ELSEVIER

Discrete Mathematics 138 (1995) 379–385

DISCRETE MATHEMATICS

Pancyclicity of hamiltonian line graphs

E. van Blanken, J. van den Heuvel, H.J. Veldman*

Faculty of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, Netherlands

Received 7 July 1993; revised 18 February 1994

Abstract

Let $f(n)$ be the smallest integer such that for every graph G of order n with minimum degree $\delta(G)>f(n)$, the line graph $L(G)$ of G is pancyclic whenever $L(G)$ is hamiltonian. Results are proved showing that $f(n)=\Theta(n^{1/3})$.

Keywords: Line graph; Hamiltonian graph; Pancyclic graph

1. Introduction

We use [4] for terminology and notation not defined here and consider finite simple graphs only. A graph of order n is *pancyclic* if it contains C_k, i.e., a cycle of length k, for each k with $3\leqslant k\leqslant n$.

A natural question is the following: how large should the minimum degree of a hamiltonian graph G be in order that G is guaranteed to be pancyclic? Amar et al. [1] answered this question for nonbipartite graphs by proving the following best possible result.

Theorem 1 (Amar et al. [1]). *Let G be a hamiltonian nonbipartite graph of order $n\geqslant 102$ with minimum degree $\delta(G)>\frac{2}{5}n$. Then G is pancyclic.*

Here we consider a similar question concerning line graphs. Specifically, let $f(n)$ be the smallest integer such that for every graph G of order n with $\delta(G)>f(n)$, the line graph $L(G)$ of G is pancyclic whenever $L(G)$ is hamiltonian. In Section 2 we obtain upper bounds for $f(n)$. In Section 3 a lower bound for $f(n)$ is derived from the construction of suitable graphs. The upper and lower bounds have the same order of magnitude: $\Theta(n^{1/3})$. In Section 4 we conjecture that the graphs constructed in Section 3 essentially determine $f(n)$.

*Corresponding author.

SSDI 0012-365X(94)00220-7

2. Upper bounds for $f(n)$

Our first result is the following.

Theorem 2. *Let G be a graph of order n with $\delta(G) \geqslant 600\ n^{1/3}$ such that $L(G)$ is hamiltonian. Then $L(G)$ is pancyclic.*

Before proving Theorem 2 we introduce some additional terminology and notation, and state a number of preliminary results.

By a *circuit* of a graph G we will mean an eulerian subgraph of G, i.e., a connected subgraph in which every vertex has even degree. Note that by this definition (the trivial subgraph induced by) a single vertex is also a circuit. If C is a circuit of G, then $I(C)$ denotes the set of edges of G incident with at least one vertex of C. We write $\imath(C)$ for $|I(C)|$.

Harary and Nash-Williams [8] characterized hamiltonian line graphs.

Theorem 3 (Harary and Nash-Williams [8]). *The line graph $L(G)$ of a graph G is hamiltonian if and only if G contains a circuit C such that $\imath(C) = |E(G)| \geqslant 3$.*

From Theorem 3 one easily proves a more general result (see, e.g., [6]).

Theorem 4. *The line graph $L(G)$ of a graph G contains a cycle of length $k \geqslant 3$ if and only if G contains a circuit C such that $|E(C)| \leqslant k \leqslant \imath(C)$.*

A key lemma for our proof of Theorem 2 is the following.

Lemma 5. *Let G be a graph of order n and minimum degree $\delta \geqslant 4$ such that $L(G)$ contains C_{m+1} but not C_m. Then*

$$m \leqslant \frac{3n - \delta - 1}{\delta + 1}.$$

Proof. Let G satisfy the hypothesis of the lemma. By Theorem 4, G contains a circuit C with $|E(C)| \leqslant m+1 \leqslant \imath(C)$. In fact $|E(C)| = m+1$, otherwise $L(G)$ contains C_m. Since C is a circuit, there exist edge-disjoint cycles $D_1, \dots, D_r$ such that

$$C = \bigcup_{i=1}^{r} D_i.$$

We distinguish the following cases.

Case 1: $r = 1$. Then C is a cycle of length $m+1$.

Case 1.1: *C has a chord.* Let C' be a longest cycle among all cycles that contain exactly one chord of C while the remaining edges belong to C. In $\sum_{v \in V(C')} d(v)$, every edge in $I(C')$ is counted at most twice. Hence

$$\imath(C') \geqslant \tfrac{1}{2}|V(C')|\delta > \tfrac{1}{4}|V(C)|\delta = \tfrac{1}{4}(m+1)\delta \geqslant m+1.$$

On the other hand, $|E(C')| \leqslant m$. Thus $L(G)$ contains C_m, a contradiction.

Case 1.2: *C has no chord.* Since $\delta \geqslant 4$, C cannot be a Hamilton cycle of G. Let u be a vertex in $V(G)\setminus V(C)$. If u is adjacent to at least four vertices of C, then G contains a cycle C' with $\frac{1}{2}|V(C)| < |V(C')| \leqslant m$ and we obtain a contradiction as in Case 1.1. Hence u is adjacent to at most three vertices of C. Defining p as the number of edges of G incident with exactly one vertex of C, we thus have

$$p \leqslant 3|V(G)\setminus V(C)| = 3(n-m-1).$$

On the other hand, since C has no chords,

$$p = \sum_{v \in V(C)} (d(v)-2) \geqslant (m+1)(\delta-2).$$

It follows that $(m+1)(\delta-2) \leqslant 3(n-m-1)$ or, equivalently, $m \leqslant (3n-\delta-1)/(\delta+1)$.

Case 2: $r \geqslant 2$. Let H be the graph with $V(H) = \{D_1, \ldots, D_r\}$ and $D_iD_j \in E(H)$ iff $V(D_i) \cap V(D_j) \neq \emptyset$ $(i \neq j)$. Since H is connected, at least two vertices of H are not cut vertices of H. Equivalently, there are at least two values of j for which $\bigcup_{1 \leqslant i \leqslant r, i \neq j} D_i$ is a connected subgraph of G, and hence a circuit of G. Assume without loss of generality that $C' = \bigcup_{i=2}^{r} D_i$ and $C'' = D_1 \cup \bigcup_{i=3}^{r} D_i$ are circuits of G. We have

$$\begin{aligned}\imath(C') &\geqslant |E(C')| + |E(D_1) \cap I(C')| + \tfrac{1}{2}|V(D_2 - V(C''))|(\delta-2)\\ &= |E(C)| - |E(D_1 - V(C'))| + \tfrac{1}{2}|V(D_2 - V(C''))|(\delta-2).\end{aligned}$$

On the other hand, since $L(G)$ does not contain C_m,

$$\imath(C') \leqslant m-1 = |E(C)|-2.$$

It follows that $|E(D_1 - V(C'))| \geqslant \frac{1}{2}|V(D_2 - V(C''))|(\delta-2)+2$ and hence, since $\delta \geqslant 4$,

$$|V(D_1 - V(C'))| \geqslant \tfrac{1}{2}|V(D_2 - V(C''))|(\delta-2)+3 > |V(D_2 - V(C''))|.$$

But then by symmetry we also have

$$|V(D_2 - V(C''))| > |V(D_1 - V(C'))|.$$

This contradiction completes the proof. □

The proof of Theorem 2 also relies on a result of Bondy and Simonovits [5].

Theorem 6 (Bondy and Simonovits [5]). *If G is a graph of order n with $|E(G)| \geqslant 100\,kn^{1+1/k}$, then G contains C_{2l} for every integer l with $k \leqslant l \leqslant kn^{1/k}$.*

Proof of Theorem 2. Let G satisfy the hypothesis of the theorem. Assuming $L(G)$ is not pancyclic, set $m = \max\{i \mid L(G)$ does not contain $C_i\}$. Then $m \leqslant |V(L(G))|-1$ and

$L(G)$ contains C_{m+1}. Clearly,

$$m > \delta \geqslant 600. \tag{1}$$

The graph G contains a cycle C of length at most 7, otherwise for any vertex u of G the subgraph induced by $\{v \in V(G) \mid d(u,v) \leqslant 3\}$ is a tree, implying that

$$n \geqslant 1 + \sum_{i=1}^{3} \delta(\delta-1)^{i-1} = \delta^3 - \delta^2 + \delta + 1 > \frac{\delta^3}{600^3} \geqslant n,$$

a contradiction. We have

$$\imath(C) \geqslant \tfrac{1}{2} |V(C)| \delta \geqslant \tfrac{3}{2}\delta \geqslant 900 n^{1/3}. \tag{2}$$

Using $|E(C)| \leqslant 7$, (1), (2) and Theorem 4 we obtain

$$m > 900 n^{1/3}. \tag{3}$$

Set $l = \lfloor 3n^{1/3} \rfloor$. Since $\delta \geqslant 600 n^{1/3}$, we have $|E(G)| \geqslant 300 n^{4/3}$. Hence by Theorem 6, G contains a cycle C' of length $2l$, which satisfies

$$\imath(C') \geqslant \tfrac{1}{2} |V(C')| \delta \geqslant (3n^{1/3} - 1) \times 600 n^{1/3} > 1200 n^{2/3}. \tag{4}$$

Using $|E(C')| \leqslant 6n^{1/3}$, (3), (4) and Theorem 4 we obtain

$$m > 1200 n^{2/3}. \tag{5}$$

On the other hand, by Lemma 5,

$$m \leqslant \frac{3n - \delta - 1}{\delta + 1} < \frac{3n}{\delta} \leqslant \frac{n^{2/3}}{200}. \tag{6}$$

(5) and (6) are contradictory, so the proof is complete. □

Theorem 2 has an equivalent formulation in terms of $f(n)$.

Corollary 7. *$f(n) < 600 n^{1/3}$ for all n.*

From results in [2] it follows that if G is a 2-edge-connected graph of order $n \geqslant 3$ with $\delta(G) > \frac{1}{3}n$, then $L(G)$ is hamiltonian; moreover, if $G \not\cong C_4, C_5$, then $L(G)$ is pancyclic. Catlin [7] improved the first part of this statement to a best possible result.

Theorem 8 (Catlin [7]). *Let G be a 2-edge-connected graph of order $n > 20$ with $\delta(G) > \frac{1}{5}n - 1$. Then G contains a spanning circuit. In particular, $L(G)$ is hamiltonian.*

Combining Theorems 2 and 8 we obtain the following.

Corollary 9. *If G is a 2-edge-connected graph of order $n \geqslant 164325$ with $\delta(G) > \frac{1}{5}n - 1$, then $L(G)$ is pancyclic.*

Corollary 9 supports Bondy's Metaconjecture (see, e.g., [3]) that almost every nontrivial condition which implies that a graph is hamiltonian also implies that the graph is pancyclic.

For large values of n we have obtained an improvement of Theorem 2. The proof, which we omit here, can be found in the internal report [9]. It uses a partial improvement of Theorem 6 obtained by suitably modifying the arguments in [5].

Theorem 10. *For every real number $\varepsilon > 0$ there exists an integer N such that if G is a graph of order $n \geqslant N$ with $\delta(G) \geqslant (96(1+\varepsilon)n)^{1/3}$, then $L(G)$ is pancyclic whenever $L(G)$ is hamiltonian.*

Corollary 11. *$f(n) < 4.6n^{1/3}$ for n sufficiently large.*

3. A lower bound for $f(n)$

We construct a family of graphs with hamiltonian but not pancyclic line graphs in order to obtain a lower bound for $f(n)$.

For any integer d with $d \geqslant 3$ and $d \not\equiv 1 \pmod 3$, define the graph G_d as follows. Set $p = \frac{1}{6}d(d+1)+1$. Then p is an integer. Let $C = u_1 u_2 \cdots u_{3p} u_1$ be a cycle of length $3p$ and let $H_1, \ldots, H_p$ be p copies of K_{d-2} such that $C, H_1, \ldots, H_p$ are pairwise disjoint. Now G_d is obtained from $C \cup \bigcup_{i=1}^{p} H_i$ by joining each vertex of H_i to u_{3i-2}, u_{3i-1} and u_{3i}, for $i = 1, \ldots, p$. We have

$$\delta(G_d) = d \tag{7}$$

and

$$|V(G_d)| = 3p + p(d-2) = p(d+1) = \tfrac{1}{6}(d^3 + 2d^2 + 7d + 6). \tag{8}$$

Furthermore, G_d is hamiltonian and hence

$$L(G_d) \text{ is hamiltonian.} \tag{9}$$

For a cycle C' of G_d with vertex set $V(H_i) \cup \{u_{3i-2}, u_{3i-1}, u_{3i}\}$ for some i we have $\iota(C') = \frac{1}{2}d(d+1)+1 = 3p-2$ and $\iota(C') \geqslant \iota(C'')$ for every circuit C'' with $|E(C'')| < |E(C)| = 3p$. We conclude that $L(G_d)$ does not contain C_{3p-1} and hence

$$L(G_d) \text{ is not pancyclic.} \tag{10}$$

Using (7)–(10) and also considering graphs obtained from G_d by deleting suitable subsets of $V(G_d)$ we reach, in particular, the following conclusion.

Theorem 12. *$f(n) > 1.8n^{1/3}$ for n sufficiently large.*

4. A conjecture

The family of graphs G_d described in Section 3 shows that the following conjecture, if true, is best possible.

Conjecture 13. Let G be a graph of order n and minimum degree δ such that $L(G)$ is hamiltonian and

$$\delta^3+2\delta^2+7\delta+6>6n. \tag{11}$$

Then $L(G)$ is pancyclic unless G is isomorphic to C_4, C_5 or the Petersen graph.

To prove Conjecture 13 for $n \geqslant 12$, it would be sufficient to show that if a graph G satisfies (11) and $L(G)$ is hamiltonian, then

$$G \text{ contains a circuit } C_0 \quad \text{with } \delta+1 \leqslant |V(C_0)| \leqslant 3\delta-2. \tag{12}$$

This can be seen by slightly refining the proof of Theorem 2, as outlined below.

Let G be a graph of order $n \geqslant 12$ and minimum degree δ such that $L(G)$ is hamiltonian and (11) and (12) are satisfied. Then $\delta \geqslant 4$. Assuming $L(G)$ is not pancyclic, set $m = \max\{i \mid L(G) \text{ does not contain } C_i\}$. Then $m \geqslant \delta+1$. A shortest cycle C of G satisfies $|E(C)| \leqslant \min\{6, \delta+1\}$ and, since C has no chords,

$$\imath(C) \geqslant |E(C)| + \sum_{v \in V(C)} (d(v)-2) \geqslant |V(C)|(\delta-1) \geqslant 3\delta-3.$$

Hence $m \geqslant 3\delta-2$. Since $L(G)$ is not pancyclic, the circuit C_0 cannot be a spanning subgraph of G. Furthermore, since $L(G)$ is hamiltonian, G is 2-edge-connected. It follows that at least two edges of G are incident with exactly one vertex of C_0, whence

$$\imath(C_0) \geqslant \tfrac{1}{2}|V(C_0)|\delta+1 \geqslant \tfrac{1}{2}\delta(\delta+1)+1.$$

Thus $m \geqslant \frac{1}{2}\delta(\delta+1)+2$. By Lemma 5, $m \leqslant (3n-\delta-1)/(\delta+1)$. Combining the last two inequalities, we obtain a contradiction with (11).

References

[1] D. Amar, E. Flandrin, I. Fournier and A. Germa, Pancyclism in hamiltonian graphs, Discrete Math. 89 (1991) 111–131.

[2] A. Benhocine, L. Clark, N. Köhler and H.J. Veldman, On circuits and pancyclic line graphs, J. Graph Theory 10 (1986) 411–425.

[3] J.A. Bondy, pancyclic graphs, in: R.C. Mullin et al., eds., Proc. 2nd Louisiana Conf. on Combinatorics, Graph Theory and Computing, Cong. Numer. III, Utilitas Mathematica, Winnipeg (1971) 80–84.

[4] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan and Elsevier, London and New York, 1976).

[5] J.A. Bondy and M. Simonovits, Cycles of even length in graphs, J. Combin. Theory Ser. B 16 (1974) 97–105.

[6] H.J. Broersma, Subgraph conditions for dominating circuits in graphs and pancyclicity of line graphs, Ars Combin. 23 (1987) 5–12.

[7] P.A. Catlin, A reduction method to find spanning eulerian subgraphs, J. Graph Theory 12 (1988) 29–44.

[8] F. Harary and C.St.J.A. Nash-Williams, On eulerian and hamiltonian graphs and line graphs, Canad. Math. Bull. 8 (1965) 701–710.

[9] E. van Blanken, J. van den Heuvel and H.J. Veldman, Pancyclicity of hamiltonian line graphs, Memorandum 1146, Faculty of Applied Mathematics, University of Twente, Enschede, Netherlands, 1993.