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A note on dominating cycles in 2-connected graphs

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Abstract

Let G be a 2-connected graph on n vertices such that $d(x)+d(y)+d(z)\geqslant n$ for all triples of independent vertices x,y,z. We prove that every longest cycle in G is a dominating cycle unless G is a spanning subgraph of a graph belonging to one of four easily specified classes of graphs.

1. Introduction

We strengthen a result in [6] concerning dominating cycles in 1-tough graphs. First we give a few definitions and some notation. We consider only finite undirected graphs with no loops or multiple edges. A good reference for any undefined terms is [15]. Let $\omega(G)$ denote the number of components of a graph G. A graph G is *t-tough* if $|S|\geqslant t\omega(G-S)$ for every subset S of the vertex set $V(G)$ with $\omega(G-S)>1$. The *toughness* of G, denoted $t(G)$, is the maximum value of t for which G is t-tough (taking $t(K_n)=\infty$ for all $n\geqslant 1$). We let $\alpha(G)$ denote the maximum cardinality of a set of independent vertices of G, and $c(G)$ be the length of a longest cycle in G. A cycle C of G is called a *dominating cycle* if every edge of G has at least one of its vertices on C. We let $d(v)$ denote the degree of vertex v. For $k\leqslant\alpha(G)$ we let $\sigma_k(G) = \min\{\sum_{i=1}^{k} d(v_i) | \{v_1,\ldots,v_k\}$ is an independent set of k vertices $\}$ and $\mathrm{NC}_k(G)$ be the minimum cardinality of the neighborhood union of any k such vertices. If G is a graph on n vertices, then for $k>\alpha(G)$ we set

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$\sigma_k(G) = k(n - \alpha(G))$ and $\mathrm{NC}_k(G) = n - \alpha(G)$. If G has a noncomplete component we let $\mathrm{NC2}(G)$ denote the minimum neighborhood union of any pair of vertices at distance two apart; otherwise $\mathrm{NC2}(G) = n - 1$. If no ambiguity can arise we sometimes write α instead of $\alpha(G)$, etc.

There are now many results in graph theory relating the toughness of a graph to its cycle structure [9,10]. However recently it was discovered [4] that for any fixed positive rational number t, it is NP-hard to recognize t-tough graphs. Nevertheless, toughness remains an important parameter in the study of hamiltonian cycles in graphs. In particular, the truth of Chvátal's well-known conjecture [16] that 2-tough graphs are hamiltonian would have a number of useful applications [1]. On the other hand, the fact that it is NP-hard to recognize 1-tough graphs makes it desirable to strengthen some theorems by replacing the assumption that a graph G is 1-tough with the weaker assumption that G is 2-connected. The idea is to try to draw the same conclusion regarding the cycle structure of G under the weaker hypothesis of 2-connectedness by specifying an easily described family of exceptional graphs for which the new theorem does not hold.

To illustrate this type of improvement consider the following theorem, established independently by Bondy [12], Bermond [11], and Linial [18].

Theorem 1. *Let G be a 2-connected graph on n vertices. Then $c(G) \geqslant \min(n, \sigma_2)$.*

An analogous theorem for 1-tough graphs was later established by Bauer and Schmeichel [8] and independently by Tian and Zhao [20].

Theorem 2. *Let G be a 1-tough graph on $n \geqslant 3$ vertices. Then $c(G) \geqslant \min(n, \sigma_2 + 2)$.*

Both of these theorems were generalized in the manner described above by the following result, presented three years ago at this conference [5]. We call a graph H a *σ_2-subgraph* of a 2-connected graph G if H is a 2-connected spanning subgraph of G with $\sigma_2(H) = \sigma_2(G)$. Define the graph $S_{h,p,q}$ to be the graph obtained from $(\{u\} + pK_h) \cup (\{v\} + qK_h)$ by adding the edge uv. Let $T_{h,p,q} = K_1 + S_{h,p,q}$.

Theorem 3. *Let G be a 2-connected nonhamiltonian graph on n vertices with $c(G) \leqslant \sigma_2 + 1$. Then G is a σ_2-subgraph of one of the following:*

(1) $K_s + tK_1$, *where* $t > s \geqslant 3$;
(2) $K_s + (K_2 \cup tK_1)$, *where* $t \geqslant s \geqslant 3$;
(3) $K_2 + (K_{1,t} \cup qK_2)$, *where* $q \geqslant 2, t \geqslant 3$;
(4) $K_3 + qK_2$, *where* $q \geqslant 4$;
(5) $K_2 + tK_h$, *where* $t \geqslant 3, h \geqslant 1$;
(6) $K_2 + (tK_h \cup K_{h-1})$, *where* $t \geqslant 2, h \geqslant 2$;
(7) $K_2 + (tK_h \cup K_{h+1})$, *where* $t \geqslant 2, h \geqslant 1$;
(8) $T_{h,p,q}$, *where* $p \geqslant 2, q \geqslant 2, h \geqslant 1$.

As another illustration, Skupień [19] was able to characterize the nonhamiltonian 2-connected graphs with $\sigma_2 \geqslant n-4$, thereby improving the following well-known theorem of Jung [17].

Theorem 4. *Let G be a 1-tough graph on $n \geqslant 11$ vertices with $\sigma_2 \geqslant n-4$. Then G is hamiltonian.*

The purpose of this note is to provide a similar improvement to the following result, established in [6].

Theorem 5. *Let G be a 1-tough graph on n vertices with $\sigma_3 \geqslant n$. Then every longest cycle in G is a dominating cycle.*

An analogous theorem for 2-connected graphs had already been established by Bondy [13].

Theorem 6. *Let G be a 2-connected graph on n vertices with $\sigma_3 \geqslant n+2$. Then every longest cycle in G is a dominating cycle.*

Our main result, proved in Section 2, generalizes the previous two theorems.

Consider the following four classes of graphs:

(1) $K_2 + (K_a \cup K_b \cup K_c)$ where $a,b,c \geqslant 2$;

(2) $K_3 + (aK_2 \cup bK_3)$, where $a,b \geqslant 0$ and $a+b=4$;

(3) $K_s + (sK_2 \cup K_3)$, where $s \geqslant 4$;

(4) $K_s + (s+1)K_2$, where $s \geqslant 4$.

Let $\mathscr{H}_i$ be the class of all spanning subgraphs of graphs in class (i), $1 \leqslant i \leqslant 4$, and let $\mathscr{H} = \cup_{i=1}^{4} \mathscr{H}_i$.

Theorem 7. *Let G be a 2-connected graph on n vertices with $\sigma_3 \geqslant n$ and suppose $G \notin \mathscr{H}$. Then every longest cycle in G is a dominating cycle.*

A number of recent results in hamiltonian graph theory rely on the knowledge that every longest cycle in a graph G is a dominating cycle [2,9,10]. Many of these theorems employ either Theorem 5 or Theorem 6, i.e., they require that either G is 1-tough with $\sigma_3 \geqslant n$ or that G is 2-connected with $\sigma_3 \geqslant n+2$. As a consequence of Theorem 7, it is no longer necessary to check the NP-hard hypothesis of Theorem 5.

2. Proof of Theorem 7

To prove Theorem 7 we make use of a stronger version of Theorem 5. In [6] it was noted that the condition that G be 1-tough in Theorem 5 can be replaced by a weaker condition. Let $\omega_1(G)$ denote the number of components of G having order at least two. A graph G is called ω_1 – ***tough*** if $\omega_1(G-S)\leqslant|S|$ for every nonempty proper subset S of the vertex set $V(G)$. The proof of the following result is implicit in the proof of Theorem 5 of [6].

Theorem 8. *Let G be an ω_1-tough graph on n vertices with $\sigma_3\geqslant n$. Then every longest cycle in G is a dominating cycle.*

The assumption that G is ω_1-tough in Theorem 8 is a natural one since a necessary condition for a graph to have a dominating cycle is that it be ω_1-tough [21]. Theorem 8 generalizes Theorem 6 and also yields the following simple proof of Theorem 7.

Proof of Theorem 7. Suppose G satisfies the hypothesis of the theorem and contains a longest cycle that is not a dominating cycle. Then by Theorem 8 there exists a nonempty set of vertices S such that $G-S$ contains at least $|S|+1$ nontrivial components. Let $s=|S|$. Since G is 2-connected, $s\geqslant 2$. Let $G_1,G_2,\ldots,G_{s+1+j}$ $(j\geqslant 0)$ be the nontrivial components of $G-S$, and let $n_i=|V(G_i)|, 1\leqslant i\leqslant s+1+j$. Let t be the number of trivial components of $G-S$. Assume, without loss of generality, that $2\leqslant n_1\leqslant n_2\leqslant\cdots\leqslant n_{s+1+j}$. If $s=2$, then $G\in\mathscr{H}_1$ (noting that our characterization is in terms of spanning subgraphs). Suppose $s\geqslant 3$. Clearly $n\geqslant n_1+n_2+n_3+\cdots+n_{s+1+j}+s+t$ and $n\leqslant\sigma_3\leqslant n_1+n_2+n_3-3+3s-\min(1,t)$. Hence $2(s+j-2)\leqslant n_4+\cdots+n_{s+1+j}\leqslant 2s-t-\min(1,t)-3$, implying that $2j\leqslant 1-t-\min(1,t)$. We conclude that $j=t=0$ and $n_{s+1}\in\{2,3\}$. Hence if $s=3, G\in\mathscr{H}_2$. Finally, if $s\geqslant 4$, then $n_s=2$, so $G\in\mathscr{H}_3$ if $n_{s+1}=3$ and $G\in\mathscr{H}_4$ if $n_{s+1}=2$. □

3. Applications

Let us now return to Theorem 5. The following strengthening of this theorem is also in [6].

Theorem 9. *Let G be a 1-tough graph on n vertices with $\sigma_3\geqslant n\geqslant 3$. Then every longest cycle in G is a dominating cycle. Moreover, if G is nonhamiltonian, G contains a longest cycle C such that* $\max\{d(v)|v\in V(G)-V(C)\}\geqslant\sigma_3/3$.

The second part of Theorem 9 is implicit in the proof of [6, Theorem 9]. The details of the argument can be found in the appendix of [7]. Using techniques analogous to those used to prove Theorem 7 we can also strengthen Theorem 9 so that its hypothesis can be checked in polynomial time.

Theorem 10. *Let G be a 2-connected nonhamiltonian graph on n vertices with $\sigma_3\geqslant n$ and suppose $G\notin\mathscr{H}$. Then G contains a longest cycle C (necessarily dominating) and a vertex $v_0\in V(G)-V(C)$ such that $d(v_0)\geqslant\sigma_3/3$ unless G is a spanning subgraph of a graph in one of the following six classes of graphs.*

(1) $K_2 + (K_1 \cup K_a \cup K_b)$, *where* $a > b \geqslant 1$;
(2) $K_3 + (K_1 \cup 2K_3 \cup K_a)$, *where* $1 \leqslant a \leqslant 3$;
(3) $K_s + (K_1 \cup K_3 \cup (s-1)K_2)$, *where* $s \geqslant 3$;
(4) $K_s + (2K_1 \cup K_3 \cup (s-2)K_2)$, *where* $s \geqslant 3$;
(5) $K_s + (K_1 \cup sK_2)$, *where* $s \geqslant 3$;
(6) $K_s + (2K_1 \cup (s-1)K_2)$, *where* $s \geqslant 3$.

As a consequence of Theorems 7 and 10 a number of recent results can be improved in the manner previously described. Three of them are given below.

Theorem 11 (Baner et al. [6]). *Let G be a 1-tough graph on n vertices with* $\sigma_3 \geqslant n \geqslant 3$. *Then* $c(G) \geqslant \min(n, n + \sigma_3/3 - \alpha)$.

Theorem 12 (Baner er al. [3]). *Let G be a 1-tough graph on n vertices with* $\sigma_3 \geqslant n \geqslant 3$. *Then* $c(G) \geqslant \min(n, 2NC2)$.

Theorem 13 (Broersma et al. [14]). . *Let G be a 1-tough graph on n vertices with* $\sigma_3 \geqslant n + r \geqslant n \geqslant 3$ *and* $n \geqslant 8k - 6r - 17$. *Then* $c(G) \geqslant \min(n, 2NC_k)$.

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