RECURSIVE ALGORITHMS FOR THE ELIMINATION OF REDUNDANT PATHS IN SPATIAL LAG OPERATORS*

Hans J. Blommestein

OECD, 2 rue André-Pascal 75755 Paris Cedex 16, France and University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

Nick A. M. Koper

Philips Semiconductors, Innovation and Engineering Group, P.O. Box 10, 9500 AA Stadskanaal, The Netherlands

ABSTRACT. Recursive algorithms for the elimination of redundant paths in spatial lag operators are introduced. It is shown that these algorithms have superior computational properties in comparison with the cumbersome procedure proposed by Ross and Harary (1952). A rigorous definition of spatial lag operators is given, while a number of mathematical results and properties are derived. Theoretical and empirical results regarding the performance of the proposed algorithms are presented.

1. INTRODUCTION

In this article we study the problem of redundant paths in spatial lag operators. Recursive algorithms for the elimination of redundant paths are presented. It is shown that these algorithms have superior computational properties in comparison with the cumbersome, non-recursive procedure proposed by Ross and Harary (1952).

Basic mathematical concepts are introduced in Section 2. The adverse consequences of redundant paths in spatial lag operators for the estimation of spatial econometric models are illustrated in Section 3. Next, we introduce in Section 4 recursive algorithms for the elimination of redundant paths. In Section 5 we study the complexity and performance of the various algorithms in both theoretical and empirical terms. Conclusions are summarized in the final Section 6.

The appendices contain details on Ross and Harary's algorithm (Appendix A) and Blommestein and Koper's algorithms (Appendices B, C and D).

2. BASIC MATHEMATICAL CONCEPTS

In spatial economics, international economics, geography, biometrics and other fields where the element of space plays an important role, the spatial lag operator L_S is a useful tool for the quantitative analysis of a wide range of problems.

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Definition 1 (Spatial Lag Operator). Let $\mathcal{V}_R$ be a R -dimensional vector space over a field $\mathcal{F}$, $\mathbf{K} = \{x_1, x_2, \dots, x_R\}$ any basis of $\mathcal{V}_R$, and $\mathbf{L}_S$ a linear transformation of $\mathcal{V}_R$ into itself, i.e., $\mathbf{L}_S: \mathcal{V}_R \rightarrow \mathcal{V}_R$. Since every vector is a linear combination of the basis vectors $\{x_r | r = 1, 2, \dots, R\}$ we have

$$(2.1) \quad \mathbf{L}_S \mathbf{x}_r = \sum_{r'=1}^R w_{rr'} \mathbf{x}_{r'} \quad (\text{for } r' \in \{1, 2, \dots, R\})$$

Once the ordered basis $\mathbf{K}$ is fixed, any linear transformation is completely and uniquely determined by the set with scalars $\{w_{rr'} | r, r' = 1, 2, \dots, R\}$ in $\mathcal{F}$. The set $\{w_{rr'}\}$ of R^2 scalars defines the spatial lag operator $\mathbf{L}_S$

$$(2.2) \quad \mathbf{L}_S = \begin{bmatrix} w_{11} & w_{12} & \cdot & \cdot & \cdot & w_{1R} \\ w_{21} & w_{22} & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ w_{R1} & w_{R2} & \cdot & \cdot & \cdot & w_{RR} \end{bmatrix}$$

Definition 1 is very general. In most applications it will be necessary to impose additional restrictions on $\mathbf{L}_S$. These restrictions may differ across applications.

(a) The real $(R \times R)$ matrix $\mathbf{L}_S = [w_{rr'}]$ is non-negative, i.e., $w_{rr'} \geq 0, \forall r \neq r' \in \{1, 2, \dots, R\}$; it is further assumed that $w_{rr} = 0, \forall r \in \{1, 2, \dots, R\}$. In some cases we require that the spatial lag operator is row stochastic. For example, in spatial econometric applications (see Example 2 in Section 3). An $(R \times R)$ matrix $\mathbf{L}_S$ is row stochastic when $\mathbf{L}_S \geq 0$ and

$$(2.3) \quad \sum_{r'=1}^R w_{rr'} = 1 \quad (\forall r \in \{1, 2, \dots, R\})$$

(b) the (non-)boolean $(R \times R)$ matrix $\mathbf{L}_S$ has one of the three general mathematical properties below

- (1) $\mathbf{L}_S$ is a real $(R \times R)$ symmetric matrix. That is: $\mathbf{L}_S = \mathbf{L}_S^T$;
- (2) $\mathbf{L}_S$ is a real $(R \times R)$ asymmetric matrix, which is non-triangular;
- (3) $\mathbf{L}_S$ is a real $(R \times R)$ upper or lower triangular matrix. That is: $w_{rr'} = 0, \forall r > r' \in \{1, 2, \dots, R\}$ for upper triangular and $w_{rr'} = 0, \forall r < r' \in \{1, 2, \dots, R\}$ for lower triangular matrices.

Remark 1. The temporal lag operator $\mathbf{L}$ (see Dhrymes, 1981) can be written as a square matrix in triangular form, that is, as a matrix containing zeros apart from ones on the first upper diagonal.

The spatial lag operator from Definition 1 can be used in the construction of many different types of space(-time) models (for an overview see Blommestein, 1985b). A common characteristic of these models is that they constitute a multivariate system which exhibits systematic dependence between the observa-

tions at each spatial unit (regions, towns, countries, etc.) and the observations at neighboring spatial units (= spatial autocorrelation). The operator $\mathbf{L}_S$ is a very parsimonious way of expressing spatial order in this multivariate system.

Let $\{w_{rr'}^p\}$ be the elements of $\mathbf{L}_S^p$ ($\mathbf{L}_S$ raised to the power p). Then, $\{w_{rr'}^p\}$ are nonzero if the spatial units r and r' are p th order neighbors. The exact specification of the form of $\{w_{rr'}^p\}$ is dependent on the spatial (economic) phenomena to be studied. A very general way of interpreting the information conveyed by $\mathbf{L}_S^p$ is to construct the incidence matrix $\mathbf{Q}$, corresponding to $\mathbf{L}_S$. Next, consider $\mathbf{Q}$ as a network representation of a spatial system consisting of R spatial units. The non-zero entries of the $(R \times R)$ matrix $\mathbf{Q}$ correspond to the vertices, or nodes, of that network.

Proposition. If $\mathbf{Q}$ is raised to a positive power ($p \geq 2$), then the off-diagonal elements ($r \neq r'$) of $\mathbf{Q}^p$ denote the total number of connecting paths (or, equivalently, chains) between each pair of vertices $(x_r, x_{r'})$.

Proof. Note first that each element of the network $\mathbf{Q}$, $q_{rr'}$ is either zero (i.e., absence of node) or one (i.e., node is present). Raise $\mathbf{Q}$ to the p th power ($p \geq 2$). The typical elements $q_{rr'}^p$ from $\mathbf{Q}^p$ can be written as

$$q_{rr'}^p = \sum_{k=1}^R q_{rk} q_{kr'}^{p-1}$$

This recursively defined summation gives then the total number of p -step connecting paths for each pair of vertices. ■

Remark 2. The entries q_{rr}^p ($r \in \{1, 2, \dots, R\}$) represent the number of p -step connecting paths of vertex x_r with itself.

Now we are in a position to state the two central definitions of this article.

Definition 2a. Let $r, r_1, r_2, \dots, r'$ be arbitrary indices from the index set $\{1, 2, \dots, R\}$ of matrix $\mathbf{L}_S$. A sequence $(r, r_1, r_2, \dots, r_{p-1}, r')$, for $p \geq 1$ and $r_0 = r$, from the index set $\{1, 2, \dots, R\}$, is said to form a p -step path (or, equivalently, a chain of length p) between the vertices or ordered pair $(x_r, x_{r'})$ if $w_{rr_1} \cdot w_{r_1 r_2} \cdot \dots \cdot w_{r_{p-2} r_{p-1}} \cdot w_{r_{p-1} r'} > 0$. ■

Definition 2b. A redundant path or chain (or, equivalently, circular route) is a path in which at least one index number repeats in the sequence of indices. ■

Remark 3. Raising (real) symmetric and asymmetric non-triangular $(R \times R)$ matrices $\mathbf{L}_S$ to powers of order $p > 1$, will in general result in redundant paths, while this is not the case for triangular $\mathbf{L}_S$ (Blommestein, 1985a).

Definition 2b can be made more transparent by considering the following example.

Example 1. Consider a system of six economic agents $(A_1, A_2, \dots, A_6)$. Each agent can trade goods in its possession with other agents. Assume further that for institutional or economical reasons every agent can only trade directly with no more than three other agents.

Consider economic agent 1 (A_1). A_1 decides to sell goods to A_2 , A_4 and A_5 . Next consider economic agent 4 (A_4). A_4 sells goods to A_1 , A_3 and A_5 . From Figure 1 it can be seen that A_1 cannot sell goods directly to A_3 , but it is possible through A_4 . Hence through an intermediate trader (i.e., in the second order) A_1 can sell goods to A_3 ($A_1 \rightarrow A_4 \rightarrow A_3$). Suppose that we wish to exclude the possibility that goods sold by A_1 are sold back to A_1 . If this would occur, we say that the goods followed a circular route or redundant path. For example, $A_1 \rightarrow A_4 \rightarrow A_1$ is a redundant path (of order 2).

In many cases these trading routes have to be ruled out as nonsensical (not in all cases!) from an economic point of view, necessitating the elimination of such redundant paths. It should be noted that goods can be sold by economic agents in alternative ways. For example, in the first order A_1 can sell goods to A_5 , but also in the second order: $A_4 \rightarrow A_5$ and $A_1 \rightarrow A_4 \rightarrow A_5$. An analysis of alternative trading routes may involve the problem of finding the shortest way to transport goods from one agent to another. We do not consider this problem here.

In Figure 1 spatial relations are described by means of a graph. Relations can also be described by means of a matrix. The matrix L_S is called a *boolean* spatial lag operator.

$$L_S = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

The first and the fourth row of the boolean spatial lag operator (set in bold face) correspond to the graph of Figure 1 (for the sake of simplicity the other rows of the matrix are not reflected in Figure 1). In the first row of the matrix L_S it can be seen that A_1 sells goods to A_2 , A_4 and A_5 , and in the fourth row of the matrix it can be seen that A_4 sells goods to A_1 , A_3 and A_5 . The columns of the matrix indicate who can buy goods from whom. For example, A_6 can buy goods from A_3 only and A_5 can

FIGURE 1: Spatial System with Six Economic Agents.

buy goods from everybody except himself (that would be redundant in the first order).

The matrix $\mathbf{L}_S$ shows the number of one-step paths that goods have to travel from one economic agent to another (first-order paths), $\mathbf{L}_S^2$ ($\mathbf{L}_S$ squared) shows the number of two-step paths (second-order paths)

$$\mathbf{L}_S^2 = \begin{bmatrix} 3 & 1 & 1 & 2 & 2 & 0 \\ 2 & 2 & 1 & 2 & 2 & 0 \\ 2 & 2 & 1 & 2 & 2 & 0 \\ 1 & 3 & 0 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 & 3 & 0 \\ 2 & 2 & 0 & 2 & 2 & 1 \end{bmatrix}$$

and $\mathbf{L}_S^n$ the number of n -step paths (paths of order n). An example of a third-order redundant path in $\mathbf{L}_S^3$ is: $A_1 \rightarrow A_5 \rightarrow A_4 \rightarrow A_1$.

Also in many spatial econometric analyses the elimination of circular routes is required. As an example we briefly discuss in the next section the importance of eliminating circular routes in Simultaneous Spatial Autoregressive Distributed lag (SSAD) models.

3. CONSEQUENCES OF REDUNDANT PATHS IN SIMULTANEOUS SPATIAL AUTOREGRESSIVE DISTRIBUTED LAG (SSAD) MODELS

In Blommestein (1985b) the following linear SSAD model was studied:

$$(3.1) \quad \mathbf{A}\mathbf{Y} - \mathbf{X}\boldsymbol{\beta} = \boldsymbol{\epsilon}$$

in which $\mathbf{A} = \mathbf{I}_R - \sum_{i=1}^p \alpha_i \mathbf{L}_S^i$, a matrix of size $(R \times R)$; $\mathbf{I}_R$ is an $(R \times R)$ identity matrix; $\mathbf{Y}$ is an $(R \times 1)$ vector with spatial observations on the dependent variable y ; $\mathbf{X}$ is an $(R \times K)$ matrix with conditioning variables; $\{\alpha_i | i = 1, \dots, p\}$; $\boldsymbol{\beta}$ are parameters to be estimated; and $\boldsymbol{\epsilon} \sim \text{NID}(0, \sigma^2 \mathbf{I}_R)$ is an $(R \times 1)$ vector with white noise error terms.

The log-likelihood corresponding to the SSAD model in (3.1) is equal to

$$(3.2) \quad \mathcal{L}(\mathbf{Y}) = (-R/2) \cdot \ln(2\pi\sigma^2) - \frac{1}{2} \sigma^2 [\mathbf{A}\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}]^T [\mathbf{A}\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}] + \ln|\mathbf{A}|$$

where $|\mathbf{A}| = |\mathbf{I}_R - \sum_{i=1}^p \alpha_i \mathbf{L}_S^i|$ is the Jacobian of the transformation from $\boldsymbol{\epsilon}$ to $\mathbf{Y}$. It has been demonstrated in Blommestein (1985b) that the ML-estimations $\hat{\alpha}_i$, $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}^2$ are contaminated by redundant information when the spatial lag operators $\mathbf{L}_S^i$ ($i = 2, \dots, p$), contain circular routes (see Definition 2b). The next example shows that failure to eliminate circular routes might seriously distort estimated parameters.

Example 2. The following SSAD model in Blommestein (1985a) was used for an econometric analysis of some spatial features of the Irish economy:

$$(3.3) \quad \mathbf{Y} = \delta_0 \mathbf{1} + \delta_1 \mathbf{X}_1 + \delta_2 \mathbf{L}_S^1 \mathbf{X}_1 + \delta_3 \mathbf{L}_S^2 \mathbf{X}_1 + \delta_4 \mathbf{L}_S^1 \mathbf{Y} + \delta_5 \mathbf{L}_S^2 \mathbf{Y} + \mathbf{E}$$

where $\mathbf{Y}$ is an $(R \times 1)$ vector with observations on the percentage of gross agricultural output (in value terms) of each of the $R (= 26)$ counties of Ireland consumed by the inhabitants of each county; $\mathbf{X}_1$ is an $(R \times 1)$ vector with values of an index of arterial road accessibility [see Cliff and Ord (1981) for details]; $\mathbf{1} = (1, \dots, 1)^T$ of size $(R \times 1)$; $\mathbf{L}_s^i$ is a i th order $(R \times R)$ spatial lag operator, where each non-zero off-diagonal element represents spatial interaction between a pair (r, r') of counties and all diagonal elements are zero; and $\mathbf{E} \sim \text{NID}(0, \sigma_e^2 \mathbf{I}_R)$ an $(R \times 1)$ vector with white noise error terms. Estimated parameters $\hat{\delta} = (\hat{\delta}_0 \hat{\delta}_1 \dots \hat{\delta}_5)^T$ are given in Table 1.

Since $\mathbf{L}_s^2$ contains redundant information, the ML-estimates $\hat{\delta} = (\hat{\delta}_0 \hat{\delta}_1 \dots \hat{\delta}_5)^T$ in the first column of Table 1 are not reliable. Indeed the difference from the estimates in the second column is striking and illustrates the necessity of eliminating circular routes in SSAD models and other spatial models [see Blommestein (1985b) for a general discussion].

4. A NEW ALGORITHM FOR THE ELIMINATION OF REDUNDANT PATHS IN SPATIAL LAG OPERATORS

Ross and Harary (1952) present a very cumbersome procedure for the elimination of redundant paths. The procedure yields separate, non-recursive formulas. The complexity of these formulas increases progressively with increasing values of the power p in $\mathbf{L}_s^p$ (see Appendix A for details). Much simpler, recursive procedures will be introduced in this paper.

Consider an $(R \times R)$ matrix $\mathbf{L}_s$. Non-triangular $\mathbf{L}_s^2$ will in general contain redundancies. A typical element from the matrix $\mathbf{L}_s^2$ can be written as

$$(4.1) \quad l_{ij}^2 = \sum_{k=1}^R l_{ik} l_{kj}$$

A redundant path occurs when $i = j$ or $j = k$. To eliminate these paths, it is sufficient to add a restriction to (4.1), namely

$$(4.2) \quad w_{ij}^2 = \sum_{k=1}^R l_{ik} l_{kj} \quad [\text{with (i) } i \neq j, \text{ (ii) } i \neq k, \text{ (iii) } j \neq k]$$

TABLE 1: Estimated Versions of Model (3.1)

	With Redundant Paths	Without Redundant Paths
δ_0	-17.32	-13.39
δ_1	+0.0008	+0.0027
δ_2	-0.0067	-0.0034
δ_3	+0.11	+0.0049
δ_4	+0.59	+0.46
δ_5	-0.11	+0.04

Note: Uses the ML-search procedure proposed by Cliff and Ord (1981) and generalized by Blommestein (1983). The derivation of the asymptotic variance-covariance matrix is given in Blommestein (1985b).

In this way redundancies are eliminated for $p = 2$. In matrices raised to some arbitrary power p ($p > 2$), redundancies can be eliminated in a similar way

$$(4.3) \quad w_{ij}^p = \sum_{k_{p-1}=1}^R (l_{ik_{p-1}} \sum_{k_{p-2}=1}^R (l_{k_{p-1}k_{p-2}} (\cdots \sum_{k_1=1}^R (l_{k_2k_1} l_{k_1j}) \cdots)))$$

with (i) $i \neq j$,

(ii) $i \neq k_m$, with $m = 1, \dots, (p-1)$,

(iii) $j \neq k_m$, with $m = 1, \dots, (p-1)$,

(iv) $k_m \neq k_n$, with $m, n = 1, \dots, (p-1)$ and $m \neq n$.

In terms of a recursion formula

$$(4.4) \quad w_{ij}^p = \sum_{k_{p-1}=1}^R (l_{ik_{p-1}} w_{k_{p-1}j}^{p-1})$$

with (i) $i \neq j$,

(ii) $i \neq k_m$, with $m = 1, \dots, (p-1)$,

(iii) $j \neq k_m$, with $m = 1, \dots, (p-1)$,

(iv) $k_m \neq k_n$, with $m, n = 1, \dots, (p-1)$ and $m \neq n$.

Equation (4.4) presented here is a simple recursion formula, which can easily be translated into a computer algorithm. The computation time of (4.4) can be significantly reduced in some cases if one uses a priori knowledge about the structure of the spatial lag operator $\mathbf{L}_S$. For example,

(i) If $\mathbf{L}_S$ is triangular, no circular routes will occur for any p so that $\mathbf{L}_S^p$ can be computed just by raising $\mathbf{L}_S$ to the p th power (Blommestein, 1985a).

(ii) When $\mathbf{L}_S$ consists of relatively few non-zero elements (a so-called sparse matrix), the computation time can be reduced by testing for zero values in $\mathbf{L}_S$. Appendix C contains an algorithm implemented in PASCAL for this case.

(iii) if $p \geq R$ (R is the dimension of the spatial lag operator $\mathbf{L}_S$) then $\mathbf{L}_S^p = 0$. In other words: in any p -step path ($p \geq R$) a circular route must have occurred or otherwise there is no p th order connection between the two vertices.

5. IMPLEMENTATION AND RELATIVE COMPUTATION SPEED OF ALGORITHMS: THEORETICAL AND EMPIRICAL RESULTS

The computation speed of an algorithm largely depends on the complexity of the algorithm. The complexity of the algorithm proposed by Ross and Harary (hereafter, R&H) and our (hereafter, B&K) algorithm will be expressed in terms of number of multiplications, number of additions and number of comparisons that must be executed by a computer. The times consumed by program flow instructions (such as JUMP) and other overhead instructions (PUSH, POP, etc.) are not taken into account in this paper.

Generally speaking, computers use different computation times for multiplication, addition and comparison, and use more time to perform a difficult multiplication (like $3.1415927 \times 2.7182818$) than a simple one (like 1.0×0.0). The same is true for additions and sometimes for comparisons. Furthermore, floating point arithmetic requires more computation time than integer arithmetic. Let us assume that for some arbitrary computer system the average time needed for a multiplica-

tion is t_m , for an addition is t_a , for comparison is t_c and for a boolean comparison is t_{bc} .

First consider square matrix multiplication. We shall express the required time in terms of t_m and t_a , assuming that no comparisons are made while multiplying matrices (the latter is actually not the case, but the total time needed for comparisons appears to be negligible in comparison with the time needed for the entire matrix multiplication).

A general $(R \times R)$ matrix $\mathbf{Q}$ consisting of typical elements q_{ij} is multiplied by itself. The computation time needed for one typical element q_{ij}^2 is $Rt_m + (R - 1)t_a$, so that the computation time required for the entire matrix $\mathbf{Q}^2$ is

$$(5.1) \quad t_M = R^2[Rt_m + (R - 1)t_a]$$

Time Consumed by the R&H Algorithm

We consider more closely the complexity of the procedure for the elimination of redundant paths as proposed by Ross and Harary (Appendix A and Ross and Harary, 1952). Separate (i.e., non-recursive) formulas are derived for the elimination of redundant paths up to six-step paths (raising the matrix to the sixth power). [The general symbolic representation of their formulas is given in Appendix A, Equation (A.4). Note that (A.4) is in non-computational form and that in this formula the term $\mathbf{C}^p$ must be derived for every p separately.] Ross and Harary derive formulas for $p = 2, \dots, 6$.

Consider $p = 2$. Equation (A.4) can be rewritten in computational form as

$$(5.2) \quad \mathbf{W}^2 = \mathbf{L}^2 - d(\mathbf{L}^2)$$

where d represents diagonalization; that is, elimination of off-diagonal elements. The computation of $\mathbf{W}^2$ will cost

$$(5.3) \quad t_{W^2} = t_M \text{ (to compute } \mathbf{L}^2) + Rt_a \text{ (to subtract the diagonal)}$$

For $p = 3$, first consider $\mathbf{C}^3$

$$(5.4) \quad \mathbf{C}^3 = [\mathbf{L}d(\mathbf{L}^2) + d(\mathbf{L}^2)\mathbf{L}] - \mathbf{S}$$

where $\mathbf{S} = \mathbf{L} \times \mathbf{L}^T$ and $\times$ denotes the element-by-element matrix multiplication (Kronecker multiplication).

To compute $\mathbf{C}^3$, $\mathbf{L}^2$ must be computed first; then we must take the diagonal out of this matrix; the following step is to calculate two matrix products; next a matrix addition takes place; to compute $\mathbf{S}$, a Kronecker matrix multiplication is performed and finally another matrix addition (a subtraction is the same as an addition of the negative) yields $\mathbf{C}^3$. Hence the time to compute $\mathbf{C}^3$ is equal to

$$(5.5) \quad t_{C^3} = 3t_M + 2t_A + t_K + t_D$$

where

$$\begin{aligned} t_M &= R^2[Rt_m + (R - 1)t_a] && \text{(time needed for matrix multiplication);} \\ t_A &= R^2t_a && \text{(time needed for matrix addition);} \\ t_K &= R^2t_m && \text{(time needed for Kronecker matrix multiplication);} \\ t_D &= Rt_a && \text{(time needed for a matrix diagonalization).} \end{aligned}$$

However, we are ultimately interested in the computation time of $\mathbf{W}^3$

$$\mathbf{W}^3 = (\mathbf{L}^3 - \mathbf{C}^3) - d(\mathbf{L}^3 - \mathbf{C}^3)$$

Since $\mathbf{L}^2$ has already been computed, the time to compute $\mathbf{L}^3$ is the time needed for one matrix multiplication (t_M). Furthermore one matrix subtraction and one matrix diagonalization (subtraction of the diagonal) have to be performed. Hence the time to compute $\mathbf{W}^3$ equals

$$(5.6) \quad t_{W^3} = 4t_M + 3t_A + t_K + 2t_D$$

In a similar fashion the computation time for $\mathbf{W}^4$, $\mathbf{W}^5$ and $\mathbf{W}^6$ can be derived.

$$(5.7) \quad \begin{aligned} t_{W^4} &= 11t_M + 8t_A + 3t_K + 3t_D \\ t_{W^5} &\approx 40t_M + 32t_A + 11t_K + 5t_D \\ t_{W^6} &\approx 140t_M + 95t_A + 55t_K + 20t_D \end{aligned}$$

These expressions are obtained by counting the multiplication signs, the addition signs, or other operators in the Ross and Harary formulas.

It can be seen that the first term in each of these expressions dominates the later terms. With increasing size of the matrix (R) this becomes even more clear. Rewriting the expressions above by using t_m and t_a , yields

$$(5.8) \quad \begin{aligned} t_{W^2} &\approx R^3(t_m + t_a) \\ t_{W^3} &\approx 4R^3(t_m + t_a) \\ t_{W^4} &\approx 11R^3(t_m + t_a) \\ t_{W^5} &\approx 40R^3(t_m + t_a) \\ t_{W^6} &\approx 140R^3(t_m + t_a) \end{aligned}$$

Time Consumed by the B&K Algorithm

Two versions of the algorithm were developed. Version I is an implementation of Equation (4.4), while Version II exploits more fully the structure of the matrix by detecting zero values.

Consider Version I of the algorithm. Assume that we are dealing with a square non-triangular matrix $\mathbf{L}_S$ of dimension $(R \times R)$. The computation time for $\mathbf{W}^p$ for any $p \in \mathbb{N}^+$ can be expressed in t_m , t_a and t_{bc}

$$(5.9) \quad \begin{aligned} t_{W^p} &= (t_m + t_a)(R^2 - R) \sum_{n=2}^p \frac{(R-2)!}{(R-n-1)!} \\ &\quad + t_{bc} \left[R^2 + (R^3 - R^2) \left(1 + \sum_{n=3}^p \frac{(R-2)!}{(R-n)!} \right) \right] \end{aligned}$$

This can easily be derived from the structure of the algorithm [implementation of Equation (4.4)] which is presented in Appendix B.

Version II of our algorithm has the “overhead cost” of testing whether elements of the matrix are equal to zero (zero-detection), but it has the important

advantage that no (expensive) multiplication has to be carried out if an element happens to be zero.

Definition 3. Let R be the dimension of the square matrix $\mathbf{L}_S$. The fraction of nonzero off-diagonal elements is defined as

$$(5.10) \quad f_{nz} = \frac{\text{Number of nonzero off-diagonal elements}}{(R-1)R}$$

(Note that $0 \leq f_{nz} \leq 1$.)

Let the time needed to perform a comparison be t_c ; then the computation time for $\mathbf{W}^p$ for any p ($\in \mathbb{N}^+$) using Version II of the B&K algorithm, expressed in t_m , t_a , t_c , t_{bc} and f_{nz} , is equal to

$$(5.11) \quad \begin{aligned} t_{W^p} = & (t_m + t_a)(R^2 - R) \sum_{n=2}^p \left(\frac{(R-2)!}{(R-n-1)!} \right) f_{nz}^p \\ & + t_{bc} \left[R^2 + (R^3 - R^2) \left(1 + \sum_{n=3}^p \left(\frac{(R-2)!}{(R-n)!} f_{nz}^{n-2} \right) \right) \right] \\ & + t_c(R^2 - R) \sum_{n=2}^p \left(\frac{(R-2)!}{(R-n-1)!} f_{nz}^{n-2} (1 + f_{nz}) \right) \end{aligned}$$

This can be derived from implementation of Version II of the B&K algorithm as shown in Appendix C.

The advantage of Version II of the algorithm for sparse matrices is evident. It is readily verified that for $f_{nz} = 1$, Version I is always faster than Version II; for $f_{nz} = 0$, Version II is always faster.

With the execution times t_m , t_a , t_c and t_{bc} given, the breakeven value for f_{nz} can be computed for a combination of R and p by equating expressions (5.11) and (5.9). If f_{nz} is larger than the break even value, B&K Version I is faster; if it is smaller, B&K Version II is faster.

It can easily be proved that B&K Version II is faster than B&K Version I for all combinations of R and p if

$$(5.12) \quad f_{nz} < 1 - \frac{t_c}{t_a + t_m}$$

Substituting the execution times listed in Table 2, Figure 2 shows results for $R = 50$. For other values of R a similar figure can be obtained. Not only the shape of the figure, but also the break-even values, are nearly the same as in Figure 2. For the computer we used in our tests, B&K Version II is faster than B&K Version I, for all values of R and p , if $f_{nz} < 0.58$.

Empirical Results

The R&H and B&K algorithms were implemented in PASCAL and executed on a Philips P2230 personal computer based on a 80286 Intel microprocessor chip.

TABLE 2: Typical *Integer* Execution Times on a Philips P2230

$t_m + t_a = 60\mu s$
$t_c = 25\mu s$
$t_{bc} = 14\mu s$

Typical execution times for $t_m + t_a$, t_c and t_{bc} are listed in Table 2. The values for $t_m + t_a$ and t_c are the execution times for operations on integer (or boolean) variables.

Since the R&H algorithm is only valid for boolean matrices, the three algorithms—R&H, B&K Version I and B&K Version II—can only be compared when they are applied to boolean matrices. Matrices of dimension ($R \times R$) ($2 \leq R \leq 100$) were raised to powers p ($2 \leq p \leq 7$), while redundant paths were eliminated by each of the three algorithms. Execution times were registered in milliseconds.

From Tables 3, 4 and 5 it can be seen that for large R and for large p the performance of the B&K algorithms—in comparison with the R&H algorithm—decreases. Further, B&K Version II performs better than B&K Version I. In Appendix D, similar results are derived for B&K Version I versus B&K Version II, when both are applied to floating point matrices. The influence of f_{nz} is demonstrated clearly.

Theoretical and empirical results roughly coincide. The reader can verify that substituting the values from Table 2 into Equations (5.8), (5.9) and (5.11) yields values very similar to those reported in Tables 3–5. Major deviations, either

FIGURE 2: f_{nz} versus p for $R = 50$. Note: In the black area, B&K Version I is faster; in the white area B&K Version II is faster.

TABLE 3: R&H Algorithm Execution Time For Raising an $(R \times R)$ Boolean Matrix to Power p and Eliminating Redundant Paths

p	2	3	4	5	6
R					
3	1				
5	4	15	40		
7	10	40	220	800	3000
10	75	120	490	200	8000
15	100	550	1600	8000	30000
20	220	1300	3600	20000	70000
30	830	4200	12000	60000	200000
50	3800	19000	54000	300000	1000000
70	10000	50000	140000	800000	3000000
100	75000	140000	440000	2000000	8000000

Note: Entries denote execution times in milliseconds. For $p = 5, 6$ the R&H algorithm was not implemented; instead the results in (5.8) were used to approximate the computation times. Thus, entries in italics were not actually measured.

positive or negative, arise from the presence or absence of extra program flow and/or indexed memory-addressing instructions.

To conclude this section, it must be emphasized that the implementation of the R&H formulas for $p = 5, 6$ is a very time consuming job, and that the derivation of the R&H formulas for $p > 6$ is an extremely time consuming and tedious task. The *implementation time* and required *work space* of the R&H algorithm is not recorded and therefore a comparison with the B&K algorithm solely based on (approximate) execution times is biased in favor of the R&H algorithm. However, it seems evident that the non-recursive R&H algorithm is not very attractive to use in empirical work.

TABLE 4: B&K's Version I Algorithm Execution Time for Raising an $(R \times R)$ Boolean Matrix to Power p and Eliminating Redundant Paths

p	2	3	4	5	6	7
R						
3	0					
5	5	15	30			
7	16	50	110	390	820	
10	60	170	1100	6200	29000	300000
15	110	880	11000	110000	300000	3000000
20	160	3100	51000	800000	3000000	4E8
30	610	17000	440000	10000000	8E8	2E10
50	2800	130000	7000000	4E8	4E10	2E12
70	80000	350000	5E7	6E9	5E11	3E13
100	24000	2400000	3E8	5E10	6E12	6E14

Note: Entries denote execution times in milliseconds. The entries in italics were approximated using Equation 5.9.

TABLE 5: B&K's Version II Algorithm Execution Time for Raising an ($R \times R$) Boolean Matrix to Power p and Eliminating Redundant Paths. $f_{nz} = 0.4$

	p	2	3	4	5	6	7
R							
3		0					
5		1	3	4			
7		5	36	21	30	40	
10		15	110	220	600	1000	2800
15		50	440	1900	7400	32000	180000
20		110	1300	8600	49000	800000	4000000
30		550	6100	72000	2000000	20000000	2E8
50		2600	50000	3000000	50000000	8E8	1E10
70		6500	500000	10000000	4E8	1E10	2E11
100		19000	2000000	90000000	3E9	1E11	5E12

Note: Entries denote execution times in milliseconds. The entries in italics were approximated using Equation 5.11.

6. FINAL REMARKS

In this paper we studied the problem of redundant paths in spatial lag operators. In addition, we discussed two procedures for the elimination of such redundant paths.

The procedure developed by Blommestein and Koper has a much simpler structure than Ross and Harary's procedure, because it is recursive. R&H's formulas must be derived for each p separately, which for large p (> 6) must be considered a tedious job. Moreover, B&K's procedure is suited for all types of matrices, while R&K's procedure is only suited for boolean matrices. Furthermore, B&K's Version II is superior for sparse matrices of all types.

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APPENDIX A: Summary and Evaluation of Ross and Harary's Procedure for the Elimination of Redundant Paths

The procedure presented by Ross and Harary (1952) is very complicated. Their procedure leads to separate, non-recursive formulas for the elimination of all orders of redundancies. Furthermore, their formulas are strictly speaking only

valid for boolean matrices, although their method could possibly be adapted for more general types of matrices. This appendix summarizes R&H's procedure. Refer to Ross and Harary (1952) for an extensive discussion. Example 1 from Section 2 is used to illustrate the R&H algorithm.

Consider an economic agent A_i and an economic agent A_j . A path from A_i to A_j is a chain of steps starting with A_i and ending with A_j . For example, $A_i \rightarrow A_k \rightarrow A_j$ is a two-step path. The elements w_{ij}^p of matrix L_i^p indicate how many p -step paths there exist from A_i to A_j .

Ross and Harary define a matrix C^p consisting of elements c_{ij}^p . The elements c_{ij}^p indicate how many p -step redundant paths there are from A_i to A_j , with $i \neq j$. Ross and Harary split the integer value p into three other integers, p_1, p_2, p_3 in such a way that

$$p_1 + p_2 + p_3 = p \text{ and } p_1, p_2, p_3 \geq 0, p_1 + p_3 < 0, p_2 > 1$$

Such a combination of p_1, p_2, p_3 is called a partition of p . Assume that p_1, p_2, p_3 is a partition of p . A redundant path from A_i to A_j satisfies this partition if

p_1 = the number of steps from the initial agent A_i to one of the places in which a repeated agent occurs other than the last appearance.

p_2 = the number of steps from this appearance of the repeated agent to a later appearance.

p_3 = the number of steps from this second appearance to the terminal or final agent A_j .

Example

$$A_i \rightarrow A_k \rightarrow A_l \rightarrow A_m \rightarrow A_k \rightarrow A_n \rightarrow A_j$$

The repeated agent is A_k , so $p_1 = 1, p_2 = 3$ and $p_3 = 2$. This is called a 132 partition. However, it is possible that some arbitrary path satisfies more than one partition.

Example

$$A_i \rightarrow A_j \rightarrow A_i \rightarrow A_j$$

Consider A_i as a repeated agent. Hence $p_1 = 0, p_2 = 2, p_3 = 1$ (a 021 partition). But we can also consider A_j as a repeated agent. Then $p_1 = 1, p_2 = 2, p_3 = 0$ (a 120 partition).

The maximum number of partitions a redundant p -step path can possibly satisfy is

$$u = 2 K(\text{Round}(p/2), 2)$$

in which

$$K(n, m) = \frac{n!}{m! (n - m)!}$$

and $\text{Round}(\)$ denotes the nearest integer such that

$$\text{Round}(0.5) = 1, \text{Round}(1) = 1, \text{Round}(1.5) = 2, \quad \text{etc.}$$

The total number of partitions of p is

$$q = K(p, 2) - 1$$

For each i, j and each p , Ross and Harary derive the number of redundant paths c_{ij}^p .

Assume that

- (i) $f_1, f_2, \dots, f_q$ is the set of all partitions of p ; remember that $q = K(p, 2) - 1$.
- (ii) $b[x_1, x_2, \dots, x_t]$ is the number of paths that satisfy f_{x_1} and f_{x_2} and $\dots$ and f_{x_t} , while $1 \leq x_p \leq q$ and $x_i \neq x_j$.

$$(A.1) \quad (iii) \quad B_t = \sum_{x_t = x_{t-1} + 1} \dots \sum_{x_2 = x_1 + 1} \sum_{x_1 = 1} b[x_1, x_2, \dots, x_t]$$

for each t , with $1 \leq t \leq u$. Define c_{ij}^p as follows:

$$(A.2) \quad c_{ij}^p = \sum_{t=1}^u [(-1)^{(t+1)} B_t]$$

Let $\mathbf{C}_t^p$ be the matrix whose i, j entry is B_t ; then

$$(A.3) \quad \mathbf{C}^p = \sum_{t=1}^u [(-1)^{(t+1)} \mathbf{C}_t^p]$$

To eliminate all redundancies from $\mathbf{L}_S^p$, the final step is to use the diagonalization operator

$$(A.4) \quad \mathbf{W}^p = (\mathbf{L}^p - \mathbf{C}^p) - d(\mathbf{L}^p - \mathbf{C}^p)$$

in which

$\mathbf{W}^p$ is the matrix $\mathbf{L}$ raised to the power p while redundant paths were eliminated;

$\mathbf{L}$ is the original matrix;

p is the power to which $\mathbf{L}$ is raised;

$\mathbf{C}^p$ is the matrix containing all p -step redundant paths except for the diagonal, and;

d is the diagonalization operator.

Formulas for the elimination of redundant paths were derived by Ross and Harary for p -values up to six. A major drawback of the Ross and Harary formulas is that they are very difficult to derive (especially for large p). Moreover, their procedure yields non-recursive and very complex formulas. R&H's formula (6) for the elimination if $p =$ six-step redundant paths occupies two printed pages in their article!

Another point of criticism is that strictly speaking the formulas are only valid for boolean matrices. For example, using Ross and Harary's formula (4) (on page 201) for eliminating redundant paths from matrices raised to the fourth power, using the matrix representation $\mathbf{L}_S$. Raising $\mathbf{L}_S$ to the fourth power yields redundant paths. In case the matrix $\mathbf{L}_S$ contains elements not equal to values 0 or 1, then the formula does not yield the desired result. For example, replace all 1's in

$\mathbf{L}_S$ on page 94 by the value 2

$$\mathbf{L}_S = \begin{bmatrix} 0 & 2 & 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 0 & 2 & 2 \\ 2 & 0 & 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 2 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 0 \end{bmatrix}$$

Eliminating redundant paths from the matrix $\mathbf{L}_S^4$, using the procedure proposed by Ross and Harary, yields

$$\mathbf{L}_S^4 = \begin{bmatrix} 0 & 16 & 24 & -32 & 0 & 32 \\ -16 & 0 & 32 & -56 & 0 & 32 \\ 96 & 8 & 0 & 120 & 40 & 0 \\ 32 & 0 & -16 & 0 & 0 & -8 \\ -32 & 0 & 24 & -40 & 0 & 32 \\ 96 & 32 & 64 & 120 & 64 & 0 \end{bmatrix}$$

Note that negative values in this matrix are not defined and therefore not allowed.

APPENDIX B: Blommestein & Koper's Algorithm Version I

```

procedure BKversionI(var l:matrix;p:integer;var w:matrix);
var i,j:integer;
    NOTinpath:array[1..R] of boolean;

function computeW(i,j,p:integer):real;
var wij:real;
    k:integer;
begin
    wij:=0;
    if p=2 then
    begin
        for k:=1 to R do
            if NOTinpath[k] then
                wij:=wij+l[i,k]*l[k,j];
        end
    else
    begin
        for k:=1 to R do
            if NOTinpath[k] then
            begin
                NOTinpath[k]:=false;
                wij:=wij+l[i,k]*computeW(k,j,p-1);
                NOTinpath[k]:=true;
            end;
        end;
        computeW:=wij;
    end;

begin
    for i:=1 to R do NOTinpath[i]:=true;
    for i:=1 to R do
        for j:=1 to R do
            if i<>j then
            begin
                NOTinpath[i]:=false; NOTinpath[j]:=false;
                w[i,j]:=computeW(i,j,p);
                NOTinpath[i]:=true; NOTinpath[j]:=true;
            end
        else
            w[i,j]:=0;
        end;
    end;
end;

```

APPENDIX C: Blommestein & Koper's Algorithm Version II

```

procedure BKversionII(var l:matrix;p:integer;var w:matrix);
var i,j:integer;
    NOTinpath:array[1..R] of boolean;

function computeW(i,j,p:integer):real;
var wij,wkj:real;
    k:integer;
begin
    wij:=0;
    if p=2 then
        begin
            for k:=1 to R do
                if NOTinpath[k] and (l[i,k]<>0.0) and (l[k,j]<>0.0) then
                    wij:=wij+l[i,k]*l[k,j];
            end
        end
    else
        begin
            for k:=1 to R do
                if NOTinpath[k] and (l[i,k]<>0.0) then
                    begin
                        NOTinpath[k]:=false;
                        wkj:=computeW(k,j,p-1);
                        if wkj<>0.0 then wij:=wij+l[i,k]*wkj;
                        NOTinpath[k]:=true;
                    end;
            end;
            computeW:=wij;
        end;
    end;

begin
    for i:=1 to R do NOTinpath[i]:=true;
    for i:=1 to R do
        for j:=1 to R do
            if i<>j then
                begin
                    NOTinpath[i]:=false; NOTinpath[j]:=false;
                    w[i,j]:=computeW(i,j,p);
                    NOTinpath[i]:=true; NOTinpath[j]:=true;
                end
            else
                w[i,j]:=0;
            end;
        end;
    end;
end;

```

APPENDIX D: Additional Results on the Performance of the B&K Algorithms

The B&K algorithms are—unlike the R&H algorithm—valid for all types of matrices. Floating point matrices of dimension R ($3 \leq R \leq 50$) with several f_{nz} values ($0 \leq f_{nz} \leq 1$) were supplied to the B&K Version I and Version II algorithm. The matrices were raised to powers p ($2 \leq p \leq 6$), while redundant paths were eliminated. Execution times were registered in milliseconds. Table D.1 presents the typical execution times $t_a + t_m$ and t_c of elementary floating point operations. Notice that t_{bc} is unchanged with respect to Table 5.1.

Table D.2 shows the execution times for B&K Version I. Tables D.3 up to D.7 display the results for B&K Version II with $f_{nz} = 1, 0.7, 0.4, 0.2$ and 0.1 .

TABLE D.1: Typical *Floating* Execution Times on a Philips P2230

$t_m + t_a = 500\mu s$
$t_c = 90\mu s$
$t_{bc} = 14\mu s$

TABLE D.2: B&K's Version I Algorithm Execution Time for Raising an $(R \times R)$ *Real* Matrix to Power p and Eliminating Redundant Paths

	p	2	3	4	5	6
R						
3		1				
5		8	25	40		
7		25	140	460	1200	1900
10		90	750	4600	24000	100000
15		350	4500	50000	500000	
20		900	16000	25000		
30		3000	90000			
50		15000	750000			

Note: Entries denote execution times in milliseconds.

TABLE D.3: B&K's Version II Algorithm Execution Time for Raising an $(R \times R)$ *Real* Matrix to Power p and Eliminating Redundant Paths— $f_{nz} = 1$

	p	2	3	4	5	6
R						
3		1				
5		10	30	55		
7		35	180	650	1500	2500
10		125	1000	6500	33000	140000
15		480	6000	68000	700000	
20		1100	21000	350000		
30		4000	120000			
50		20000				

Note: See Note to Table D.2.

TABLE D.4: B&K's Version II Algorithm Execution Time for Raising an $(R \times R)$ Real Matrix to Power p and Eliminating Redundant Paths— $f_{nz} = 0.7$

	p	2	3	4	5	6
R						
3		1				
5		6	15	19		
7		22	80	200	350	460
10		75	420	1900	7100	22000
15		280	2600	20000	150000	
20		710	9000	100000		
30		2500	50000			
50		12000	410000			

Note: See Note to Table D.2.

TABLE D.5: B&K's Version II Algorithm Execution Time for Raising an $(R \times R)$ Real Matrix to Power p and Eliminating Redundant Paths— $f_{nz} = 0.4$

	p	2	3	4	5	6
R						
3		0				
5		4	6	6		
7		12	28	48	62	65
10		40	150	400	890	1700
15		160	850	4000	16000	61000
20		380	2800	19000	120000	
30		1300	16000	160000		
50		6500	130000			

Note: See Note to Table D.2.

TABLE D.6: B&K's Version II Algorithm Execution Time for Raising an $(R \times R)$ Real Matrix to Power p and Eliminating Redundant Paths— $f_{nz} = 0.2$

	p	2	3	4	5	6
R						
3		0				
5		2	3	3		
7		8	14	17	18	18
10		26	61	100	140	180
15		99	330	820	1800	3700
20		250	1100	3600	12000	33000
30		850	5500	30000	150000	
50		4200	43000	400000		

Note: See Note to Table D.2.

From Tables D.2–D.7 it can be concluded that in practice: (1) Version II of the B&K algorithm performs better than Version I, (2) the superior performance of Version II versus Version I increases with decreasing values of f_{nz} , and (3) only for $f_{nz} = 1$ Version I is always faster.

TABLE D.7: B&K's Version II Algorithm Execution Time for Raising an $(R \times R)$ Real Matrix to Power p and Eliminating Redundant Paths— $f_{nz} = 0.1$

	p	2	3	4	5	6
R						
3		0				
5		2	2	2		
7		7	9	10	10	10
10		22	37	46	51	55
15		83	190	290	390	490
20		210	550	1100	1900	3100
30		730	2600	7700	20000	51000
50		3500	20000	95000	430000	

Note: See Note to Table D.2.