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# ACCURATE CALCULATION METHODS FOR NATURAL FREQUENCIES OF PLATES WITH SPECIAL ATTENTION TO THE HIGHER MODES

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Various computational methods have been studied with respect to their suitability for obtaining very accurate solutions of plate vibration problems, especially for the higher modes. Because of the interest in the higher modes, also higher order effects such as transverse shear deformation and rotational inertia are considered. The Rayleigh–Ritz method with global trial functions appeared to be a suitable choice. To reach a high convergence rate in order to obtain accurate solutions, the complementary boundary conditions formulated by Baruh and Tadikonda should be satisfied. This can be accomplished when polynomials are used as trial functions. When the polynomials are not properly chosen, the algorithm is not numerically stable. It is shown that orthogonalization of the polynomials by means of the Gram–Schmidt process results in a numerical stable process. For free–free boundary conditions, these orthogonal polynomials are the well known Legendre polynomials. For other boundary conditions the resulting polynomials are very similar to the Legendre polynomials. Because of the very high convergence rates, these methods are suitable for obtaining accurate solutions. The numerical stability guarantees that also the higher modes can be calculated.

## 1. INTRODUCTION

Sometimes it is desirable to determine an exact or a very accurate solution of one or more sample problems, in order to validate numerical methods such as the finite element method in respect to accuracy and convergence rates.

In this paper, methods are examined for calculating the natural frequencies and mode shapes of rectangular plates with clamped, simply supported and free boundaries. The number of problems which can be solved in closed form is very limited. For rectangular plates these are the cases in which two opposite sides are simply supported. The solution of these problems is known as the "Levy solution" [1]. When the problem cannot be solved in closed form, one has to resort to approximate methods. Because the emphasis is focussed on very accurate solutions, these must be methods with a sufficiently high convergence rate.

The plate theory used for the lower modes is Kirchhoff plate theory. Because the attention is focussed on the higher modes, also higher order effects such as transverse shear deformation and rotational inertia, should be considered. For this, several higher order theories are available. One of these will be used to include the higher order effects.

All of the existing methods for calculating natural frequencies are based on either the strong formulation or on the weak formulation of the mechanics problem. Before going into detail in choosing the most suitable method, a short description of both methods is given. Hereafter an explanation is given as to why the Rayleigh–Ritz method with global trial functions is chosen as the most suitable method for the problem considered. As will



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become clear, the choice of the trial functions in the Rayleigh–Ritz method is very important with respect to convergence rate and numerical stability. The influence of trial functions on the convergence rate has been examined by Baruh and Tadikonda [2]. They define the concept of complementary boundary conditions. This will be further examined and used to choose suitable trial functions. The second topic, numerical stability, has received much attention by the authors. A numerically stable process guarantees that a large number of trial functions can be used, which is needed to obtain very accurate solutions. Both topics will be illustrated by a simple one-dimensional problem.

## 2. STRONG AND WEAK FORMULATIONS

### 2.1. STRONG FORMULATION

A vibration problem can be formulated as a differential equation together with boundary conditions, the so-called strong formulation. For Kirchhoff plate theory this yields the equation (a list of symbols is given in Appendix B)

$$D\nabla^4 w + \rho h\, \partial^2 w/\partial t^2 = q, \tag{1a}$$

with the boundary conditions

$$w = 0 \qquad \text{or} \qquad V_n = D(\partial^3 w/\partial n^3 + (2 - \nu)\, \partial^3 w/\partial s^2\, \partial n) = 0, \tag{1b}$$

$$\partial w/\partial n = 0 \qquad \text{or} \qquad M_n = -D(\partial^2 w/\partial n^2 + \nu\, \partial^2 w/\partial s^2) = 0. \tag{1c}$$

$V_n$ is the Kirchhoff shear force and $M_n$ is the bending moment.

For free vibration calculations ($q = 0$), the transverse displacement, $w$, is the product of a spatial function and a harmonic time function, i.e., $w = W\, e^{i\omega t}$. This leads to the eigenvalue problem

$$D\nabla^4 W - \rho h\omega^2 W = 0, \qquad \text{w} = 0 \quad \text{or} \quad V_n = 0, \qquad \partial w/\partial n = 0 \qquad \text{or} \qquad M_n = 0. \tag{2a–c}$$

The construction of a solution is possible by writing the displacement $W$ as a sum of solutions of the eigenvalue equation (2a). Satisfying the boundary conditions results in a homogeneous system of equations. Approximation is possible by truncating the series. Non-trivial solutions are possible when the determinant is zero. The boundary conditions will then be satisfied approximately. Claassen and Thorne [3] (Fourier series), and Gorman [4] (superposition method) have used this approach, which results in very accurate solutions.

### 2.2. WEAK FORMULATION

The weak formulation is often used in a variational form. For vibration problems, this is the Rayleigh principle:

$$\omega^2 = \min\,(V_{max}/T^*). \tag{3}$$

The maximum elastic energy of a plate is

$$V_{max} = \frac{D}{2}\iint\left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 2\nu\,\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2\right] \mathrm{d}x\,\mathrm{d}y, \tag{4}$$

where $D$ is the plate bending stiffness, $D = Eh^3/12(1 - \nu^2)$. The expression for $T^*$ is

$$T^* = \tfrac{1}{2}\rho \iint W^2\, \mathrm{d}x\,\mathrm{d}y. \tag{5}$$

An approximation can be made by applying a direct variational method, such as the Ritz method. The displacement $W$ is written as a sum of trial functions $\varphi_i$:

$$W = \sum_{i=1}^{N} a_i \varphi_i. \tag{6}$$

The conditions which must be satisfied in order to let the Ritz approximation converge to the solution of the eigenvalue problem (2) are as follows [5]:

the functions $\varphi_i$ are linearly independent, (7a)

the functions $\varphi_i$ are complete in the energy norm, (7b)

the functions $\varphi_i$ satisfy the geometric boundary conditions, (7c)

the functions $\varphi_i$ have at least derivatives up to half of the order of the differential equation. (7d)

Functions which satisfy these conditions are the so-called admissible functions.

The effectiveness of the Rayleigh–Ritz method depends on the choice of the trial functions for a particular problem. The convergence rate depends on how well the function and its derivatives can be approximated. For example, when beam functions are used in cases in which free edges are involved, the convergence rate is low.

Bassily and Dickinson [6] showed that when beam functions are used as trial functions, on a free edge the bending moment cannot become zero, irrespective of the number of trial functions in the Rayleigh–Ritz approximation. They used the so-called degenerated beam functions to solve this problem and obtained accurate natural frequencies.

In 1985, Bhat [7] proposed the use of orthogonal polynomials as trial functions in the Rayleigh–Ritz method. Since then, these functions have frequently been used for calculating the lowest modes of various problems. When compared with the use of the so-called degenerated beam functions, the results obtained are even better [8]. The superposition method [4] gives solutions which have the same accuracy as the Rayleigh–Ritz method with orthogonal polynomials as trial functions [9].

In published papers on the use of orthogonal polynomials [7, 8, 10] results have been presented for the lower modes. One of the papers reported numerical problems with the calculation of higher modes [10]. This is examined in more detail in section 3.3 of this paper.

### 2.3. CHOICE OF THE COMPUTATIONAL METHOD

For an accurate calculation of the natural frequencies of a thin plate with clamped, simply supported or free boundary conditions, there are several suitable methods. When the strong formulation is used, the Fourier series method of Claassen and Thorne [3] and the superposition method of Gorman [4] can be used. For calculation of the higher modes, the asymptotic method of Bolotin [11] is also interesting.

When the weak formulation is used (for example, the Rayleigh–Ritz method), the choice of the trial functions is important, because this choice already determines some characteristics of the approximation. In particular, when accurate solutions are required, this should lead to an appropriate choice of the trial functions.

If the plate has no free boundaries, beam functions can be used. When orthogonal polynomials are used, highly accurate solutions can be obtained for all the boundary conditions (including free boundaries).

When the plate theory is extended with higher order effects such as transverse shear and rotational inertia, the Rayleigh–Ritz method can easily be modified to include these effects.

This can be done by adding the corresponding expressions for the shear deformation and rotational energies to the elastic and kinetic energy expressions.

This study is concentrated on the possibilities for the accurate calculation of the higher modes with account taken of the effects of transverse shear and rotational inertia. For this purpose the Rayleigh–Ritz method has been chosen, for the following reasons: with a suitable choice of trial functions, accurate approximations can be obtained; inclusion of higher order effects is easy; extension from rectangular to polygonal plates can be done by a simple transformation.

### 2.4. THE RAYLEIGH–RITZ METHOD AND THE COMPLEMENTARY BOUNDARY CONDITIONS

Until 1976, beam functions were the most frequently used trial functions in the Rayleigh–Ritz method for solving plate bending problems. A notable characteristic was the very slow convergence when free edges were involved. As Bassily and Dickinson pointed out, the cause of this problem is that the natural boundary conditions cannot be satisfied. One of their conclusions was that this set of functions is not complete.

Baruh and Tadikonda [2], introduced the complementary boundary conditions (CBC). These are conditions which should be satisfied by the trial functions in order that the Rayleigh–Ritz approximation and its derivatives can converge uniformly to the solution of the differential equation.

The convergence problems which arise when beam functions are used in the Rayleigh–Ritz method, and when the plate has free edges, are related to these CBC. The beam functions individually satisfy the zero moment (second derivative) and zero force (third derivative) end condition (of a beam), so the products of these beam functions cannot satisfy the boundary conditions of the plate on a free edge. These are (see equations 1(b) and (c))

$$V_n = D(\partial^3 w/\partial n^3 + (2 - \nu)\, \partial^3 w/\partial s^2\, \partial n) = 0, \qquad M_n = -D(\partial^2 w/\partial n^2 + \nu\, \partial^2 w/\partial s^2) = 0.$$

Substituting the boundary conditions of the beam with free ends ($\partial^2 w/\partial n^2 = 0$ and $\partial^3 w/\partial n^3 = 0$) in these expressions yields

$$\partial^3 w/\partial s^2\, \partial n = 0, \qquad \partial^2 w/\partial s^2 = 0. \tag{8a, b}$$

The second condition says that the curvature on an edge in the direction of the edge is zero. For an arbitrary mode this will not be the case. Also, the first condition will not be satisfied for an arbitrary mode. As a result of this, the products of beam functions do not satisfy the CBC.

In the next section the completeness of these functions and the type of convergence when used in the Rayleigh–Ritz method will be examined. This will be done with the help of one of the examples that Baruh and Tadikonda used to show the influence of the CBC.

## 3. THE ONE-DIMENSIONAL PROBLEM

### 3.1. COMPLEMENTARY BOUNDARY CONDITIONS

The system that Baruh and Tadikonda examined is a bar clamped at one side and at the other end connected to a spring of stiffness $k$ (see Figure 1). The axial natural vibrations will be approximated by the Rayleigh–Ritz method. Baruh and Tadikonda used for this example eigenfunctions of the clamped–free vibration problem: i.e.,

$$\psi_r = \sin\{(2r - 1)(\pi x/2l)\}, \qquad r = 1, 2, \ldots, N. \tag{9}$$

These functions satisfy the conditions (7a)–7(d) and so they are admissible. The functions $\psi_r$ do not satisfy the CBC of $x = l$. Baruh and Tadikonda [2] showed that with the use

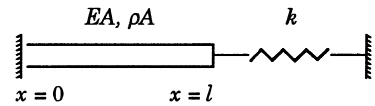


Figure 1. The example problem: a clamped bar connected with a spring.

of the eigenfunctions $\psi_r$ there is no uniform convergence to the boundary conditions at $x = l$. This results in very slow convergence of the natural frequencies.

In the next sections we will look more closely at this convergence, and it will be demonstrated that the natural frequencies converge to the exact values.

### 3.2. CONVERGENCE BEHAVIOUR

To examine the convergence behaviour the calculation has been performed with a large number of functions in the Rayleigh–Ritz approximation (see Table 1). The variation in the convergence rate for the first natural frequency as the number of trial functions increases is shown in Figure 2. The difference between the Rayleigh–Ritz approximation and the exact value versus the number of functions is shown in Figure 2 (double logarithmic scales). From this graph it can be determined that the error is approximately proportional to $1/N$. This can also be shown by writing the first derivative in a Fourier series (see Appendix A). From inspection of this analysis, it also becomes clear what the nature of this convergence is. Because Fourier functions are complete in the norm of the space $L_2$ ($L_2$ is the space of quadratic integrable functions), there exists convergence in the mean (convergence in the norm of the space $L_2$). In Figures 3 and 4, plots of the approximate displacement function and its derivative (for $N = 50$) are shown. Because the function has different values at $x = 0$ and $x = l$, the periodic function which is approximated by a Fourier series is discontinuous. When spectral methods such as the Rayleigh–
Ritz method are used, it has been established that discontinuities in the function to be

TABLE 1

*Natural frequencies ($\psi_r$ as trial functions) $k' = kl/EA = 1000$ ($k'$ is the ratio of spring stiffness $k$ and the stiffness of the bar $EA/l$)*

| | $N = 7$ | $N = 20$ | $N = 30$ | $N = 50$ | Exact |
|---|---|---|---|---|---|
| $\omega_1$ | 3·232198 | 3·170559 | 3·159780 | 3·151214 | 3·138454 |
| $\omega_2$ | 6·467149 | 6·341235 | 6·319597 | 6·302436 | 6·276908 |
| $\omega_3$ | 9·708281 | 9·512147 | 9·479487 | 9·453676 | 9·415363 |

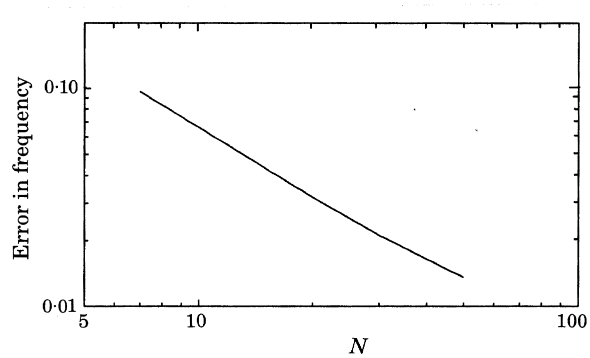


Figure 2. The convergence rate of the first natural frequency.

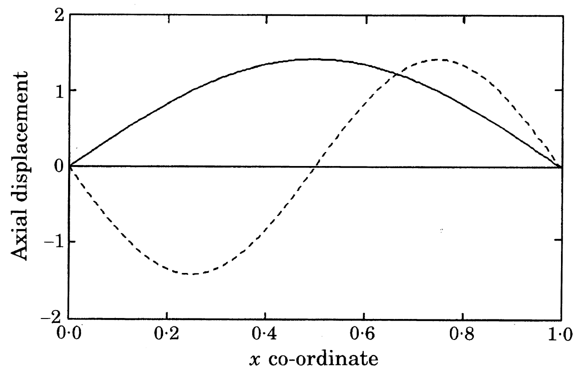


Figure 3. The displacements of the first (——) and second (– – –) modes.

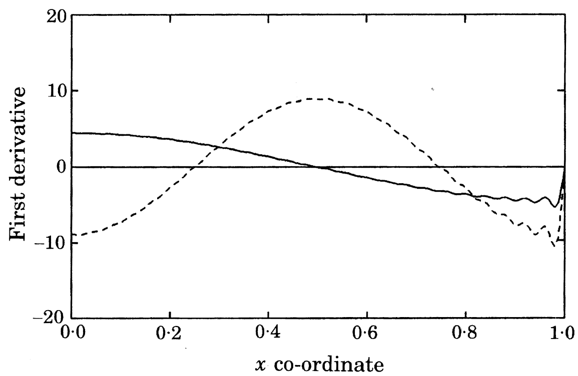


Figure 4. The derivatives of the displacements of the first (——) and second (– – –) modes.

approximated result in algebraic convergence. This type of convergence is characterized by a truncation error which is proportional with $1/N^\alpha$. The exponent $\alpha$ depends on the order of the derivative in which the discontinuity appears [12]. From reference [12] it becomes clear that when, for non-periodic problems, non-periodic trail functions such as Chebyshev or Legendre polynomials (orthogonal polynomials) are used, convergence is exponential rather than algebraic. Now the truncation error diminishes faster than $1/N^\alpha$ for any $\alpha$ (the coefficient $a_i$ in the approximating series is $O[\exp(-qi^r)]$ for $i \gg 1$, $q$ is a constant and $r > 0$). This convergence behaviour is qualitatively different from the algebraic convergence. Other names used for exponential convergence are infinite order or spectral convergence [12].

With this convergence it is possible to construct solutions with many significant digits, so these can be considered to be exact. This will be demonstrated by using the example problem of section 3.1.

### 3.3. USE OF POLYNOMIALS IN THE RAYLEIGH–RITZ METHOD

When polynomials are used in the Rayleigh–Ritz method, the non-uniform convergence with respect to the derivatives is avoided. To demonstrate this, Baruh and Tadikonda [2] used the simple polynomials

$$\psi_{1r} = x^r, \qquad r = 1, 2, \ldots, N. \tag{10}$$

TABLE 2

*Natural frequencies when polynomials are used as trial functions*

| | Simple polynomials | | | |
|---|---|---|---|---|
| | All powers $N = 7$ | Only odd powers $N = 7$ | Legendre polynomials $N = 7$ | Exact |
| $\omega_1$ | 3·138454 | 3·138454 | 3·138454 | 3·138454 |
| $\omega_2$ | 6·276913 | 6·276909 | 6·276909 | 6·276908 |
| $\omega_3$ | 9·433533 | 9·415859 | 9·415859 | 9·415363 |

However, when these functions are used the Rayleigh–Ritz process becomes numerically unstable. In single precision, calculations can be performed with a maximum of five trial functions. In double precision the maximum is 11 trial functions. The mass matrix is a Hilbert matrix, and the stiffness matrix has similar characteristics: "With these matrices almost no calculation is possible" [13].

For the determination of very accurate solutions and for the calculation of the higher modes, it is necessary that a large number of trial functions should be used in the Rayleigh–Ritz approximation. This requires that the Rayleigh–Ritz process is numerically stable.

A necessary and sufficient condition for this numerical stability is that the trial functions are strongly minimal in the energy space of the relevant operator [14]. In reference [14], a proof is given that simple polynomials do not satisfy this condition. Polynomials which do satisfy this stability condition are Legendre polynomials. Simple polynomials can be transformed to Legendre polynomials by a linear transformation. This transformation does not change the subspace spanned by these functions. Therefore, the Rayleigh–Ritz approximation is the same. The transformation only changes the basis of trial functions in the same subspace.

For an illustration of this, the first three natural frequencies were calculated with the simple polynomials and with Legendre polynomials as trial functions. Because of the geometric boundary condition at $x = 0$, only the odd Legendre polynomials can be used. For comparison with the simple polynomials, here the odd ones also should be used; see Table 2.

The numerical stability of the Rayleigh–Ritz process when Legendre polynomials are used will be demonstrated for the axial vibration problem. This calculation is performed with 50 trial functions (see Table 3). There is no numerical instability. It is also clear that approximately one half of the natural frequencies are calculated very accurately (at least to seven significant digits).

For the problems considered here, Legendre polynomials can be used because they are admissible functions for these problems. Another way to look at these polynomials is that they can be constructed by starting with the polynomial with the lowest degree which is admissible for the problem (it should satisfy the geometric boundary conditions). The others can be constructed by a Gram–Schmidt orthogonalization process. For problems with other boundary conditions, this procedure can be used to derive a set of orthogonal polynomials which are admissible for that problem. This is the procedure used by Bhat [7] and others. For the orthogonalization the standard inner product† is used. As a result, the mass matrix that results from the Rayleigh–Ritz discretization is a diagonal matrix.

† $(f, g) = \int fg \, dx$.

TABLE 3

*Natural frequencies*; $k' = 1000$

| | Eigenfunctions $\sin\{(2r-1)(\pi x/2l)\}$ | Legendre functions | Exact |
|---|---|---|---|
| $\omega_1$ | 3·151214 | 3·138454 | 3·138454 |
| $\omega_2$ | 6·302436 | 6·276908 | 6·276908 |
| $\omega_3$ | 9·453676 | 9·415363 | 9·415363 |
| ⋮ | | | |
| $\omega_{26}$ | 81·960416 | 81·599989 | 81·599989 |
| $\omega_{27}$ | 85·115518 | 84·738467 | 84·738465 |
| $\omega_{28}$ | 88·271019 | 87·877005 | 87·876943 |
| $\omega_{29}$ | 91·426957 | 91·016942 | 91·015422 |
| $\omega_{30}$ | 94·583372 | 94·174616 | 94·153902 |
| $\omega_{31}$ | 97·740311 | 97·443728 | 97·292385 |
| $\omega_{32}$ | 100·897825 | 101·053528 | 100·430870 |
| ⋮ | | | |
| $\omega_{48}$ | 151·642752 | 731·479885 | 150·646925 |
| $\omega_{49}$ | 154·889221 | 1251·250303 | 153·785450 |
| $\omega_{50}$ | 329·833986 | 3837·886055 | 156·923978 |

## 4. PLATE BENDING CALCULATIONS

### 4.1. INTRODUCTION

First, the Kirchhoff plate theory will be used. This theory is applicable when the plate is thin with respect to the wave length of the vibrational modes.

When a plate has free boundaries and products of beam functions are used as trial functions, comparable convergence problems as for the one-dimensional problem arise. This is due to the fact that the CBC are not satisfied. As a result, one of the second derivatives will not converge uniformly to the exact solution. The convergence of this second derivative is convergence in the mean. This does not imply that these trial functions are not complete for the used variational principle. The conditions for the Rayleigh–Ritz approximation to converge to the exact solution are conditions (7a)–(7d).

When these conditions are satisfied, the sequence of Rayleigh–Ritz approximations is a minimizing sequence. The Rayleigh–Ritz approximation converges in the energy norm (of the relevant operator, which will be denoted by $B$, in this case the biharmonic operator) to the exact solution:

$$\lim_{N\to\infty} \| W_N - W \|_B = 0. \tag{11}$$

The energy norm for the plate bending problem is twice the elastic energy [5]:

$$\| W \|_B = D \iint \left[ \left( \frac{\partial^2 W}{\partial x^2} \right)^2 + \left( \frac{\partial^2 W}{\partial y^2} \right)^2 + 2\nu \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} + 2(1-\nu) \left( \frac{\partial^2 W}{\partial x \, \partial y} \right)^2 \right] \mathrm{d}x \, \mathrm{d}y. \tag{12}$$

Convergence in this energy norm guarantees that at least the second derivatives converge in the mean to the exact solution. When the selected trial functions are well chosen, uniform convergence of the second derivatives is possible, which implies that the convergence rate is increased.

In the literature, a free edge with beam functions used in the Rayleigh–Ritz method is often described as an "over-restrained edge". For this problem it means that numerical over-restraining occurs, which in the limit ($N \to \infty$) vanishes. Physical over-restraining (no free boundary), which results in a higher limit value for the natural frequencies, does not occur.

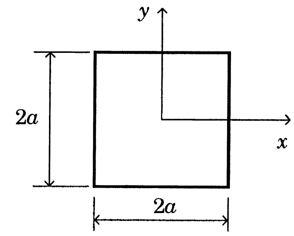


Figure 5. The square plate.

When eigenfunctions of a related problem are used as trial functions (such as the functions $\sin\{(2r-1)(\pi x/2l)\}$ in the one-dimensional problem and also the products of beam functions in the plate bending problem), these convergence problems often occur. The advantages of the use of this type of functions are that with only a few trial functions a very good approximation is obtained and that these functions guarantee that the Rayleigh–Ritz process is stable [14].

### 4.2. IMPROVEMENT OF THE CONVERGENCE

The first attempts to improve the convergence of the Rayleigh–Ritz method when the plate has free boundaries were made by Bassily and Dickinson [6]. They introduced the degenerated beam functions to improve the convergence. The accuracy obtained with this method is good. For a square plate (see Figure 5) with all edges free, the natural frequencies of the antisymmetric–symmetric modes (AS–S) were calculated by the authors. This was done with the Bassily and Dickinson method and compared with the results obtained by Gorman's superposition method [15]; see Table 4. From this table it can be concluded that the first four digits are the same for both methods.

When calculating the natural frequencies with the degenerated beam functions as trial functions in the Rayleigh–Ritz method, it became clear that these functions are not suitable for calculating the higher modes. Numerical problems arose, which are comparable with those encountered when using simple polynomials. These are caused by the hyperbolic functions which form part of the beam functions. The set of degenerated beam functions consists of these hyperbolic functions together with the trigonometric functions.

TABLE 4

*Natural frequencies of AS–S vibrations*; $\nu = 0{\cdot}333$; $N_x$, $N_y$ = *numbers of trial functions in x- and y-directions, respectively*; *the total number of trial functions is* $N_xN_y$

| | $\omega a^2\sqrt{(\rho h/D)}$, $a/b = 1{\cdot}0$ | | |
|---|---|---|---|
| | | Degenerated beam functions | |
| Mode | Gorman [15] | $N_x = N_y = 5$ | $N_x = N_y = 7$ |
| 1 | 8·558 | 8·558 | 8·558 |
| 2 | 15·23 | 15·233 | 15·233 |
| 3 | 26·05 | 26·051 | 26·051 |
| 4 | 32·68 | 32·684 | 32·684 |
| 5 | 49·44 | 49·445 | 49·442 |
| 6 | 53·41 | 53·410 | 53·407 |

TABLE 5

*Condition number of the mass matrix for degenerated beam functions*

| $N_x, N_y$ | Condition number, $\kappa$ Symmetrical | Anti-symmetrical |
|---|---|---|
| 1 | 1·0 | 1·0 |
| 2 | 4·1 | 2·19 |
| 3 | $1{\cdot}18 \times 10^3$ | $1{\cdot}30 \times 10^5$ |
| 4 | $8{\cdot}41 \times 10^3$ | $5{\cdot}63 \times 10^5$ |
| 5 | $1{\cdot}59 \times 10^8$ | $1{\cdot}28 \times 10^9$ |
| 6 | $1{\cdot}76 \times 10^9$ | $9{\cdot}69 \times 10^9$ |
| 7 | $5{\cdot}78 \times 10^{13}$ | $2{\cdot}58 \times 10^{14}$ |

The condition number of the mass matrix can be used to show that numerical problems occur. This number ($\kappa = \lambda_N/\lambda_1$, maximum eigenvalue/minimum eigenvalue) is a measure of the stability with respect to matrix inversion, but it also gives an indication of the stability of the Rayleigh–Ritz process. The condition number of the mass matrix (Gram matrix of the trial functions) is shown in Table 5. It was necessary to use double precision to perform the calculations. Without normalization of the degenerated beam functions it is possible to use a maximum of five functions (for the SS–SS) and seven functions (for the other symmetry classes) in each co-ordinate direction. With normalization with respect to the mass matrix, hardly any calculations can be performed. This implies that the numerical stability not only depends on the choice of the trial functions, but also on the normalization. The Rayleigh–Ritz approximation is independent of the normalization,

TABLE 6

*Natural frequencies for a square plate with free edges; AS–AS modes;* $\nu = 0{\cdot}3$

| Mode | | $\omega a^2\sqrt{(\rho h/D)}$ | | | |
|---|---|---|---|---|---|
| | | Legendre polynomials | | | Beam functions, |
| No. | Symmetry with respect to diagonal | $N_x = N_y = 7$ | $N_x = N_y = 11$ | $N_x = N_y = 15$ | $N_x = N_y = 15$ |
| 1 | S | 13·4682 | 13·4682 | 13·4682 | 13·472 |
| 2 | AS | 69·2654 | 69·2654 | 69·2654 | 66·445 |
| 3 | S | 77·1717 | 77·1717 | 77·1717 | 77·298 |
| 4 | S | 152·845 | 152·8449 | 152·845 | 153·03 |
| 5 | AS | 204·142 | 204·141 | 204·141 | 204·75 |
| 6 | S | 213·947 | 213·947 | 213·947 | 214·43 |
| 7 | AS | 291·865 | 291·865 | 291·865 | 292·28 |
| 8 | S | 298·519 | 298·519 | 298·519 | 298·90 |
| 9 | AS | 419·595 | 418·875 | 418·875 | 420·17 |
| 10 | S | 429·807 | 428·943 | 428·943 | 429·99 |
| ⋮ | | | | | |
| 23 | | 1275·52 | 1086·09 | 1086·03 | 1089·24 |
| 24 | | 1286·52 | 1098·38 | 1098·28 | 1100·60 |
| 25 | | 1665·23 | 1173·82 | 1173·70 | 1175·47 |
| ⋮ | | | | | |
| 30 | | 2260·98 | 1347·20 | 1346·91 | 1348·72 |
| 31 | | 2299·50 | 1516·29 | 1516·16 | 1517·26 |
| 32 | | 2542·24 | 1556·53 | 1539·13 | 1542·42 |

because the spanned subspace does not change. When the normalization factor is not the same for every function, a strongly minimal system can change into a minimal one, which has the consequence that the stability of the Rayleigh–Ritz process is lost [14].

The results obtained when Legendre polynomials are used for the one-dimensional problem, also suggest that good results may be expected when these polynomials are used for plate vibration problems. To test the numerical stability, the natural frequencies of a totally free plate have been calculated with the Legendre polynomials as trial functions. Indeed, at least up to 15 functions in each co-ordinate direction could be used (the total number of trial functions is $15 \times 15 = 225$) without numerical problems. The lowest 25% of the natural frequencies can be considered to be very accurate. This means that, for this symmetry class, 50 modes were calculated with a high accuracy (see Table 6).

### 4.3. HIGHER-ORDER THEORIES

To include the influence of transverse shear deformation and rotational inertia, a higher order plate theory should be used. There are several higher order plate theories. Mindlin–Uyfland theory [16] is the oldest and most frequently used. This is a first order theory (Kirchhoff theory can be regarded as a zero order theory). Third order theories have also been developed. In these theories, the zero shear force boundary conditions on the surface of the plate is satisfied. One of these theories has been developed by Reddy [17]. Lim *et al*. [18] have simplified this method by eliminating one of the variables. In this theory the unknown variables are two displacements (two rotations have been replaced by one displacement). They showed that this simplification gives natural frequencies almost equal to the values from the theory of Reddy. Therefore it seems justified to use this simplified theory. To include effects of transverse shear deformation and rotational inertia, the method of Lim *et al*. has been used in the present calculations.

The Rayleigh–Ritz method was used with Legendre polynomials as trial functions to approximate the natural frequencies and mode shapes. Just as for a Kirchhoff plate, an increase in the number of trial functions was used to determine the obtained accuracy (see Table 7). The two independent displacements were both approximated by a linear combination of the same trial functions. The total number of unknowns therefore was twice the number of trial functions used.

TABLE 7

*Natural frequencies of a square plate with free edges*; AS–AS *modes*; $\nu = 0{\cdot}3$, $h/2a = 0{\cdot}03$

| Mode | | $\omega a^2\sqrt{(\rho h/D)}$, Legendre polynomials | | |
|---|---|---|---|---|
| No | Symmetry with respect to diagonal | $N_x = N_y = 7$ | $N_x = N_y = 9$ | $N_x = N_y = 11$ |
| 1 | S | 13·4414 | 13·4414 | 13·4414 |
| 2 | AS | 68·5430 | 68·5430 | 68·5430 |
| 3 | S | 76·2510 | 76·2510 | 76·2510 |
| 4 | S | 149·338 | 149·338 | 149·338 |
| 5 | AS | 197·781 | 197·781 | 197·781 |
| 6 | S | 207·007 | 207·007 | 207·007 |
| 7 | AS | 279·468 | 279·468 | 279·468 |
| 8 | S | 285·380 | 285·379 | 285·379 |
| 9 | AS | 293·957 | 393·333 | 393·333 |
| 10 | S | 403·151 | 402·398 | 402·398 |

It is shown in Table 7 that with seven functions in each direction (total number of trial functions is 49, total number of unknowns is 98), eight natural frequencies of this symmetry class can be calculated with an accuracy of five digits. For the four symmetry classes (SS, SA, AS and AA) this implies a total of 32 modes.

An explanation as to why the simplification in Lim's method gives such good results (as compared with the theory of Reddy), can be given by considering the boundary layers for Mindlin plates, as given by Häggblad and Bathe [19].

## 5. CONCLUSIONS

It has been shown that when the complementary boundary conditions are not satisfied, in most cases the natural frequencies do converge to the exact values. The low convergence rate, which is a result of not satisfying the CBC, has similarities with known convergence characteristics when discontinuities are approximated by a Fourier series.

The most important advantage of using orthogonal polynomials as trial functions in the Rayleigh–Ritz method appears not to be the resulting standard eigenvalue problem instead of a generalized eigenvalue problem, but the numerical stability of the Rayleigh–Ritz process. As a consequence, a great number of trial functions can be used in the Rayleigh–Ritz approximation. Because of the high convergence rates, convergence studies can be used to obtain "exact" solutions.

By using a higher order plate theory, higher order effects such as transverse shear and rotational inertia can be included. When using orthogonal polynomials as trial functions, the resulting numerical stable Rayleigh–Ritz process provides the opportunity to calculate the frequencies and mode shapes of the higher modes, where the higher order effects become significant, with great accuracy.

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## APPENDIX A: FOURIER ANALYSIS OF AN EXAMPLE PROBLEM

The example from section 3.1 is considered; see Figure A1.

Suppose that the bar is clamped at $x = l$, instead of being connected with a spring. This is a geometrical boundary condition, $u = 0$. This also can be seen as the boundary condition with the spring connected at $x = l$ (it is then a dynamical boundary condition) in the limit for $k \to \infty$.

The exact solution for the lowest mode with the bar clamped at $x = l$ is

$$u_1 = a_1 \sin(\pi x / l). \tag{A1}$$

When the eigenfunctions of a clamped–free beam are used as trial functions, the Rayleigh–Ritz approximation is

$$u_n = \sum_{r=1,3}^{N} a_r \sin(r\pi x / 2l). \tag{A2}$$

The Rayleigh–Ritz approximation for the lowest mode is the projection of this function (with the energy inner product) on the subspace spanned by the trial functions. Because the spring constant $k$ is infinite, the energy contained in the spring is zero and the Fourier coefficients are determined by expanding the derivative of $u_N$ in a Fourier series:

$$\frac{n}{l} \cos\left(\frac{\pi x}{l}\right) = \sum_{r=1,3}^{N} a_r \frac{r\pi}{2l} \cos\left(\frac{r\pi x}{2l}\right), \tag{A3}$$

or

$$\cos\left(\frac{\pi x}{l}\right) = \sum_{r=1,3}^{N} b_r \cos\left(\frac{r\pi x}{2l}\right). \tag{A4}$$

The coefficients $b_r$ are

$$b_1 = \frac{4}{3\pi}, \quad r = 1, \qquad b_r = \frac{2}{\pi}\left(\frac{1}{(r+2)} + \frac{1}{(r-2)}\right), \quad r = 3, 7, 11, \ldots,$$

$$b_r = -\frac{2}{\pi}\left(\frac{1}{(r+2)} + \frac{1}{(r-2)}\right), \quad r = 5, 9, 13, \ldots . \tag{A5a–c}$$

f

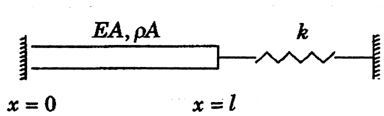


Figure A1. The example problem.

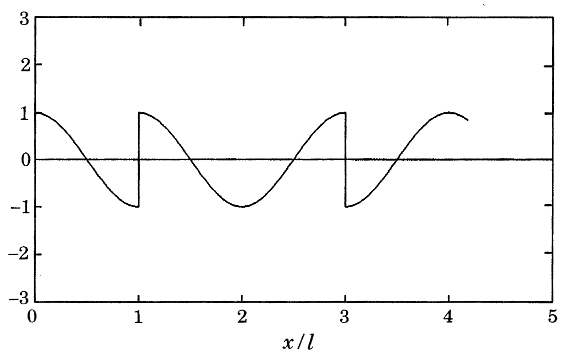


Figure A2. A periodic function with discontinuities.

The Fourier functions are periodic, and so outside the interval $(0, l)$ a periodic function is approximated. This periodic function has discontinuities at $x = \pm l, \pm 3l \ldots$; a plot of this function is shown in Figure A2.

From the theory of Fourier series, it is known that the Fourier series converges in the norm of the space of quadratic integrable functions (the space $L_2$). This means that the derivative of the displacement (the strain and thus also the stress) converges in the mean to the exact solution. The convergence rate of the Fourier coefficients is of order $1/r$ as expressions (A5a)–(A5c) show.

## APPENDIX B: LIST OF SYMBOLS

| | |
|---|---|
| $D$ | plate bending stiffness |
| $w$ | transverse displacement |
| $W$ | transverse displacement as a function of spatial co-ordinates |
| $\rho$ | mass density |
| $h$ | plate thickness |
| $q$ | external load |
| $\nabla$ | gradient operator |
| $V_n$ | Kirchhoff shear force on the edge of a plate |
| $M_n$ | Bending moment on the edge of a plate |
| $\omega$ | natural frequency |
| $V_{max}$ | maximum potential energy |
| $T^*$ | expression for kinetic energy with the displacement substituted for the velocity |
| $x$ | spatial $x$ co-ordinate |
| $y$ | spatial $y$ co-ordinate |
| $E$ | Young's modulus |
| $\nu$ | Poisson ratio |
| $\varphi_i$ | trial function |
| $\psi_r$ | trial function |
| $a_i$ | coefficient in Rayleigh–Ritz approximation |
| $A$ | cross-sectional area |
| $l$ | length of the bar |
| $k$ | spring stiffness |
| $k'$ | dimensionless spring stiffness |
| $L_2$ | space of quadratic integrable functions |
| $\| \ \|$ | norm of a function |
| $u$ | axial displacement of the bar |
| $U$ | axial displacement of the bar as a function of the spatial co-ordinate |
| $\| \ \|_B$ | energy norm with respect to the operator $B$ |
| S | symmetric |
| AS | antisymmetric |
| AS–S | mode antisymmetric in $x$-direction and symmetric in $y$-direction |

| | |
|---|---|
| S–S | mode symmetric in $x$-direction and symmetric in $y$-direction |
| S–AS | mode symmetric in $x$-direction and antisymmetric in $y$-direction |
| AS–AS | mode antisymmetric in $x$-direction and antisymmetric in $y$-direction |
| $a$ | half-length of plate |
| $b$ | half-width of plate |
| $N_x$ | number of functions used in $x$-direction |
| $N_y$ | number of functions used in $y$-direction |
| $N$ | total number of trial functions |
| $\kappa$ | condition number of a matrix |
| $\lambda_N$ | maximum eigenvalue |
| $\lambda_1$ | minimum eigenvalue |
| $a_k$ | coefficient in a Fourier series |
| $b_k$ | coefficient in a Fourier series |