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Hamiltonian properties of graphs with large neighborhood unions

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Abstract

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Let G be a graph of order n, $\sigma_k = \min\{\sum_{i=1}^{k} d(v_i): \{v_1, \ldots, v_k\}$ is an independent set of vertices in $G\}$, $\mathrm{NC} = \min\{|N(u) \cup N(v)|: uv \notin E(G)\}$ and $\mathrm{NC2} = \min\{|N(u) \cup N(v)|: d(u, v) = 2\}$. Ore proved that G is hamiltonian if $\sigma_2 \geq n \geq 3$, while Faudree et al. proved that G is hamiltonian if G is 2-connected and $\mathrm{NC} \geq \frac{1}{3}(2n-1)$. It is shown that both results are generalized by a recent result of Bauer et al. Various other existing results in hamiltonian graph theory involving degree-sums or cardinalities of neighborhood unions are also compared in terms of generality. Furthermore, some new results are proved. In particular, it is shown that the bound $\frac{1}{3}(2n-1)$ on NC in the result of Faudree et al. can be lowered to $\frac{1}{3}(2n-3)$, which is best possible. Also, G is shown to have a cycle of length at least $\min\{n, 2(\mathrm{NC2})\}$ if G is 2-connected and $\sigma_3 \geq n+2$. A D_λ-cycle (D_λ-path) of G is a cycle (path) C such that every component of $G - V(C)$ has order smaller than λ. Sufficient conditions of Lindquester for the existence of Hamilton cycles and paths involving NC2 are extended to D_λ-cycles and D_λ-paths.

1. Introduction

We use [4] for terminology and notation not defined here and consider simple graphs only.

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Throughout, let G be a graph of order n. If G has a Hamilton cycle (a cycle containing every vertex of G), then G is called *hamiltonian*. The graph G is *traceable* if G has a Hamilton path (a path containing every vertex of G), and *Hamilton-connected* if every two vertices of G are connected by a Hamilton path. The number of vertices in a maximum independent set of G is denoted by $\alpha(G)$ and the set of vertices adjacent to a vertex v by $N(v)$. We denote by $\sigma_k(G)$ the minimum value of the degree-sum of any k pairwise non-adjacent vertices if $k \leqslant \alpha(G)$; if $k > \alpha(G)$, we set $\sigma_k(G) = k(n-1)$. Instead of $\sigma_1(G)$ we use the more common notation $\delta(G)$. If G is noncomplete, then $\mathrm{NC}(G)$ denotes

$$\min\{|N(u) \cup N(v)|: uv \notin E(G)\};$$

if G is complete, we set $\mathrm{NC}(G) = n-1$. If G has a noncomplete component, then $\mathrm{NC2}(G)$ denotes

$$\min\{|N(u) \cup N(v)|: d(u, v) = 2\},$$

where $d(u, v)$ is the distance between u and v; otherwise, $\mathrm{NC2}(G) = n-1$. If no ambiguity can arise we sometimes write α instead of $\alpha(G)$, σ_k instead of $\sigma_k(G)$, etc.

We mention two classical results in order of increasing generality.

Theorem 1 [7]. *If $\delta(G) \geqslant \frac{1}{2}n > 1$, then G is hamiltonian.*

Theorem 2 [17]. *If $\sigma_2(G) \geqslant n \geqslant 3$, then G is hamiltonian.*

In recent literature on hamiltonian graph theory many results appear in which certain vertex sets are required to have large neighborhood unions instead of large degree-sums. Two such results are the following.

Theorem 3 [9]. *If G is 2-connected and $\mathrm{NC}(G) \geqslant \frac{1}{3}(2n-1)$, then G is hamiltonian.*

Theorem 4 [8]. *If G is 2-connected and $\mathrm{NC}(G) \geqslant n - \delta(G)$, then G is hamiltonian.*

Theorems 3 and 2 are incomparable in the sense that neither theorem implies the other. Theorem 3 is not even comparable to Theorem 1. It is easily seen that Theorem 4 is more general than Theorem 1, but Theorems 4 and 2 are incomparable again.

Sufficient conditions in terms of neighborhood unions were also obtained for other hamiltonian properties.

Theorem 5 [9]. *If G is 3-connected and $\mathrm{NC}(G) \geqslant \frac{1}{3}(2n+1)$, then G is Hamilton-connected.*

Theorem 6 [9]. *If G is 2-connected and* $\mathrm{NC}(G) \geq \frac{1}{2}(n-1)$, *then G is traceable.*

The following three results are due to Lindquester.

Theorem 7 [16]. *If G is 2-connected and* $\mathrm{NC2}(G) \geq \frac{1}{3}(2n-1)$, *then G is hamiltonian.*

Theorem 8 [16]. *If G is 3-connected and* $\mathrm{NC2}(G) \geq \frac{2}{3}n$, *then G is Hamilton-connected.*

Theorem 9 [16]. *If G is 2-connected and* $\mathrm{NC2}(G) \geq \frac{1}{3}(2n-4)$, *then G is traceable.*

Since $\mathrm{NC2}(G) \geq \mathrm{NC}(G)$, Theorem 7 is more general than Theorem 3 and Theorem 8 is more general than Theorem 5. Theorem 9 does not imply Theorem 6, but it was conjectured in [16] that Theorems 6 and 9 admit the following improvement.

Conjecture 10 [16]. If G is 2-connected and $\mathrm{NC2}(G) \geq \frac{1}{2}(n-1)$, then G is traceable.

In addition to establishing some new results we also compare a number of existing results in terms of generality.

In Section 2 it is shown that a recent result in [2] is a common generalization of Theorems 2, 3 and 4 (and Theorem 1). Using another result in [2] it is further shown that the bound $\frac{1}{3}(2n-1)$ in Theorem 3 can be lowered to $\frac{1}{3}(2n-3)$, which is best possible for all $n \geq 5$. A new common generalization of Theorems 2, 3 and 4 is also established.

Section 3 is concerned with hamiltonian properties of $K_{1,3}$-free graphs, i.e., graphs containing no induced subgraph isomorphic to $K_{1,3}$. It is shown that several results and a conjecture in [10] are implied by results of Broersma [5] and Zhang [19].

In Section 4 the results in [16] are extended to so-called D_λ-cycles and D_λ-paths. As in [18], a cycle C of G is called a *D_λ-cycle* if every component of $G-V(C)$ has order smaller than λ. D_1-cycles are Hamilton cycles, while D_2-cycles are sometimes called *dominating cycles*. The definition of a *D_λ-path* is analogous to that of a D_λ-cycle. Section 4 also contains an extension of Theorem 6 to D_λ-paths.

2. Long cycles

We start by showing that Theorems 2, 3 and 4 are all generalized by the following recent result, where $c(G)$ denotes the length of a longest cycle in G.

Theorem 11 [2]. *If G is 2-connected and $\sigma_3(G) \geqslant n+2$, then $c(G) \geqslant \min\{n, n+\frac{1}{3}\sigma_3(G) - \alpha(G)\}$.*

A key lemma for our observations is the following.

Lemma 12. $\sigma_3(G) \geqslant 3\mathrm{NC}(G) - n + 3$ *for* $n \geqslant 3$.

Proof. We assume $\alpha \geqslant 3$. Let $\{v_1, v_2, v_3\}$ be an independent set of three vertices in G such that $\sum_{i=1}^{3} d(v_i) = \sigma_3$. Then

$$|N(v_1) \cup N(v_2)| \geqslant \mathrm{NC}, \qquad |N(v_1) \cup N(v_3)| \geqslant \mathrm{NC} \quad \text{and} \quad |N(v_2) \cup N(v_3)| \geqslant \mathrm{NC}.$$

Setting

$$d = |N(v_1) \cap N(v_2)| + |N(v_1) \cap N(v_3)| + |N(v_2) \cap N(v_3)|$$

and adding the three inequalities, we obtain

$$2\sigma_3 - d \geqslant 3\mathrm{NC}. \tag{1}$$

Set $t = |N(v_1) \cap N(v_2) \cap N(v_3)|$. Using (1) we have

$$n - 3 \geqslant |N(v_1) \cup N(v_2) \cup N(v_3)| = \sigma_3 - d + t \geqslant \tfrac{1}{2}(3\mathrm{NC} + d) - d + t,$$

whence

$$d \geqslant 3\mathrm{NC} + 2t - 2n + 6 \geqslant 3\mathrm{NC} - 2n + 6. \tag{2}$$

Combination of (1) and (2) completes the proof. □

We need two more lemmas. Their proofs are simple and are hence omitted.

Lemma 13. $\sigma_3(G) \geqslant \frac{3}{2}\sigma_2(G)$.

Lemma 14. $\alpha(G) \leqslant n - \mathrm{NC}(G)$.

Proposition 15. *If G satisfies the hypothesis of Theorem 2, then G is hamiltonian by Theorem 11.*

Proof. Assume $\sigma_2 \geqslant n \geqslant 3$. Then G is 2-connected and hence $\sigma_3 \geqslant \sigma_2 + 2 \geqslant n + 2$. By Lemmas 13 and 14 and the obvious fact that $\mathrm{NC} \geqslant \frac{1}{2}\sigma_2$,

$$n + \tfrac{1}{3}\sigma_3 - \alpha \geqslant n + \tfrac{1}{2}\sigma_2 - (n - \mathrm{NC}) = \tfrac{1}{2}\sigma_2 + \mathrm{NC} \geqslant \sigma_2 \geqslant n.$$

Hence G is hamiltonian by Theorem 11. □

Proposition 16. *If G satisfies the hypothesis of Theorem 3, then G is hamiltonian by Theorem 11.*

Proof. Assume G is 2-connected and $\mathrm{NC} \geqslant \frac{1}{3}(2n-1)$. By Lemma 12, $\sigma_3 \geqslant n+2$. Using Lemma 14 we obtain

$$n + \tfrac{1}{3}\sigma_3 - \alpha \geqslant n + \tfrac{1}{3}(n+2) - (n - \mathrm{NC}) \geqslant \tfrac{1}{3}(n+2) + \tfrac{1}{3}(2n-1) > n.$$

Hence G is hamiltonian by Theorem 11. □

Proposition 17. *If G satisfies the hypothesis of Theorem 4, then G is hamiltonian by Theorem 11.*

Proof. Assume G is 2-connected and $\mathrm{NC} \geqslant n - \delta$. If $\delta \leqslant \frac{1}{3}(n+1)$, then $\mathrm{NC} \geqslant \frac{1}{3}(2n-1)$ and we are done by Proposition 16. Hence assume $\delta \geqslant \frac{1}{3}(n+2)$. Then $\sigma_3 \geqslant 3\delta \geqslant n+2$ and, by Lemma 14,

$$n + \tfrac{1}{3}\sigma_3 - \alpha \geqslant n + \tfrac{1}{3}\sigma_3 - (n - \mathrm{NC}) \geqslant \tfrac{1}{3}\sigma_3 + n - \delta \geqslant n.$$

Hence G is hamiltonian by Theorem 11. □

We now show that the bound $\frac{1}{3}(2n-1)$ in Theorem 3 can be lowered to $\frac{1}{3}(2n-3)$ by using a result in [2] which is closely related to Theorem 11. The graph G is called *1-tough* if $\omega(G-S) \leqslant |S|$ for every subset S of $V(G)$ such that $\omega(G-S) > 1$, where $\omega(G-S)$ denotes the number of components of $G-S$.

Theorem 18 [2]. *If G is 1-tough and $\sigma_3(G) \geqslant n \geqslant 3$, then $c(G) \geqslant \min\{n, n + \frac{1}{3}\sigma_3(G) - \alpha(G)\}$.*

Corollary 19. *If G is 2-connected and $\mathrm{NC}(G) \geqslant \frac{1}{3}(2n-3)$, then G is hamiltonian.*

Proof. Let G be 2-connected with $\mathrm{NC} \geqslant \frac{1}{3}(2n-3)$. It is easily checked that G is hamiltonian if $n \leqslant 6$. We assume $n \geqslant 7$ and show that G is then hamiltonian by Theorem 18.

We first prove that G is 1-tough. Assuming the contrary, let S be a subset of $V(G)$ such that $\omega(G-S) \geqslant |S|+1$, G_1 a smallest component of $G-S$ and G_2 a smallest component of $G - (S \cup V(G_1))$. Then

$$|V(G_1)| + |V(G_2)| \leqslant 2\left(\frac{n-|S|}{|S|+1}\right) \quad \text{and} \quad 2 \leqslant |S| \leqslant \tfrac{1}{2}(n-1).$$

If $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, then

$$\mathrm{NC} \leqslant |N(v_1) \cup N(v_2)| \leqslant 2\left(\frac{n-|S|}{|S|+1}\right) - 2 + |S| \leqslant \tfrac{1}{3}(2n-4).$$

This contradiction shows that G is 1-tough.

By Lemmas 12 and 14,

$$n + \tfrac{1}{3}\sigma_3 - \alpha \geqslant n + \tfrac{1}{3}n - (n - \mathrm{NC}) \geqslant n - 1.$$

From Theorem 18 we conclude that G is hamiltonian unless

$$\sigma_3 = n, \qquad \mathrm{NC} = \tfrac{1}{3}(2n-3) \quad \text{and} \quad \alpha = \tfrac{1}{3}(n+3). \tag{3}$$

The proof is now completed by showing that G cannot satisfy (3). Suppose (3) holds. Let S be an independent set with $|S| = \alpha = \frac{1}{3}(n+3)$ and set $T = V(G) - S$. Let $\{v_1, v_2, v_3\}$ be an independent set with $\sum_{i=1}^{3} d(v_i) = \sigma_3 = n$. For any $u \in T$,

$$u \text{ is adjacent to all but at most one vertex of } S, \tag{4}$$

otherwise $\mathrm{NC} < |T| = \frac{1}{3}(2n-3)$. In particular, every vertex of T has degree at least $\frac{1}{3}n$. We now derive contradictions in four cases.

Case 1: $v_1, v_2, v_3 \in T$.

Then $d(v_1) = d(v_2) = d(v_3) = \frac{1}{3}n$, implying that $N(v_i) \subseteq S$ $(i = 1, 2, 3)$. Hence, since $n \geqslant 7$,

$$\mathrm{NC} \leqslant |N(v_1) \cup N(v_2)| \leqslant |S| = \tfrac{1}{3}(n+3) < \tfrac{1}{3}(2n-3),$$

a contradiction.

Case 2: $v_1, v_2 \in T$ *and* $v_3 \in S$.

Then $N(v_1) \cap N(v_2) \supseteq S - \{v_3\}$. Hence

$$\begin{aligned} d(v_1) + d(v_2) &= |N(v_1) \cup N(v_2)| + |N(v_1) \cap N(v_2)| \\ &\geqslant \tfrac{1}{3}(2n-3) + |S| - 1 = n - 1. \end{aligned}$$

It follows that $d(v_3) \leqslant 1$, contradicting the fact that G is 2-connected.

Case 3: $v_1 \in T$ *and* $v_2, v_3 \in S$.

Then v_1 is adjacent to at most $|S| - 2$ vertices of S, contradicting (4).

Case 4: $v_1, v_2, v_3 \in S$.

By (4), every vertex of T is adjacent to at least two vertices in $\{v_1, v_2, v_3\}$. Since $n \geqslant 7$, we obtain

$$\sum_{i=1}^{3} d(v_i) \geqslant 2\,|T| = \tfrac{2}{3}(2n-3) > n,$$

contradicting (3). □

For a real number r, let $\lceil r \rceil$ denote the smallest integer greater than or equal to r. Corollary 19 is best possible in the sense that for every $n \geqslant 5$ there exists a nonhamiltonian graph of order n with $\mathrm{NC} = \lceil \frac{1}{3}(2n-3) \rceil - 1$. For $n \geqslant 5$, define the graph G_n as the join of K_2 and the graph of order $n-2$ consisting of three disjoint complete subgraphs, the orders of which pairwise differ by at most one. G_n is nonhamiltonian and $\mathrm{NC}(G_n) = \lceil \frac{1}{3}(2n-3) \rceil - 1$.

Note that the graph G_n is not 1-tough. For $n \geqslant 7$ and $n \not\equiv 2 \pmod 3$ there also exist extremal graphs for Corollary 19 which are 1-tough. For $n \geqslant 7$, construct the graph H_n from the join of K_1 and the graph of order $n-1$ consisting of three disjoint complete subgraphs, the orders of which pairwise differ by at most one, by choosing a vertex in each of the three complete subgraphs and adding the edges of a triangle between the three vertices. The graph H_n is 1-tough and nonhamiltonian and, if $n \not\equiv 2 \pmod 3$, $\mathrm{NC}(H_n) = \lceil \frac{1}{3}(2n-3) \rceil - 1$. Another 1-tough extremal graph for Corollary 19 is the Petersen graph.

For $n \equiv 2 \pmod 3$ the graph H_n shows that the bound on NC in Corollary 19 cannot be lowered by two if G is 1-tough.

For future reference we describe another class of 1-tough non-hamiltonian graphs. For $n \geq 9$, construct the graph J_n of order n from three disjoint complete graphs A_1, A_2, A_3 with

$$||V(A_i)| - |V(A_j)|| \leq 1 \quad \text{for } 1 \leq i < j \leq 3$$

by choosing, for $i = 1, 2, 3$, two distinct vertices x_i and y_i in A_i and adding the edges x_1x_2, x_1x_3, x_2x_3, y_1y_2, y_1y_3, y_2y_3. If $n \equiv 0 \pmod 3$, then $\mathrm{NC2}(J_n) = \frac{1}{3}(2n-3)$. In Section 4 (Corollary 38) it is shown that in the hypothesis of Corollary 19 NC can be replaced by NC2 unless $n \equiv 0 \pmod 3$ and G is isomorphic to J_n. Note that for $n \equiv 0 \pmod 3$ the graph J_n is also an extremal graph for Corollary 19.

Corollary 19 is also contained in the following recent result, stated in [11] as a consequence of a more general result.

Theorem 20 [11]. *If G is 2-connected and, for every pair of non-adjacent vertices u and v, $3|N(u) \cup N(v)| + \max\{2, |N(u) \cap N(v)|\} \geq 2n - 1$, then G is hamiltonian.*

We have observed that Theorem 11 implies Theorems 2, 3 and 4. Using Lemmas 21 and 22 below we establish another common generalization of Theorems 2, 3 and 4. We need some additional notation. If C is a cycle of G, we denote by $\vec{C}$ the cycle C with a given orientation. If $u, v \in V(C)$, then $u\vec{C}v$ denotes the consecutive vertices on C from u to v in the direction specified by $\vec{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We will consider $u\vec{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. We use u^+ to denote the successor of u on $\vec{C}$ and u^- to denote its predecessor. If $A \subseteq V(C)$, then $A^+ = \{v^+ : v \in A\}$. The set A^- is analogously defined. In Section 4 we use similar notation for paths instead of cycles.

A central lemma in [2] is the following.

Lemma 21 [2]. *Assume $\delta(G) \geq 2$, $\sigma_3(G) \geq n$ and G contains a longest cycle $\vec{C}$ which is a D_2-cycle. If $u \in V(G) - V(C)$ and $A = N(u)$, then $(V(G) - V(C)) \cup A^+$ is an independent set.*

The first part of the next lemma is a result of Bondy [3]; the second part is implicit in the proof of [2, Theorem 10].

Lemma 22. *Assume G is 2-connected and $\sigma_3(G) \geq n + 2$. Then every longest cycle of G is a D_2-cycle. Moreover, G contains a longest cycle C such that $\max\{d(v) : v \in V(G) - V(C)\} \geq \frac{1}{3}(n+2)$.*

Suppose G satisfies the hypothesis of one of Theorems 2, 3 and 4. Then $\sigma_3(G) \geq n+2$, as observed in the proofs of Propositions 15, 16 and 17. Furthermore, it is easily seen that $\mathrm{NC2}(G) \geq \frac{1}{2}n$. Hence G is hamiltonian by the following result.

Theorem 23. *If G is 2-connected and $\sigma_3(G) \geq n+2$, then $c(G) \geq \min\{n, 2(\mathrm{NC2}(G))\}$.*

Proof. Assume G is 2-connected and $\sigma_3 \geq n+2$. We are done if G is hamiltonian. Otherwise, by Lemma 22, G contains a longest cycle $\vec{C}$ such that C is a D_2-cycle and $V(G)-V(C)$ contains a vertex u with $d(u) \geq \frac{1}{3}(n+2)$. Set $A = N(u)$. Clearly, $A \cap A^+ = \emptyset$. Since $d(u) > \frac{1}{3}|V(C)|$, C contains a vertex v such that $v^+, v^- \in A$. In particular, $d(u, v) = 2$. Set $B = N(u) \cup N(v)$. By Lemma 21, $B \subseteq V(C)$. We claim that $B \cap B^+ = \emptyset$. Assuming the contrary, let w be a vertex in $B \cap B^+$. It is clearly impossible that $w \in N(u)$ and $w^- \in N(u)$. If $w \in N(u)$ and $w^- \in N(v)$, then the cycle $v^+uw\vec{C}vw^-\overleftarrow{C}v^+$ is longer than C, a contradiction. If $w \in N(v)$ and $w^- \in N(u)$, then the cycle $v^-uw^-\overleftarrow{C}vw\vec{C}v^-$ contradicts the choice of C. Finally, if $w \in N(v)$ and $w^- \in N(v)$, then the cycle $v^-uv^+\vec{C}w^-vw\vec{C}v^-$ contradicts the choice of C. Hence, indeed, $B \cap B^+ = \emptyset$. It follows that $|V(C)| \geq 2|B| \geq 2(\mathrm{NC2})$. □

Theorem 23 is best possible in two different senses. We first note that if G is a complete bipartite graph, then $c(G) = 2(\mathrm{NC2}(G))$, so the conclusion of Theorem 23 cannot be strengthened. We next observe that the condition $\sigma_3(G) \geq n+2$ cannot be relaxed: the graph G_n (defined after Corollary 19) has $\sigma_3(G_n) = n+1$, while $c(G_n) = \mathrm{NC2}(G_n)+2$ if $n \equiv 2 \pmod 3$ and $c(G_n) = \mathrm{NC2}(G_n)+3$ otherwise. In this context we mention the following result.

Theorem 24 [13]. *Let G be 2-connected and noncomplete. Then $c(G) \geq \mathrm{NC2}(G)+2$. If $\mathrm{NC2}(G)$ is odd and $n > \mathrm{NC2}(G)+3$, then $c(G) \geq \mathrm{NC2}(G)+3$.*

The following result is closely related to Theorem 23 and will appear elsewhere.

Theorem 25 [6]. *If G is 2-connected and G contains a D_2-cycle, then $c(G) \geq \min\{n, 2\mathrm{NC}(G)\}$ unless G is the Petersen graph.*

A variation of Theorem 23 for 1-tough graphs is the following.

Theorem 26. *If G is 1-tough and $\sigma_3(G) \geq n \geq 3$, then $c(G) \geq \min\{n, 2(\mathrm{NC2}(G))\}$.*

The proof of Theorem 26 is omitted, since it is almost identical to the proof of Theorem 23. We note that Theorem 26, too, implies Corollary 19: if G is 2-connected and $\mathrm{NC}(G) \geq \frac{1}{3}(2n-3)$, then, as in the proof of Corollary 19, G is 1-tough and $\sigma_3(G) \geq n$, so that, by Theorem 26,

$$\begin{aligned} c(G) &\geq \min\{n, 2(\mathrm{NC2}(G))\} \geq \min\{n, 2\mathrm{NC}(G)\} \\ &\geq \min\{n, \lceil \tfrac{2}{3}(2n-3) \rceil\} \geq n \quad (n \geq 4). \end{aligned}$$

We conjecture that Theorem 26 admits the following improvement.

Conjecture 27. If G is 1-tough and $\sigma_3(G) \geq n \geq 3$, then $c(G) \geq \min\{n, 2(\mathrm{NC2}(G)) + 4\}$.

If true, Conjecture 27 would imply the following recent improvement of Jung's Theorem [15].

Theorem 28 [1]. *If G is 1-tough, $\sigma_3(G) \geq n \geq 3$ and, for all vertices x, y, $d(x, y) = 2$ implies $\max\{d(x), d(y)\} \geq s$, then $c(G) \geq \min\{n, 2s + 4\}$.*

3. Hamilton cycles and paths in $K_{1,3}$-free graphs

We state a result occurring in [10]. The graph G is *homogeneously traceable* if for every vertex v of G there exists a Hamilton path of G starting at v. Clearly, every hamiltonian graph is homogeneously traceable.

Theorem 29 [10]. *Let G be 3-connected and $K_{1,3}$-free. If $\mathrm{NC}(G) \geq \frac{1}{3}(2n-4)$, then G is homogeneously traceable.*

It is conjectured in [10] that, under the hypothesis of Theorem 29, G is in fact hamiltonian.

Conjecture 30 [10]. Let G be 3-connected and $K_{1,3}$-free. If $\mathrm{NC}(G) \geq \frac{1}{3}(2n-4)$, then G is hamiltonian.

The following result was independently obtained by Broersma and Zhang.

Theorem 31 [5, 19]. *Let G be 2-connected and $K_{1,3}$-free. If $\sigma_3(G) \geq n-2$, then G is hamiltonian.*

More generally, Zhang [19] proved that G is hamiltonian if G is k-connected and $K_{1,3}$-free with $\sigma_{k+1}(G) \geq n-k$ $(k \geq 2)$. The following consequence of Theorem 31 and Lemma 12 improves Theorem 29 and Conjecture 30.

Corollary 32. *Let G be 2-connected and $K_{1,3}$-free. If* $\mathrm{NC}(G) \geq \frac{1}{3}(2n-5)$, *then G is hamiltonian.*

The graph J_n is $K_{1,3}$-free and $\mathrm{NC}(J_n) = \lceil \frac{1}{3}(2n-5) \rceil - 1$, showing that Corollary 32 is best possible for all $n \geq 9$.

Another improvement of Conjecture 30 was recently obtained by Li and Virlouvet.

Theorem 33 [14]. *Let G be 3-connected and $K_{1,3}$-free. If* $\mathrm{NC}(G) > \frac{11}{21}(n-7)$, *then G is hamiltonian.*

Broersma proved an analogoue of Theorem 31 for traceable graphs.

Theorem 34 [5]. *Let G be connected and $K_{1,3}$-free. If $\sigma_3(G) \geq n-2$, then G is traceable.*

Combination with Lemma 12 yields the following.

Corollary 35. *Let G be connected and $K_{1,3}$-free. If* $\mathrm{NC}(G) \geq \frac{1}{3}(2n-5)$, *then G is traceable.*

A weaker version of Corollary 35, with $\frac{1}{3}(2n-5)$ replaced by $\frac{1}{3}(2n-3)$, occurs in [10]. The connected nontraceable graph J'_n obtained from J_n by deleting the edges y_1y_2, y_1y_3, y_2y_3 is $K_{1,3}$-free and $\mathrm{NC}(J'_n) = \lceil \frac{1}{3}(2n-5) \rceil - 1$, showing that Corollary 35 is best possible.

4. D_λ-cycles and D_λ-paths

In order to extend Theorems 6–9 to results on D_λ-cycles and D_λ-paths we introduce some additional terminology and notation. Let H, H_1, H_2 be subgraphs of G and t, λ positive integers. By $N(H)$ we denote the set of vertices in $V(G) - V(H)$ that are adjacent to at least one vertex of H. The vertices in $N(H)$ are called *neighbors* of H. The *distance* $d(H_1, H_2)$ between H_1 and H_2 is the length of a shortest path in G starting at a vertex of H_1 and ending at a vertex of H_2. We call H_1 and H_2 *remote* if $d(H_1, H_2) \geq 2$. If H is connected and u and v are neighbors of H, then uHv denotes a (u, v)-path of length at least 2 with all internal vertices in H; uH denotes a nontrivial path starting at u such that all other vertices of the path are in H; Hu is analogously defined. By $\omega_t(H)$ we denote the number of components of H with at least t vertices; in particular, $\omega_1(H) = \omega(H)$. If G contains two remote connected subgraphs of order λ, then $\mathrm{NC}_\lambda(G)$ denotes $\min\{|N(H_1) \cup N(H_2)| : H_1$ and H_2 are remote connected subgraphs of order $\lambda\}$; otherwise we set $\mathrm{NC}_\lambda(G) = n - 2\lambda + 1$. If G contains two connected subgraphs of order λ at distance 2, then $\mathrm{NC2}_\lambda(G)$ denotes $\min\{|N(H_1)\cup N(H_2)|: H_1$ and H_2 are connected subgrapns of order λ with $d(H_1, H_2)=2\}$; otherwise, $\mathrm{NC2}_\lambda(G)=n-2\lambda+1$. In particular, $\mathrm{NC}_1(G)=\mathrm{NC}(G)$ and $\mathrm{NC2}_1(G)=\mathrm{NC2}(G)$. The following lemma is easily established; we omit its proof.

Lemma 36. *If* $t\geqslant\lambda$, *then* $\mathrm{NC}_t(G)\geqslant \mathrm{NC}_\lambda(G)-2(t-\lambda)$ *and* $\mathrm{NC2}_t(G)\geqslant \mathrm{NC2}_\lambda(G)-2(t-\lambda)$.

If $\vec{H}$ is an oriented cycle or path and $v\in V(H)$, then we call H_1 an *$(\vec{H}, v, t)$-subgraph* if each of the following requirements holds:

(i) H_1 is connected and has order t,

(ii) $\emptyset\neq V(H_1)\cap V(H)=v\vec{H}w$ for some vertex $w\in V(H)$,

(iii) if H_2 satisfies (i) and (ii), then $V(H_1)\cap V(H)\subseteq V(H_2)\cap V(H)$.

An *$(\overleftarrow{H}, v, t)$-subgraph* is similarly defined. (In (ii), replace $v\vec{H}w$ by $v\overleftarrow{H}w$). We are now ready to state and prove the following result.

Theorem 37. *If G is 2-connected and* $\mathrm{NC2}_\lambda(G)\geqslant\frac{1}{3}(2n+3)-2\lambda$, *then G contains a D_λ-cycle unless* $n\equiv 0 \pmod 3$, $n\geqslant 3\lambda+6$ *and G is a spanning subgraph of* J_n.

Proof. Assume G satisfies the conditions of the theorem, but G contains no D_λ-cycle. Set $t+1=\min\{i: G$ has a D_i-cycle$\}$, so that $t\geqslant\lambda$. Let $\vec{C}$ be a D_{t+1}-cycle of G for which $\omega_t(G-V(C))$ is minimum. Since G has no D_t-cycle, $G-V(C)$ has a component H_0 of order t. Let $v_1,\ldots,v_k$ be the neighbors of H_0, occurring on $\vec{C}$ in the order of their indices. Since G is 2-connected, $k\geqslant 2$. As in the proof of [18, Theorem 2] there exists, for $i=1,\ldots,k$, a $(\vec{C}, v_i^+, t)$-subgraph H_i such that $H_0, H_1,\ldots,H_k$ are pairwise remote. Let u_i be the first vertex on $v_i^+\vec{C}v_{i+1}$ such that $u_i\notin V(H_i)$ ($i=1,\ldots,k$; indices mod k). Set

$$U=V(G)-(V(H_0)\cup V(H_1)\cup V(H_k))\quad\text{and}\quad W=U-\{u_1,v_1\}.$$

Define the function $f: W\to U$ by

$$f(v)=\begin{cases} v^- & \text{if } v\in u_1^+\vec{C}v_k,\\ v^+ & \text{if } v\in u_k\vec{C}v_1^-,\\ v & \text{if } v\notin V(C).\end{cases}$$

We show that

$$\text{if } v\in(N(H_1)\cap W)-\{v_1^-\}, \text{ then } f(v)\notin N(H_0)\cup N(H_k). \tag{5}$$

Assuming the contrary to (5), let v be a vertex in $(N(H_1)\cap W)-\{v_1^-\}$ such that $f(v)\in N(H_0)\cup N(H_k)$. If $v\in u_1^+\vec{C}v_k$ and $f(v)\in N(H_0)$, then $v^-=v_i$ for some $i\in\{2,\ldots,k-1\}$ and we obtain the contradiction that H_1 and H_i are not remote. If $v\in u_1^+\vec{C}v_k$ and $f(v)\in N(H_k)$, then G contains the cycle

$$C'=v_1H_0v_k\overleftarrow{C}vH_1u_1\vec{C}v^-H_ku_k\vec{C}v_1.$$

By the way H_1 and H_k were chosen and the fact that H_1 and H_k are remote, we have $\omega_t(G - V(C')) < \omega_t(G - V(C))$, contradicting the choice of C. If $v \in u_k\vec{C}v_1^{--}$, then, since $u_k\vec{C}v_1^-$ contains no neighbors of $H_0, f(v) \in N(H_k)$. But then the cycle

$$v_1 H_0 v_k \overleftarrow{C} u_1 H_1 v \overleftarrow{C} u_k H_k v^+ \vec{C} v_1$$

contradicts the choice of C. If $v \notin V(C)$, then clearly $f(v) \notin N(H_0)$, whence $f(v) \in N(H_k)$; but then the cycle $v_1 H_0 v_k \overleftarrow{C} u_1 H_1 v H_k u_k \vec{C} v_1$ contradicts the choice of C. Thus (5) holds.

Note that $d(H_0, H_k) = 2$ and f is an injection. Combining these facts with (5) and Lemma 36 we conclude that

$$\begin{aligned}\tfrac{1}{3}(2n+3) - 2t &\leq |N(H_0) \cup N(H_k)| \\ &\leq n - |V(H_0) \cup V(H_1) \cup V(H_k)| - (|N(H_1) \cap W| - \varepsilon) \\ &= n - 3t - (|N(H_1)| - 2 - \varepsilon), \end{aligned} \tag{6}$$

where $\varepsilon = 0$ if $v_1^- \notin N(H_1) \cap W$ and $\varepsilon = 1$ if $v_1^- \in N(H_1) \cap W$. Hence

$$|N(H_1)| \leq \tfrac{1}{3}(n+3) - t + \varepsilon. \tag{7}$$

Since $|N(H_0) \cup N(H_1)| \geq \frac{1}{3}(2n+3) - 2t$ and $|N(H_0) \cap N(H_1)| \geq 1$, we obtain

$$k = |N(H_0)| \geq \tfrac{1}{3}(2n+3) - 2t - (\tfrac{1}{3}(n+3) - t + \varepsilon) + 1 = \tfrac{1}{3}(n+3) - t - \varepsilon. \tag{8}$$

We now distinguish two cases, the first of which will turn out to yield a contradiction.

Case 1: *For some* $i \in \{1, \ldots, k\}$, $v_i^- \notin N(H_i)$.

Assume without loss of generality that $v_1^- \notin N(H_1)$. Then $\varepsilon = 0$ in (6), (7) and (8). The fact that $H_0, H_1, \ldots, H_k$ are pairwise remote and (8) imply

$$\begin{aligned}|N(H_0) \cup N(H_1)| &\leq n - (k+1)t \leq n - (\tfrac{1}{3}(n+3) - t + 1)t \\ &= \tfrac{1}{3}(2n+3) - 2t + (t-1)(t - \tfrac{1}{3}(n-3)). \end{aligned} \tag{9}$$

Since G is 2-connected, we have $|N(H_1)| \geq 2$. Hence by (7), $t \leq \frac{1}{3}(n-3)$. From (9) it now follows that $t = 1$ or $t = \frac{1}{3}(n-3)$.

Case 1.1: $t = 1$.

Let u be the vertex of H_0 and set $R = V(G) - V(C)$, $S = \{v \in V(C): v^-, v^+ \in N(u)\}$. Using (8) we have

$$n \geq 2|S| + 3(|N(u)| - |S|) + |R| = 3|N(u)| + |R| - |S| \geq n + |R| - |S|,$$

implying that $|R| \leq |S|$. Since every vertex in R can be adjacent to at most one vertex in S while u is adjacent to no vertex in S, there exists a vertex v in S such that $N(v) \cap R = \emptyset$. Hence, if we set $B = N(u) \cup N(v)$, we have $B \subseteq V(C)$. Arguing as in the proof of Theorem 23 we obtain $B \cap B^+ = \emptyset$. But then

$$n - 1 \geq |V(C)| \geq 2|B| \geq 2\text{NC}2 \geq \tfrac{2}{3}(2n-3),$$

whence $n \leq 3$, a contradiction.

Case 1.2: $t = \frac{1}{3}(n-3)$.

Since (9) holds with equality, (8) also holds with equality. In particular, $|N(H_0) \cap N(H_1)| = 1$, implying that

$$V(G) = V(H_0) \cup V(H_1) \cup V(H_2) \cup \{v_1, u_1, v_2\}.$$

The vertex u_1 is not in $N(H_2)$, otherwise the cycle $v_1 H_0 v_2 H_2 u_1 \overleftarrow{C} v_1$ contradicts the choice of C. It follows that

$$|N(H_0) \cup N(H_2)| = 2 < 3 = \tfrac{1}{3}(2n+3) - 2t,$$

a contradiction.

Case 2: *For all* $i \in \{1, \ldots, k\}$, $v_i^- \in N(H_i)$.

Since H_i and H_{i+1} are remote and $v_{i+1}^- \in N(H_{i+1})$, we have $u_i \neq v_{i+1}$ ($i = 1, \ldots, k$; indices mod k). Also, $u_i \neq v_{i+1}^-$, otherwise the cycle

$$v_i H_0 v_{i+1} u_i H_{i+1} u_{i+1} \vec{C} v_i^- H_i v_i$$

would contradict the choice of C. It follows that

$$n \geqslant t + k(t+3). \tag{10}$$

Combining (10) and (8), now with $\varepsilon = 1$, yields

$$n \geqslant t + (\tfrac{1}{3}n - t)(t+3),$$

whence $n \leqslant 3t+6$. On the other hand, (10) implies $n \geqslant 3t+6$, since $k \geqslant 2$. We conclude that $n = 3t+6$, $k = 2$ and

$$V(G) = V(H_0) \cup V(H_1) \cup V(H_2) \cup \{v_1, u_1, u_1^+, v_2, u_2, u_2^+\}.$$

To prove that G is a spanning subgraph of J_n, we first observe that $N(u_i) \subseteq V(H_i) \cup \{u_i^+\}$ ($i = 1, 2$). Assuming the contrary, one easily finds a cycle that contradicts the choice of C.

We next show that v_2^+ is the only vertex of H_2 adjacent to u_1^+. Assuming the contrary, let x be a vertex of H_2 with $u_1^+ x \in E(G)$ and $x \neq v_2^+$ and let $\vec{P}$ be a (v_2^+, x)-path in H_2. Consider the cycle

$$C' = v_1 H_0 v_2 v_2^+ \vec{P} x u_1^+ u_1 H_1 u_2^+ v_1$$

and let H be the component of $G - V(C')$ that contains u_2. Then

$$V(H) \subseteq \{u_2\} \cup (V(H_2) - \{x, v_2^+\}),$$

so $|V(H)| < t$. Since the other two components of $G - V(C')$ also have fewer than t vertices, C' contradicts the choice of C. Hence, indeed, v_2^+ is the only vertex of H_2 adjacent to u_1^+. Similarly, v_1^+ is the only vertex of H_1 adjacent to u_2^+.

Finally, $u_1^+ u_2^+ \notin E(G)$, $u_1^+ v_1 \notin E(G)$, $u_2^+ v_2 \notin E(G)$, $v_1 \notin N(H_1 - v_1^+)$, $v_2 \notin N(H_2 - v_2^+)$, $v_1 \notin N(H_2)$ and $v_2 \notin N(H_1)$ by similar arguments. Thus G is a spanning subgraph of J_n, where

$$V(A_1) = V(H_0) \cup \{v_1, v_2\}, \quad V(A_2) = V(H_1) \cup \{u_1, u_1^+\} \text{ and}$$
$$V(A_3) = V(H_2) \cup \{u_2, u_2^+\}. \quad \square$$

If $n \geq 3\lambda + 2$, then the graph G_n has no D_λ-cycle while $\mathrm{NC2}_\lambda(G_n) = \lceil \frac{1}{3}(2n+3) \rceil - 2\lambda - 1$. Hence Theorem 37 is best possible.

Substituting $\lambda = 1$ and noting that $\mathrm{NC2}(H) < \frac{1}{3}(2n-3)$ for every proper spanning subgraph H of J_n, we obtain the following improvement of Theorem 7.

Corollary 38. *If G is 2-connected and $\mathrm{NC2}(G) \geq \frac{1}{3}(2n-3)$, then G is hamiltonian unless $n \equiv 0 \pmod 3$, $n \geq 9$ and G is isomorphic to J_n.*

Note that Corollary 19, too, is implied by Corollary 38.

Theorem 37 also improves the following extension of Theorem 3, which is a special case of a more general theorem for k-connected graphs ($k \geq 2$).

Corollary 39 [12]. *If G is 2-connected and $\mathrm{NC}_\lambda(G) \geq \frac{1}{3}(2n+5) - 2\lambda$, then G contains a D_λ-cycle.*

Theorems 40 and 41 below extend Theorems 8 and 9, respectively. The proofs are similar to the proof of Theorem 37 and are hence omitted.

Theorem 40. *If G is 3-connected and $\mathrm{NC2}_\lambda(G) \geq \frac{2}{3}n + 2 - 2\lambda$, then every two vertices are connected by a D_λ-path.*

For $\lambda = 1$ we obtain Theorem 8. For $n \geq 6$ define the 3-connected graph L_n as the join of K_3 and the graph of order $n-3$ consisting of three disjoint complete graphs, the orders of which pairwise differ by at most one. If $n \equiv 0 \pmod 3$ and $n \geq 3\lambda + 3$, then $\mathrm{NC2}_\lambda(L_n) = \frac{2}{3}n + 1 - 2\lambda$ while L_n contains two vertices which are not connected by a D_λ-path. Hence Theorem 40 is best possible for $n \equiv 0 \pmod 3$. For $n \not\equiv 0 \pmod 3$ the graph L_n shows that the bound on $\mathrm{NC2}_\lambda$ cannot be lowered by two.

Theorem 41. *If G is 2-connected and $\mathrm{NC2}_\lambda(G) \geq \frac{1}{3}(2n+2) - 2\lambda$, then G contains a D_λ-path.*

For $\lambda = 1$ we obtain Theorem 9. We believe that Theorem 41 admits the following improvement, extending Conjecture 10.

Conjecture 42. If G is 2-connected and $\mathrm{NC2}_\lambda(G) \geq \frac{1}{2}(n+3) - 2\lambda$, then G contains a D_λ-path.

We provide some evidence for Conjecture 42 by extending Theorem 6.

Theorem 43. *If G is 2-connected and $\mathrm{NC}_\lambda(G) \geq \frac{1}{2}(n+3) - 2\lambda$, then G contains a D_λ-path.*

Proof. Assume G satisfies the conditions of the theorem, but G contains no D_λ-path. Set $t+1=\min\{i\colon G$ has a D_i-path$\}$, so that $t \geq \lambda$. Let $\vec{P}$ be a D_{t+1}-path of G for which $\omega_t(G-V(P))$ is minimum and H_0 be a component of $G-V(P)$ of order t. Set $k=|N(H_0)|$. Then $k \geq 2$. Let $v_2, \ldots, v_{k+1}$ be the neighbors of H_0, occurring on $\vec{P}$ in the order of their indices, let v_1 be the first and w the last vertex of $\vec{P}$. A straightforward variation of the argument in the proof of Theorem 37 shows that

> there exists a $(\vec{P}, v_1, t)$-subgraph H_1, a $(\overleftarrow{P}, w, t)$-subgraph H_{k+1} and, for $i=2, \ldots, k$, a $(\vec{P}, v_i^+, t)$-subgraph H_i such that $H_0, H_1, \ldots, H_{k+1}$ are pairwise remote.

Let u_i be the first vertex on $v_i^+\vec{P}v_{i+1}$ such that $u_i \notin V(H_i)$ for $1 \leq i \leq k$, and let u_{k+1} be the last vertex on $v_{k+1}\vec{P}w$ such that $u_{k+1} \notin V(H_{k+1})$. Set

$$U=V(G)-(V(H_0)\cup V(H_1)\cup V(H_2)\cup V(H_{k+1})) \quad \text{and} \quad W=U-\{v_2, u_{k+1}\}.$$

Define the function $f\colon W \to U$ by

$$f(v)=\begin{cases} v^+ & \text{if } v \in V(P), \\ v & \text{if } v \notin V(P). \end{cases}$$

We show that

$$\text{if } v \in (N(H_0)\cup N(H_{k+1}))\cap W, \text{ then } f(v) \notin N(H_0)\cup N(H_2). \tag{11}$$

Assuming the contrary to (11), let v be a vertex in $(N(H_0)\cup N(H_{k+1}))\cap W$ such that $f(v) \in N(H_0)\cup N(H_2)$.

First suppose $v \in u_1\vec{P}v_2^-$. Then $v \notin N(H_0)$, so $v \in N(H_{k+1})$. Also, $v \neq v_2^-$, otherwise the path $v_1\vec{P}v_2^-H_{k+1}u_{k+1}\overleftarrow{P}v_2H_0$ would contradict the choice of P. It follows that $f(v) \notin N(H_0)$, so $f(v) \in N(H_2)$. But then the path

$$v_1\vec{P}vH_{k+1}u_{k+1}\overleftarrow{P}u_2H_2v^+\vec{P}v_2H_0$$

contradicts the choice of P.

Next suppose $v \in u_2\vec{P}u_{k+1}^-$. If $v \in N(H_0)$, then $f(v) \notin N(H_0)$, so $f(v) \in N(H_2)$. But then the path $v_1\vec{P}v_2H_0v\overleftarrow{P}u_2H_2v^+\vec{P}w$ contradicts the choice of P. If $v \in N(H_{k+1})$ and $f(v) \in N(H_0)$, then the path $v_1\vec{P}vH_{k+1}u_{k+1}\overleftarrow{P}v^+H_0$ contradicts the choice of P. If $v \in N(H_{k+1})$ and $f(v) \in N(H_2)$, then the path

$$v_1\vec{P}v_2H_0v_{k+1}\overleftarrow{P}v^+H_2u_2\vec{P}vH_{k+1}u_{k+1}\overleftarrow{P}v_{k+1}^+$$

(where $u_{k+1}\overleftarrow{P}v_{k+1}^+$ is understood to be empty if $u_{k+1}=v_{k+1}$) contradicts the choice of P.

Finally suppose $v \notin V(P)$. Clearly $v \notin N(H_0)$, so $v \in N(H_{k+1})$ and $f(v)=v \in N(H_2)$. But then the path

$$v_1\vec{P}v_2H_0v_{k+1}\overleftarrow{P}u_2H_2vH_{k+1}u_{k+1}\overleftarrow{P}v_{k+1}^+$$

contradicts the choice of P. Hence (11) holds.

Using (11), Lemma 36 and the fact that f is an injection we conclude that

$$\begin{aligned}\tfrac{1}{2}(n+3)-2t &\leqslant |N(H_0)\cup N(H_2)| \\ &\leqslant n-|V(H_0)\cup V(H_1)\cup V(H_2)\cup V(H_{k+1})| \\ &\quad -|(N(H_0)\cup N(H_{k+1}))\cap W| \\ &= n-4t-|N(H_0)\cup N(H_{k+1})|+2 \\ &\leqslant n-4t+2-(\tfrac{1}{2}(n+3)-2t)=\tfrac{1}{2}(n+1)-2t.\end{aligned}$$

This contradiction completes the proof. □

For $n \geqslant 6$ define the 2-connected graph M_n as the join of K_2 and the graph of order $n-2$ consisting of four disjoint complete graphs, the orders of which pairwise differ by at most one. If $n \geqslant 4\lambda+2$ and $n \not\equiv 0 \pmod 4$, then $\mathrm{NC}_\lambda(M_n)=\lceil\frac{1}{2}(n+3)\rceil-2\lambda-1$ while M_n contains no D_λ-path. If $n \geqslant 6$ and n is even, then $\mathrm{NC}(K_{n/2-1,n/2+1})=\lceil\frac{1}{2}(n+3)\rceil-2-1$ while $K_{n/2-1,n/2+1}$ contains no D_1-path. Hence Theorem 43 is best possible if either $\lambda \geqslant 2$ and $n \not\equiv 0 \pmod 4$ or $\lambda=1$. The graph M_n shows that the bound on NC_λ cannot be lowered by two if $\lambda \geqslant 2$ and $n \equiv 0 \pmod 4$.

References

[1] D. Bauer, H.J. Broersma and H.J. Veldman, On generalizing a theorem of Jung, 1988, preprint.

[2] D. Bauer, A. Morgana, E.F. Schmeichel and H.J. Veldman, Long cycles in graphs with large degree sums, Discrete Math. 79 (1989/90) 59–70.

[3] J.A. Bondy, Longest paths and cycles in graphs of high degree, Research Report CORR80-16, Dept. of Combinatorics and Optimization, Univ. of Waterloo, Waterloo, Ont., Canada, 1980.

[4] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, London, 1976) and (Elsevier, Amsterdam, 1976).

[5] H.J. Broersma, Hamilton cycles in graphs and related topics, 1988, thesis.

[6] H.J. Broersma and H.J. Veldman, Long dominating cycles and paths in graphs with large neighborhood unions, 1989, preprint.

[7] G.A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. 2 (1952) 69–81.

[8] R.J. Faudree, R.J. Gould, M.S. Jacobson and L. Lesniak, Neighborhood unions and highly hamiltonian graphs, J. Graph Theory 15 (1991) 29–38.

[9] R.J. Faudree, R.J. Gould, M.S. Jacobson and R.H. Schelp, Neighborhood unions and hamiltonian properties in graphs, J. Combin. Theory Ser. B 47 (1989) 1–9.

[10] R.J. Faudree, R.J. Gould and T.E. Lindquester, Hamiltonian properties and adjacency conditions in $K(1,3)$-free graphs, in: Proc. 6th Internat. Conf. Theory and Applications of Graphs, Kalamazoo, MI (1988).

[11] E. Flandrin, H.A. Jung and H. Li, Hamiltonism, degree sum and neighborhood intersections, Discrete Math. 90 (1991) 41–52.

[12] P. Fraisse, D_λ-cycles and their applications for hamiltonian graphs, 1987, preprint.

[13] R.J. Gould and T.E. Lindquester, Some extremal problems involving adjacency conditions for vertices at distance two, in: V.R. Kulli, ed., Recent Studies in Graph Theory (Vishwa, Gulbarga, India, 1989) 140–148.

[14] H. Li and C. Virlouvet, Neighborhood conditions for claw-free hamiltonian graphs, Ars Combin. 29A (1990) 109–116.

[15] H.A. Jung, On maximal circuits in finite graphs, Ann. Discrete Math. 3 (North-Holland, Amsterdam, 1978) 129–144.
[16] T.E. Lindquester, The effects of distance and neighborhood union conditions on hamiltonian properties in graphs, J. Graph Theory 13 (1989) 335–352.
[17] O. Ore, Note on Hamilton circuits, Amer. Math. Monthly 67 (1960) 55.
[18] H.J. Veldman, Existence of D_λ-cycles and D_λ-paths, Discrete Math. 44 (1983) 309–316.
[19] C.-Q. Zhang, Hamilton cycles in claw-free graphs, J. Graph Theory 12 (1988) 209–216.