



Exponential Stabilization of an Underactuated Autonomous Surface Vessel*

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Key Words—Asymptotic stability; exponential convergence; discontinuous feedback control.

Abstract—This paper studies the problem of controlling the planar position and orientation of an autonomous surface vessel using two independent thrusters. It is first shown that although the system is not asymptotically stabilizable to a given configuration using a time-invariant continuous feedback, it is strongly accessible and small-time locally controllable at any equilibrium. Time-invariant discontinuous feedback laws are then constructed to asymptotically stabilize the system to the desired configuration with exponential convergence rates. A simulation example is included to demonstrate the results. © 1997 Elsevier Science Ltd.

1. Introduction

In the past few years, there has been a considerable amount of interest in the control of underactuated mechanical systems, i.e. systems with fewer inputs than degrees of freedom. The possibility of controlling a system with fewer than the typical number of actuators is indeed appealing, for it allows to reduce cost, weight as well as the occurrence of component failures. However, in general, underactuated systems present challenges which are not found in systems with full control. Controllability, for instance, which is usually implied in systems with full control, is not easy to determine in an underactuated system. Control synthesis for an underactuated system is also more complex than it is for a system with full control. While many interesting techniques and results have been presented for underactuated systems (Orinolo and Nakamura, 1991; Leonard, 1995; Egeland *et al.*, 1996; Reyhanoglu *et al.*, 1996; Spong, 1996), the control of these systems still remains an open problem.

This paper considers the problem of controlling the planar position and orientation of an autonomous surface vessel (marine vehicle) using two independent thrusters. The dynamics of this underactuated system is complex enough to yield a rich source of control problems, yet simple enough to permit a complete mathematical analysis. It has been shown (Wichlund *et al.*, 1995) that underactuated vehicles do not satisfy Brockett's necessary condition (Brockett, 1983) if the unactuated dynamics contain no gravitational field component and, hence, in this case, these vehicles are not asymptotically stabilizable to a desired equilibrium solution using time-invariant continuous feedback laws. In this paper, it is first shown that although the system is not asymptotically stabilizable to a given equilibrium solution using a time-invariant continuous feedback, it is strongly accessible and small-time locally controllable at any equilibrium and, hence, the system is asymptotically stabilizable to a desired equilibrium using time-invariant discontinuous feedback laws. Discontinuous feedback laws are then constructed to

achieve asymptotic stabilization. The methodology followed in the construction of the discontinuous feedback laws is based on first transforming the system into a discontinuous one in which the design of feedback laws is easily carried out. Then, transforming back into the original coordinates yields discontinuous feedback laws which asymptotically stabilize the original system with exponential convergence rates. The discontinuous coordinate transformation employed here constitutes an example of a σ -process (Arnold, 1983), which has proved useful in the stabilization of a special class of nonholonomic systems (Astolfi, 1995). A preliminary version of this paper can be found in Reyhanoglu (1996).

The organization of this paper is as follows. In Section 2, the nonlinear control system describing the dynamics of a surface vessel with two independent thrusters is introduced. Controllability and stabilizability results are presented in Section 3. Discontinuous feedback laws are derived in Section 4. In Section 5 a simulation example is included. Finally, Section 6 consists of a summary of the main results and concluding remarks about future research.

2. Mathematical model

Consider the problem of controlling the Cartesian position and orientation of a surface vessel with two independent propellers as shown in Fig. 1. The kinematic model which describes the geometrical relationship between the earth-fixed (I-frame) and the vehicle-fixed (B-frame) motion is given as

$$\dot{x} = v_x \cos \psi - v_y \sin \psi, \quad (1)$$

$$\dot{y} = v_x \sin \psi + v_y \cos \psi, \quad (2)$$

$$\dot{\psi} = \omega_z, \quad (3)$$

where (x, y) denotes the I-frame position of the center of mass of the vehicle, ψ denotes the orientation angle; (v_x, v_y) and ω_z are the linear and angular velocities of the vehicle in the B-frame. For simplicity the origin of the B-frame is assumed to be located at the center of the mass of the vehicle. It is also assumed that the vehicle is neutrally buoyant. Then the dynamic equations of motion of the vehicle can be expressed in the B-frame as

$$M\dot{v} + C(v)v + D(v)v = \tau, \quad (4)$$

where $v = (v_x, v_y, \omega_z)^T$ denotes the velocity vector, $\tau = (F_x, 0, T_z)^T$ denotes the vector of external force and torque generated by the two propellers. $M \in \mathbb{R}^{3 \times 3}$ is the inertia matrix, including hydrodynamic added mass; and $C(v) \in \mathbb{R}^{3 \times 3}$ and $D(v) \in \mathbb{R}^{3 \times 3}$ denote the Coriolis/centrifugal and the damping matrices, also including hydrodynamic added mass effects, respectively. The reader is referred to Fossen (1994) for the relevant concepts and the detailed formulation of the dynamics of marine vehicles. The following simplified model can be obtained by assuming that both the inertia matrix M and the damping matrix D are constant and diagonal:

$$m_{11}\dot{v}_x - m_{22}v_y\omega_z + d_{11}v_x = F_x, \quad (5)$$

$$m_{22}\dot{v}_y + m_{11}v_x\omega_z + d_{22}v_y = 0, \quad (6)$$

$$m_{33}\dot{\omega}_z + (m_{22} - m_{11})v_xv_y + d_{33}\omega_z = T_z, \quad (7)$$

where m_{ii} , d_{ii} , $i = 1, 2, 3$, are positive constants.

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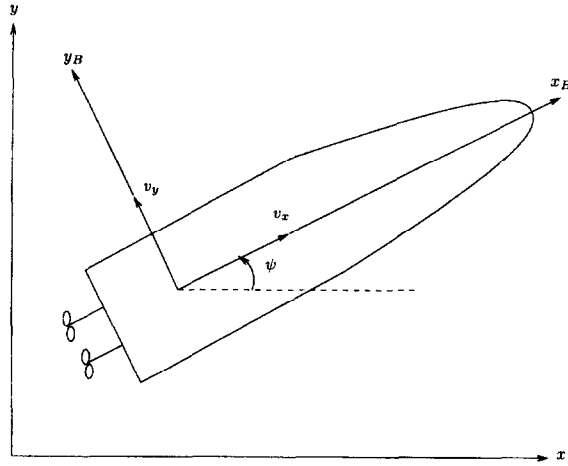


Fig. 1. Model of a surface vessel with two propellers.

Remark 1. Equation (6) can be rewritten in earth-fixed coordinates as

$$m_{22}(\ddot{x} \sin \psi - \ddot{y} \cos \psi) + (m_{22} - m_{11})\dot{\psi}(\dot{x} \cos \psi + \dot{y} \sin \psi) + d_{22}(\dot{x} \sin \psi - \dot{y} \cos \psi) = 0.$$

This equation represents a nonintegrable relation involving not only the generalized coordinates and velocities but also the generalized accelerations and, hence, it can be viewed as a second-order nonholonomic constraint. In contrast to the first-order nonholonomic case (Bloch *et al.*, 1992), a second-order nonholonomic constraint does not reduce the dimension of the state space. A set of three independent configuration variables and three velocity variables is required to completely specify the state of the system.

Remark 2. The surface vessel considered in this paper has no side thruster, i.e., $F_y = 0$, but the controllability analysis and control synthesis of this paper can be easily extended to the cases where $\tau = (F_x, F_y, 0)^T$ or $\tau = (0, F_y, T_z)^T$. Note also that, for simplicity, both the inertia matrix M and the damping matrix D have been assumed to be diagonally dominant and constant. This assumption is satisfied, for instance, when (1) the vehicle body has two perpendicular axes of symmetry (e.g. an elliptic body), (2) hydrodynamic damping terms of order higher than one are negligible, and (3) changes in inertia are negligible. The subsequent development of this paper can be modified to account for more complex cases.

Define the state variables

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (\psi, x \cos \psi + y \sin \psi, -x \sin \psi + y \cos \psi, v_y, \omega_z, v_x)$$

so that the state equations are given by

$$\dot{x}_1 = x_5, \quad (8)$$

$$\dot{x}_2 = x_6 + x_3 x_5, \quad (9)$$

$$\dot{x}_3 = x_4 - x_2 x_5, \quad (10)$$

$$\dot{x}_4 = -\alpha x_4 - \beta x_5 x_6, \quad (11)$$

$$\dot{x}_5 = u_1, \quad (12)$$

$$\dot{x}_6 = u_2, \quad (13)$$

where $\alpha = d_{22}/m_{22}$, $\beta = m_{11}/m_{22}$ and

$$u_1 = (T_z - d_{33}\omega_z + (m_{11} - m_{22})v_x v_y)/m_{33}, \quad (14)$$

$$u_2 = (F_x + m_{22}v_x \omega_z - d_{11}v_x)/m_{11}. \quad (15)$$

Note that the variables (x_1, x_2, x_3) correspond to the orientation angle of the vehicle and the B-frame coordinates of the center of

mass of the vehicle, which parameterize the three-dimensional configuration space of the vehicle; and the variables (x_4, x_5, x_6) correspond to the B-frame velocity components.

In the next section, we will consider the system [equations (8)–(13)] and study its controllability and stabilizability properties.

3. Controllability and stabilizability results

Equations (8)–(13) define a nonlinear control system of the form

$$\dot{x} = f(x) + \sum_{i=1}^2 g_i(x)u_i, \quad (16)$$

where $x = (x_1, x_2, x_3, x_4, x_5, x_6)^T \in \mathbf{M} = S \times R^5$ is the state, and f and g_i , $i = 1, 2$, are the drift and control vector fields given by

$$f = x_5 \frac{\partial}{\partial x_1} + (x_6 + x_3 x_5) \frac{\partial}{\partial x_2} + (x_4 - x_2 x_5) \frac{\partial}{\partial x_3} - (\alpha x_4 + \beta x_5 x_6) \frac{\partial}{\partial x_4},$$

$$g_1 = \frac{\partial}{\partial x_5}, \quad g_2 = \frac{\partial}{\partial x_6}.$$

Note that the set of equilibrium solutions corresponding to $u = 0$ is given by the equilibrium manifold

$$\mathbf{M}^e = \{x \in \mathbf{M} | x_4 = x_5 = x_6 = 0\}.$$

It is easily verified that the linearization of equations (8)–(13) about an equilibrium x^e has an uncontrollable eigenvalue at the origin. This implies that a nonlinear analysis is necessary in order to characterize the controllability and stabilizability properties of the system. Note that since the linearization of equations (8)–(13) is not stabilizable, the system cannot be exponentially stabilized at an equilibrium using smooth feedback (Zabczyk, 1989). Moreover, it is easy to see that the system does not satisfy Brockett's necessary condition (Brockett, 1983) and, hence, it is not asymptotically stabilizable to a desired equilibrium solution using time-invariant continuous feedback.

We now consider the nonlinear control system [equations (8)–(13)] and employ certain results of nonlinear control theory. We refer the reader to Nijmeijer and van der Schaft (1990) and Sussmann (1987) for the relevant controllability definitions and results used in the subsequent development.

The following results characterize the controllability and stabilizability properties of the underactuated vehicle dynamics described by equations (8)–(13).

Proposition 1. The underactuated vehicle dynamics described by equations (8)–(13) is strongly accessible on \mathbf{M} .

Proof. Consider the system [equations (8)–(13)]. Since the vector fields

$$g_1, g_2, [f, g_1], [f, g_2], [g_2, [f, g_1]], [[f, g_2], [f, g_1]]$$

span a six-dimensional space at any point $x \in \mathbf{M}$, the strong accessibility Lie algebra rank condition is satisfied at any point. Hence, the system is strongly accessible on \mathbf{M} . \square

Proposition 2. The underactuated vehicle dynamics described by equations (8)–(13) is small-time locally controllable at any equilibrium $x^e \in \mathbf{M}^e$.

Proof. Consider the system [equations (8)–(13)]. Now, following Sussmann (1987), let $\text{Br}(X)$ denote the smallest Lie algebra of vector fields containing f, g_1 , and g_2 and let B denote any bracket in $\text{Br}(X)$. Let $\delta^0(B)$, $\delta^1(B)$ and $\delta^2(B)$ denote the number of times f, g_1 and g_2 , respectively, occur in the bracket B . The degree of B is equal to the value of $\sum_{i=0}^2 \delta^i(B)$. The Sussmann condition for small-time local controllability is essentially that the so-called bad brackets, the brackets with $\delta^0(B)$ odd and $\delta^1(B)$, $\delta^2(B)$ even, must be a linear combination of good (i.e. not

of the bad type) brackets of lower degree at the equilibrium. The degree of a bad bracket must necessarily be odd. The only bad bracket of degree one is f which vanishes at any equilibrium. The bad brackets of degree three are brackets with $\delta^0 = 1$ and $\delta^i = 2, i = 1$ or 2 , and all are identically zero vector fields. It follows that the Sussmann condition is satisfied at x^e . Hence, the system is small-time locally controllable at x^e .

Since the system is real analytic, the above controllability results imply the existence of piecewise analytic feedback laws (Sussmann, 1979) which asymptotically stabilize the closed-loop system to a given x^e . In the next section, guided by these results, we will focus on designing asymptotically stabilizing discontinuous feedback laws for the system. \square

Remark 3. Note that in the absence of damping, after a variable reordering, equations (8)–(13) take the form of those used by Coron and Kerai (1996) for the construction of locally stabilizing time-varying almost-continuous feedback laws for a rigid spacecraft with only two control torques. These time-varying feedback laws can be used, with minor modifications, to locally asymptotically stabilize the system [equations (8)–(13)]. It is also possible to use geometric phase ideas which proved useful for stabilizing mechanical systems with first-order nonholonomic constraints (Bloch *et al.*, 1992; Reyhanoglu and McClamroch, 1992). It should be remarked that the geometric phase approach yields discontinuous feedback laws with many control switchings. In this paper, we will consider construction of discontinuous feedback laws which yield global asymptotic stabilization with at most one switching.

4. Discontinuous feedback control laws

In this section, we will consider the problem of designing feedback control laws of the form $u = u(x)$ for the system [equations (8)–(13)]. As discussed in the previous section, the system cannot be asymptotically stabilized to an equilibrium using any time-invariant continuous feedback. Therefore, we restrict our consideration to designing time-invariant discontinuous feedback laws.

Note that the problem of stabilizing the system to a given equilibrium $x^e \in M^c$ can be reduced to the problem of stabilizing the system to the origin via an appropriate state transformation. Hence, without loss of generality, we focus only on the problem of feedback stabilization to the origin, i.e. $x^e = 0$.

We will first study the problem of stabilizing the following reduced-order system, which is obtained by considering the subsystem [equations (8)–(11)] and letting (x_5, x_6) be the control variables (v_1, v_2) :

$$\dot{x}_1 = v_1, \tag{17}$$

$$\dot{x}_2 = v_2 + x_3 v_1, \tag{18}$$

$$\dot{x}_3 = x_4 - x_2 v_1, \tag{19}$$

$$\dot{x}_4 = -\alpha x_4 - \beta v_1 v_2. \tag{20}$$

4.1. Stabilization of the reduced system. The idea that will be employed is based on first transforming the reduced system [equations (17)–(20)] into a discontinuous one by applying a discontinuous coordinate transformation, e.g. by applying a σ -process (see Arnold, 1983). From the analytical point of view, the σ -process, also termed as the process of resolution of singularities, consists of a rational coordinate transformation.

Consider the reduced system [equations (17)–(20)]. Restricting consideration to $x_1 \neq 0$, apply the σ -process

$$y = x_1, \quad z_1 = x_2, \quad z_2 = \frac{x_3}{x_1}, \quad z_3 = \frac{x_4}{x_1}$$

to obtain

$$\dot{y} = v_1, \tag{21}$$

$$\dot{z}_1 = v_2 + y z_2 v_1, \tag{22}$$

$$\dot{z}_2 = z_3 - \frac{z_1 + z_2}{y} v_1, \tag{23}$$

$$\dot{z}_3 = -\alpha z_3 - \frac{z_3 + \beta v_2}{y} v_1. \tag{24}$$

The feedback control law

$$v_1 = -ky, \tag{25}$$

$$v_2 = -lz, \tag{26}$$

where $z = (z_1, z_2, z_3)^T$, and $k > 0$ and $l = (l_1 \ l_2 \ l_3)$ are the gains, yields the reduced closed-loop system

$$\dot{y} = -ky, \tag{27}$$

$$\dot{z}_1 = -l_1 z_1 - l_2 z_2 - l_3 z_3 - ky^2 z_2, \tag{28}$$

$$\dot{z}_2 = kz_1 + kz_2 + z_3, \tag{29}$$

$$\dot{z}_3 = -k\beta l_1 z_1 - k\beta l_2 z_2 + (k - \alpha - k\beta l_3) z_3. \tag{30}$$

The z -dynamics can be rewritten as

$$\dot{z} = (A_1 + A_2(t))z, \tag{31}$$

where

$$A_1 = \begin{pmatrix} -l_1 & -l_2 & -l_3 \\ k & k & 1 \\ -k\beta l_1 & -k\beta l_2 & k - \alpha - k\beta l_3 \end{pmatrix}, \tag{32}$$

$$A_2(t) = \begin{pmatrix} 0 & -ky_0^2 e^{-2kt} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{33}$$

It can be easily seen that if $k \neq \alpha$, the spectrum of the matrix A_1 can be assigned arbitrarily through the gain matrix l . Clearly, the y -dynamics is globally exponentially stable at $y = 0$. Moreover, since the matrix $A_2(t)$ given by equation (33) goes to zero as $t \rightarrow \infty$ and

$$\int_0^\infty \|A_2(t)\| dt < \infty,$$

the z -dynamics can also be rendered globally exponentially stable at the origin $z = 0$ by selecting $l = (l_1 \ l_2 \ l_3)$ such that the matrix A_1 given by equation (32) is a Hurwitz matrix (see Slotine and Li, 1991, Section 4.2.2).

Note that in the (x_1, x_2, x_3, x_4) coordinates the controls [equations (25) and (26)] take the form

$$v_1(x_1, x_2, x_3, x_4) = -kx_1, \tag{34}$$

$$v_2(x_1, x_2, x_3, x_4) = -l_1 x_2 - l_2 \frac{x_3}{x_1} - l_3 \frac{x_4}{x_1} \tag{35}$$

and the reduced closed-loop system is given by

$$\dot{x}_1 = -kx_1, \tag{36}$$

$$\dot{x}_2 = -l_1 x_2 - l_2 \frac{x_3}{x_1} - l_3 \frac{x_4}{x_1} - kx_1 x_3, \tag{37}$$

$$\dot{x}_3 = x_4 + kx_1 x_2, \tag{38}$$

$$\dot{x}_4 = -k\beta l_2 x_3 - (k\beta l_3 + \alpha)x_4 - k\beta l_1 x_1 x_2. \tag{39}$$

We now present the following result.

Proposition 3. Consider the reduced closed-loop system [equations (36)–(39)] with $k > 0, k \neq \alpha$, and $l = (l_1 \ l_2 \ l_3)$ selected such that the matrix A_1 given by equation (32) is a Hurwitz matrix. Let $(x_{10}, x_{20}, x_{30}, x_{40})$ denote an initial condition with $x_{10} \neq 0$. Then the following hold.

(i) The trajectory $(x_1(t), x_2(t), x_3(t), x_4(t))$ is bounded for all $t \geq 0$ and converges exponentially to zero.

(ii) The control $(v_1(t), v_2(t))$ is bounded for all $t \geq 0$ and converges exponentially to zero.

Proof. Consider the reduced closed-loop system [equations (36)–(39)] and let $(x_{10}, x_{20}, x_{30}, x_{40})$ denote an initial condition with $x_{10} \neq 0$.

(i) We have $x_1(t) = x_{10} e^{-kt}$. Since $x_{10} \neq 0, x_1(t) \neq 0, \forall t \in [0, \infty)$. Under the stated assumptions, the reduced closed-loop system in the transformed coordinates is globally

exponentially stable. Thus, $(x_1(t), x_2(t), x_3(t)/x_1(t), x_4(t)/x_1(t))$ converges exponentially to zero. Hence, the trajectory $(x_1(t), x_2(t), x_3(t), x_4(t))$ is bounded for all $t \geq 0$ and converges exponentially to zero.

(ii) From the above discussion, each term in the control v given by equations (34) and (35) consists of bounded terms which converge exponentially to zero. Hence, the result follows.

Remark 4. The above result demonstrates that for initial conditions satisfying $x_{10} \neq 0$, the feedback control law [equations (34) and (35)] is well defined for all $t \geq 0$. Moreover, it drives the system [equations (36)–(39)] to the origin, while avoiding the set

$$N = \{(x_1, x_2, x_3, x_4) \mid x_1 = 0, (x_1, x_2, x_3, x_4) \neq 0\}.$$

Note that one can use a finite-time feedback control law (Haimo, 1986) to move the system away from N .

4.2. Stabilization of the complete system. We now return to the problem of asymptotic stabilization of the system [equations (8)–(13)] with u_1 and u_2 , instead of x_5 and x_6 , as control inputs. Since $u_1 = \dot{x}_5$ and $u_2 = \dot{x}_6$, the problem corresponds to the classical situation where integrators are added at the input level. It should be remarked that the integrator back-stepping approach developed for smooth systems (Krstić *et al.*, 1995) cannot be directly applied here due to the discontinuous nature of the system.

Again restrict consideration to $x_1 \neq 0$ and consider the following controller:

$$u_1(x) = -K(x_5 - v_1(x_1, x_2, x_3, x_4)) + s_1(x), \quad (40)$$

$$u_2(x) = -L(x_6 - v_2(x_1, x_2, x_3, x_4)) + s_2(x), \quad (41)$$

where $v_1(x_1, x_2, x_3, x_4)$ and $v_2(x_1, x_2, x_3, x_4)$ denote the feedback controls [equations (34) and (35)] for the reduced system; and $s_1(x)$ and $s_2(x)$ correspond to their time derivatives along the trajectories of the system [equations (8)–(13)]

$$s_1(x) = -kx_5,$$

$$s_2(x) = -l_1(x_6 + x_3x_5) - l_2 \frac{x_4 - x_2x_5}{x_1} + l_3 \frac{\alpha x_4 + \beta x_5x_6}{x_1} + l_2 \frac{x_3x_5}{x_1^2} + l_3 \frac{x_4x_5}{x_1^2}.$$

Now assume that the control parameters are selected such that k, l_1, l_2 and l_3 satisfy the conditions of Proposition 3 and $K > k, L > 0$. The main idea behind the proposed control law is to implement the control law [equations (34) and (35)] through the integrators by choosing the gains K and L appropriately, while avoiding the set

$$N' = \{x \in M \mid x_1 = 0, x \neq 0\}. \quad (42)$$

Consider the coordinate transformation

$$y = x_1, \quad z_1 = x_2, \quad z_2 = \frac{x_3}{x_1}, \quad z_3 = \frac{x_4}{x_1},$$

$$w_1 = x_5 + kx_1, \quad w_2 = x_6 + l_1x_2 + l_2 \frac{x_3}{x_1} + l_3 \frac{x_4}{x_1}.$$

It can be shown that in the above coordinates the closed-loop system can be written as

$$\dot{y} = -ky + w_1, \quad (43)$$

$$\dot{z} = (A_1 + \tilde{A}_2(t))z + h(t), \quad (44)$$

$$\dot{w}_1 = -Kw_1, \quad (45)$$

$$\dot{w}_2 = -Lw_2, \quad (46)$$

where A_1 is the matrix given by equation (32) and

$$\tilde{A}_2(t) = \begin{pmatrix} 0 & r_1(t) & 0 \\ -r_2(t) & r_2(t) & 0 \\ \beta l_1 r_2(t) & \beta l_2 r_2(t) & (\beta l_3 - 1)r_2(t) \end{pmatrix},$$

$$r_1(t) = \left(e^{-kt}y_0 + \frac{e^{-kt} - e^{-Kt}}{K - k}w_{10} \right) \times \left(-ke^{-kt}y_0 + \frac{ke^{-kt} - Ke^{-Kt}}{k - K}w_{10} \right),$$

$$r_2(t) = \left(e^{-kt}y_0 + \frac{e^{-kt} - e^{-Kt}}{K - k}w_{10} \right)^{-1} e^{-Kt}w_{10}.$$

$$h(t) = (1 \ 0 \ \beta(k - r_2(t)))^T e^{-Lt}w_{20}.$$

The (y, w_1, w_2) -dynamics is globally exponentially stable at $(y, w_1, w_2) = (0, 0, 0)$. Moreover, it can be easily shown that if $y_0 w_{10} \geq 0$ (or, equivalently, $x_{10}(x_{50} + kx_{10}) \geq 0$), then $\tilde{A}_2(t)$ and $h(t)$ go to zero as $t \rightarrow \infty$ and

$$\int_0^\infty \|A_2(t)\| dt < \infty, \quad \int_0^\infty \|h(t)\| dt < \infty.$$

Thus, for any initial condition $(y_0, z_0, w_{10}, w_{20})$ satisfying $y_0 \neq 0$ and $y_0 w_{10} \geq 0$, both the trajectory $(y(t), z(t), w_1(t), w_2(t))$ and the control $(u_1(t), u_2(t))$ are bounded for all $t \geq 0$ and converge exponentially to zero.

We now present the following result.

Proposition 4. Consider the system [equations (8)–(13)] with the feedback controls (40) and (41), where the control parameters are selected such that k, l_1, l_2 and l_3 satisfy the conditions of Proposition 3 and $K > k, L > 0$. Let $(x_{10}, x_{20}, x_{30}, x_{40}, x_{50}, x_{60})$ denote an initial condition satisfying $x_{10} \neq 0$ and $x_{10}(x_{50} + kx_{10}) \geq 0$. Then the following hold.

(i) The trajectory $(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ is bounded for all $t \geq 0$ and converges exponentially to zero.

(ii) The control $(u_1(t), u_2(t))$ is bounded for all $t \geq 0$ and converges exponentially to zero.

Remark 5. Note that the above choice of the feedback control guarantees that

$$x_1(t) = e^{-kt}x_{10} + \frac{e^{-kt} - e^{-Kt}}{K - k}(x_{50} + kx_{10}).$$

It can be easily seen that if $x_{10} \neq 0$ and $x_{10}(x_{50} + kx_{10}) \geq 0$, then $x_1(t) \neq 0, \forall t \in [0, \infty)$. Thus, for all initial conditions satisfying $x_{10} \neq 0$ and $x_{10}(x_{50} + kx_{10}) \geq 0$, the feedback control law [equations (40) and (41)] is well defined for all $t \geq 0$. Moreover, it drives the system [equations (8)–(13)] to the origin, while avoiding the set [equation (42)]. Clearly, one can use a finite-time feedback control law to move the system to a state satisfying the conditions of Proposition 4, e.g.

$$u_1 = -|x_1 - \varepsilon|^a \text{sign}(x_1 - \varepsilon) - |x_5|^b \text{sign}(x_5), \quad (47)$$

$$u_2 = 0, \quad (48)$$

where $b \in (0, 1), a > b/(2 - b)$ and $\varepsilon \neq 0$ are constants, can be used to transfer the system to a state satisfying the conditions of Proposition 4 in finite time.

5. Example

We illustrate the results of the paper with a simulation example of a surface-vessel model with two independent propellers as shown in Fig. 1. The model parameters are given by

$$m_{11} = 200 \text{ kg}, \quad m_{22} = 250 \text{ kg}, \quad m_{33} = 80 \text{ kg m}^2,$$

$$d_{11} = 70 \text{ kg s}^{-1}, \quad d_{22} = 100 \text{ kg s}^{-1}, \quad d_{33} = 50 \text{ kg m}^2 \text{ s}^{-1}.$$

A computer implementation of the discontinuous feedback control law specified in Proposition 4 was used to asymptotically stabilize the origin. The results of the simulation for a sample initial condition given by $x_0 = (0, 1, 1, 0, 1, 0)$ (or equivalently $(x, y, \psi, v_x, v_y, \omega_z) = (1, 1, 0, 0, 0, 1)$) are shown in Figs. 2-5. Note that this initial condition does not satisfy the conditions of Proposition 4. Thus, the controls given by equations (47) and (48) with $\varepsilon = 0.5$ and $a = b = \frac{1}{3}$ were first used to move the system to a state satisfying the conditions of Proposition 4.

Then, at $t = 5.25$ s, the controls were switched to those specified in Proposition 4 with

$$k = 0.5, \quad l = (5.00, 2.50, -7.25), \quad K = 1, \quad L = 1.$$

Note that the above choice of the gain matrix l locates the eigenvalues of the matrix A_1 given by equation (32) at $(-0.5, -0.5, -0.5)$. Figure 2 shows the time responses of the configuration variables x , y and ψ . The time responses for the

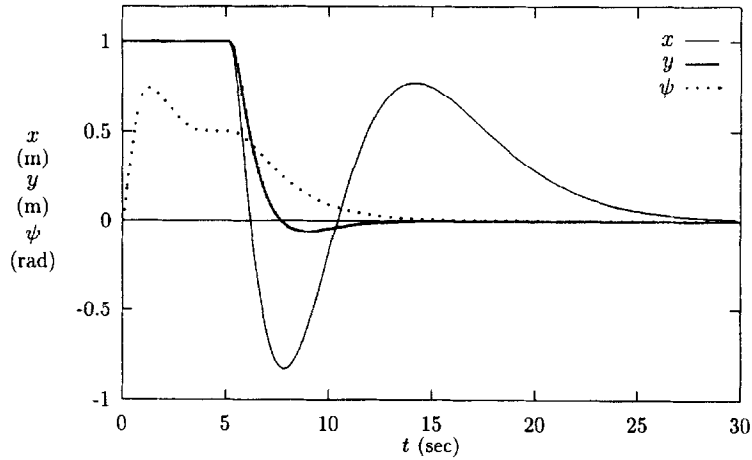


Fig. 2. Time responses for configuration variables.

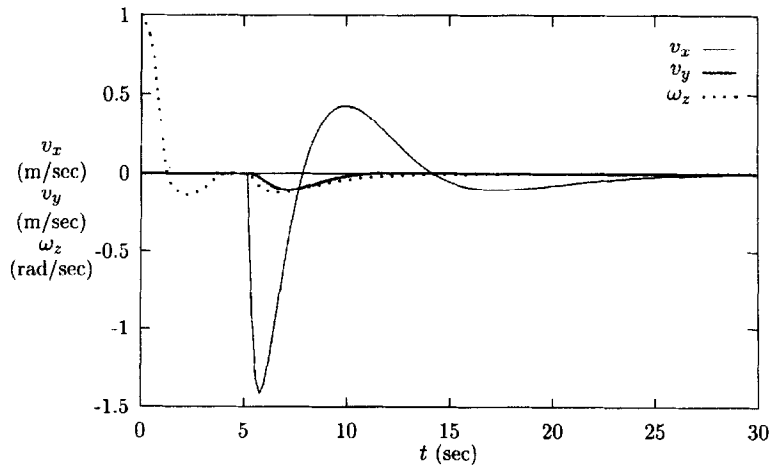


Fig. 3. Time responses for velocity variables.

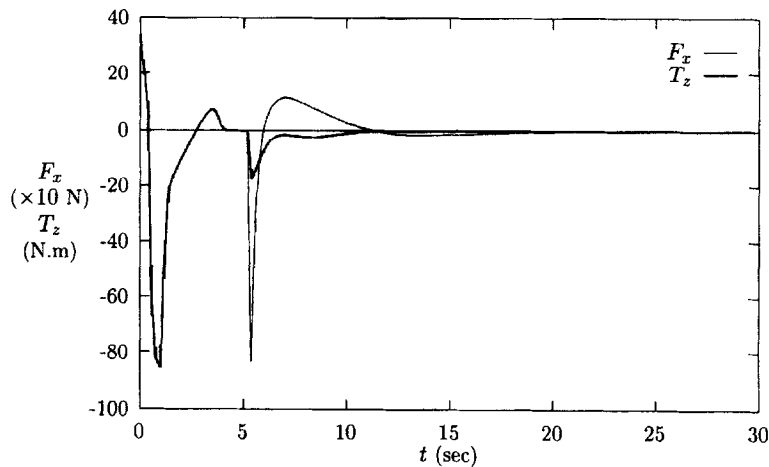


Fig. 4. Controls F_x and T_z .

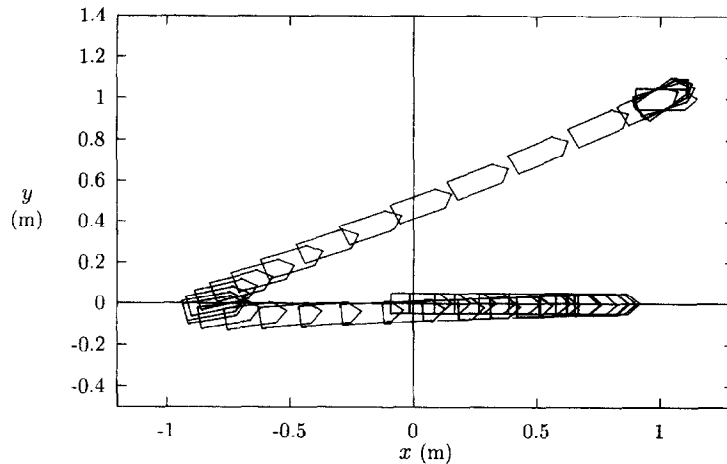


Fig. 5. Surface vessel configuration.

velocities v_x , v_y and ω_z and the controls F_x and T_z are shown in Figs. 3 and 4, respectively. Exponential convergence of the closed-loop state and control trajectories can be observed. Figure 5 illustrates the configuration of the surface vessel for a sequence of time instants. The motion until the control switching occurs is a pure yaw motion, which changes the orientation of the vehicle from $\psi = 0$ to 0.5 while keeping the I-frame position of the vehicle constant. After the control switching, the vehicle approaches the origin with monotonically decreasing ψ .

Remark 6. The simulation results indicate that, with the proposed control law, a significant amount of control effort may be required at certain stages of the motion. A modification of this control law is certainly essential to account for both saturation constraints and rate limits that may be imposed on the control inputs. Such a modification seems to be possible using the ideas introduced in Tsiotras and Luo (1996).

6. Conclusions

Following a controllability and stabilizability analysis, time-invariant discontinuous feedback control laws have been derived for the asymptotic stabilization of a surface vessel with two independent thrusters. Boundedness and convergence of the closed-loop state and control trajectories have been demonstrated. The effectiveness of the proposed feedback control laws has been illustrated through a simulation example.

In this paper, a number of assumptions have been made about the surface vessel dynamics to simplify the presentation of the results. Future research in this area includes relaxation of certain of these assumptions as mentioned in Remark 2.

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