$

$$\pi_2(t) = V^{T*} \Sigma_v(t)^{-1} V^T. \quad (5.27c)$$

The kernel $g(t; \cdot, \cdot)$ of the operator $G(t)$ which maps the measurement $\{dy_s^c, s \in [0, t]\}$ into $\{C(s)\hat{x}_{s|t}^c; s \in [0, t]\}$ can be determined by solving the system of equations (5.24a), (5.24b) and (5.26a), (5.26b). Let

$$\begin{aligned} W^0(t) &\triangleq \begin{pmatrix} \pi_0(t) & -I \\ \pi_1(t) & 0 \end{pmatrix}; \quad W^T(t) \triangleq \begin{pmatrix} \pi_1(t) & 0 \\ \pi_2(t) & I \end{pmatrix} \\ \underline{A}(s) &\triangleq \begin{pmatrix} A(s) & B(s)B(s)^* \\ C(s)^*C(s) & -A(s)^* \end{pmatrix}; \quad \underline{B}(s) \triangleq \begin{pmatrix} 0 \\ -C(s)^* \end{pmatrix} \\ \underline{C}(s) &\triangleq (C(s) \ 0); \quad \hat{z}_{s|t} \triangleq \begin{pmatrix} \hat{x}_{s|t}^c \\ \hat{p}_{s|t}^c \end{pmatrix}. \end{aligned} \quad (5.28)$$

Then the process $\{\hat{z}_{s|t}; s \in [0, t]\}$ may be described by the following TPBVP model:

$$\begin{aligned} d\hat{z}_{s|t} &= \underline{A}(s)\hat{z}_{s|t}ds + \underline{B}(s)dy_s^c \\ W^0(t)\hat{z}_{0|t} + W^T(t)\hat{z}_{t|t} &= 0. \end{aligned} \quad (5.29)$$

We now determine the solution $\hat{z}_{s|t}$ of (5.29). Let Ψ be the transition matrix of $\underline{A}(t)$; $\Psi(t) \triangleq \Psi(t, 0)$ and let

$$F(t) \triangleq W^0(t)\Psi(t)^{-1} + W^T(t). \quad (5.30)$$

It follows then that

$$\begin{aligned} \hat{z}_{s|t} &= \Psi(t, s)^{-1} F(t)^{-1} (W^0(t) \int_0^s \Psi(u)^{-1} \underline{B}(u) dy_u^c \\ &\quad - W^T(t) \int_s^t \Psi(t, u) \underline{B}(u) dy_u^c). \end{aligned} \quad (5.31)$$

Since $C(s)\hat{x}_{s|t}^c \equiv \underline{C}(s)\hat{z}_{s|t}$, the kernel of the operator $G(t)$ is given by

$$g(t; s, u) = \begin{cases} \underline{C}(s)\Psi(t, s)^{-1} F(t)^{-1} W^0(t) \Psi(u)^{-1} \underline{B}(u), & 0 \leq u \leq s \\ -\underline{C}(s)\Psi(t, s)^{-1} F(t)^{-1} W^T(t) \Psi(t, u) \underline{B}(u), & s < u \leq t. \end{cases} \quad (5.32)$$

It follows then that the kernel of the Volterra operator L is given by

$$\begin{aligned} l(t, s) &\equiv g(t; t, s) = \underline{C}(t)F(t)^{-1} W^0(t) \Psi(s)^{-1} \underline{B}(s); \\ &0 \leq s \leq t \leq T. \end{aligned} \quad (5.33)$$

We have seen in (5.2) that to determine the likelihood ratio, we

have to calculate $\int_0^T \text{Tr. } [l(t, t)] dt$. From (5.33) it follows that:

$$\int_0^T \text{Tr. } (l(t, t)) dt = \int_0^T \text{Tr. } [\underline{C}(t)F(t)^{-1} W^0(t) \Psi(t)^{-1} \underline{B}(t)] dt. \quad (5.34)$$

Using (5.30), we have

$$\begin{aligned} \int_0^T \text{Tr. } (l(t, t)) dt &= \int_0^T \text{Tr. } [\underline{C}(t)(I - F(t)^{-1} W^T(t)) \underline{B}(t)] dt \\ &= \int_0^T \text{Tr. } [\underline{C}(t)\{F(t)^{-1}\}_{1,2} C(t)^*] dt. \end{aligned} \quad (5.35)$$

This integral may be evaluated by means of (5.30), (5.28), (5.27a)–(5.27c), (5.12), and (5.10a). Note also that $\pi_i(t)$, $i = 0, 1, 2$, satisfy the differential equations

$$\dot{\pi}_0(t) = \pi_1(t)B(t)B(t)^*\pi_1(t)^*, \quad \pi_0(T) = V^{0*} \Sigma_v^{-1} V^0 \quad (5.36a)$$

$$\begin{aligned} \dot{\pi}_1(t) &= \pi_1(t)B(t)B(t)^*\pi_2(t) - \pi_1(t)A(t), \\ \pi_1(T) &= V^{0*} \Sigma_v^{-1} V^T \end{aligned} \quad (5.36b)$$

$$\begin{aligned} \dot{\pi}_2(t) &= \pi_2(t)B(t)B(t)^*\pi_2(t) - \pi_2(t)A(t) - A(t)^*\pi_2(t), \\ \pi_2(T) &= V^{T*} \Sigma_v^{-1} V^T. \end{aligned} \quad (5.36c)$$

We can, therefore, evaluate (5.35) by using (5.30), (5.28), and (5.36a)–(5.36c) as well.

We are now in a position to give the complete formula for the likelihood ratio in terms of the system parameters. Going back to (3.9), we get

$$\begin{aligned} LF(y) &= \exp. \left(-\frac{1}{2} \int_0^T \text{Tr. } [\underline{C}(t)\{F(t)^{-1}\}_{1,2} C(t)^*] dt \right. \\ &\quad + \frac{1}{2} \int_0^T [C(t)\hat{x}_t^c, dy_t^c] + \int_0^T [C(t)x^u(t), dy_t^c] \\ &\quad \left. + \frac{1}{2} \int_0^T [C(t)x^u(t), C(t)x^u(t)] dt \right). \end{aligned} \quad (5.37)$$

VI. FILTERING EQUATIONS FOR TPBVP MODELS AND THE LIKELIHOOD RATIO

We have, so far, worked with the smoothing equations. If we instead worked with the filtered estimate $\hat{x}_t^{cf}$ as defined in (5.8), we could follow Kailath [13] to obtain the well-known formula for the likelihood ratio:

$$\begin{aligned} LF(y) &= \exp. \left(-\frac{1}{2} \int_0^T [C(t)\hat{x}_t^{cf}, C(t)\hat{x}_t^{cf}] dt \right. \\ &\quad + \int_0^T [C(t)\hat{x}_t^{cf}, dy_t^c] + \int_0^T [C(t)x^u(t), dy_t^c] \\ &\quad \left. + \frac{1}{2} \int_0^T [C(t)x^u(t), C(t)x^u(t)] dt \right). \end{aligned} \quad (6.1)$$

We note that the “determinant term” does not appear in this expression. We can determine $\hat{x}_t^{cf}$ from $\hat{z}_{t|t}$ given in (5.29).

Denoting the $n \times n$ unit matrix by I and the $n \times n$ zero matrix by 0 , we see from (5.31) that

$$\hat{z}_{t|t} = F(t)^{-1} W^0(t) \int_0^t \Psi(u)^{-1} \underline{B}(u) dy_u^c. \quad (6.2)$$

Using (5.30), we can rewrite this as

$$\hat{z}_{t|t} = \left[-F(t)^{-1} W^T(t) + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right] \int_0^t \Psi(t, u) \underline{B}(u) dy_u^c. \quad (6.3)$$

Define the $2n$ -dimensional vector η_t by

$$\eta_t = \begin{pmatrix} \eta_{1t} \\ \eta_{2t} \end{pmatrix} = \int_0^t \Psi(t, u) \underline{B}(u) dy_u^c. \quad (6.4)$$

The n -dimensional processes η_{1t} and η_{2t} satisfy

$$d\eta_{1t} = A(t)\eta_{1t}dt + B(t)B(t)^*\eta_{2t}dt, \quad \eta_{10} = 0 \quad (6.5a)$$

$$d\eta_{2t} = C(t)^*C(t)\eta_{1t}dt - A(t)^*\eta_{2t}dt - C(t)^*dy_t^c, \quad \eta_{20} = 0. \quad (6.5b)$$

Clearly

$$\begin{aligned} \hat{x}_t^{cf} &\triangleq \hat{x}_{t|t}^c = (I \ 0) \hat{z}_{t|t} \\ &= -(I \ 0) F(t)^{-1} W^T \begin{pmatrix} \eta_{1t} \\ \eta_{2t} \end{pmatrix} + \eta_{1t}. \end{aligned} \quad (6.6)$$

This is, of course, no real simplification to the likelihood ratio formula obtained in the previous section. Adams [5] proposed a different method to derive the filtering equations (6.5a), (6.5b). It has been shown in [5, sect. 4.2] that the TPBVP model, which is a reciprocal process, can be embedded into a larger model with state space of dimension $2n$ where it is indeed Markov. In this larger model, one can use the Kalman filtering technique to derive the desired filtering equations for the TPBVP model. This is consistent with the fact that the filtering equations (6.5a), (6.5b) also have dimension $2n$.

We now specialize to the situation when the TPBVP model reduces to the Markov case; that is, when $V^0 \equiv I$ and $V^T \equiv 0$. For simplicity, we also assume that the initial condition is given by $x_0^c = 0$. The boundary conditions for the smoothing equations, (5.25a), (5.25b), then become

$$\hat{x}_{0|t}^c = 0 \text{ and } \hat{p}_{t|t}^c = 0. \quad (6.7)$$

Comparing to (5.29), we see that $W^0(t)$ and $W^T(t)$ in this case may be chosen as

$$W^0(t) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \text{ and } W^T(t) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \quad (6.8)$$

With $\Psi(t)$ denoting, as before, the transition matrix for $\underline{A}(t)$, let

$$\Psi(t)^{-1} = \begin{pmatrix} e_{11}(t) & e_{12}(t) \\ e_{21}(t) & e_{22}(t) \end{pmatrix}. \quad (6.9)$$

Then

$$F(t) = W^0(t) \Psi(t)^{-1} + W^T(t) = \begin{pmatrix} e_{11}(t) & e_{12}(t) \\ 0 & I \end{pmatrix}. \quad (6.10)$$

It follows then that

$$F(t)^{-1} = \begin{pmatrix} e_{11}(t)^{-1} & -e_{11}(t)^{-1}e_{12}(t) \\ 0 & I \end{pmatrix}. \quad (6.11)$$

From (5.33), we get

$$l(t, s) = -C(t)e_{11}(t)^{-1}e_{12}(s)C(s)^*. \quad (6.12)$$

Let us now define

$$P(t) \triangleq -e_{11}(t)^{-1}e_{12}(t). \quad (6.13)$$

It is easy to verify that $P(t)$ satisfies

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)^* - P(t)C(t)^*C(t)P(t) + B(t)B(t)^* \quad (6.14a)$$

$$P(0) = 0 \quad (6.14b)$$

the familiar Riccati equation appearing in the Kalman filtering.

Using (6.2), we get

$$\begin{aligned} \begin{pmatrix} \hat{x}_{t|t}^c \\ \hat{p}_{t|t}^c \end{pmatrix} &= F(t)^{-1} W^0(t) \int_0^t \Psi(u)^{-1} \underline{B}(u) dy_u^c \\ &= \begin{pmatrix} -\int_0^t e_{11}(t)^{-1}e_{12}(u)C(u)^* dy_u^c \\ 0 \end{pmatrix}. \end{aligned} \quad (6.15)$$

Using the fact that $\Psi(t, s)$ is the state transition matrix and the identity $\hat{x}_t^{cf} \equiv \hat{x}_{t|t}^c$, we get

$$d\hat{x}_t^{cf} = (A(t) - P(t)C(t)^*C(t))\hat{x}_t^{cf}dt + P(t)C(t)^*dy_t^c \quad (6.16a)$$

which is precisely the Kalman filter.

Going back to the determinant term in the expression for the likelihood ratio, we see that

$$\int_0^T \text{Tr.} (l(t, t)) dt = \int_0^T \text{Tr.} [C(t)\{F(t)^{-1}\}_{1,2}C(t)^*] dt.$$

In the special case of the Markov model, (6.11) and (6.13) yields

$$\int_0^T \text{Tr.} (l(t, t)) dt = \int_0^T \text{Tr.} [C(t)P(t)C(t)^*] dt \quad (6.17)$$

which is the well-known correction term in the Itô integral discussed in detail in [12].

In fact, even for the TPBVP model, the matrix $C(t)\{F(t)^{-1}\}_{1,2}C(t)^*$ may be interpreted as an error covariance. One can show, by direct calculation, that the covariance operator of the process $\{C(t)(x_t^c - \hat{x}_t^{cf}), t \in [0, T]\}$ is identical to $(L + L^*)$. Therefore

$$E[C(t)(x_t^c - \hat{x}_t^{cf})(x_s^c - \hat{x}_s^{cf})^*C(s)^*] = \begin{cases} l(t, s), & 0 \leq s \leq t \\ l(s, t)^*, & t < s \leq T \end{cases} \quad (6.18)$$

and, in particular,

$$E[C(t)(x_t^c - \hat{x}_t^{cf})(x_t^c - \hat{x}_t^{cf})^*C(t)^*] = C(t)\{F(t)^{-1}\}_{1,2}C(t)^*. \quad (6.19)$$

This is a consequence of the fact that, although the TPBVP $\{x_t^c\}$ is not Markov, the “innovations process”

$$y_t^c - \int_0^t C(s)\hat{x}_s^{cf} ds \triangleq z_t^f \quad (6.20)$$

is still a white noise process in the sense that

$$E \left(\int_0^T [e_1(t), dz_t^f] \int_0^T [e_2(t), dz_t^f] \right) = [e_1, e_2] \quad (6.21)$$

for any $e_1, e_2 \in L_2^m[0, T]$.

VII. CONCLUSION

The extension of the results of this paper to general random fields, e.g., to multiparameter processes satisfying stochastic elliptic partial differential equations is being investigated at present. One difficulty is the extension of the Krein factorization theorem to the multiparameter case. The likelihood ratio formula can also be used for estimating parameters of the TPBVP model. In this context the statistical questions connected with the (maximum likelihood) parameter estimators become important. In practice, one could make n independent experiments and, based on the measurements $y_1^t, \dots, y_n^t$, $0 \leq t \leq T$, obtain the maximum likelihood estimate by maximizing the likelihood functional with respect to the unknown parameters. Statistical analysis of the parameter estimators are performed in [14].

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