SOME RECENT RESULTS IN INFINITE DIVISIBILITY*

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Abstract. In this paper, a survey is given of some recent developments in infinite divisibility. There are three main topics: (i) the occurrence of infinitely divisible distributions in applied stochastic processes such as queueing processes and birth–death processes, (ii) the construction of infinitely divisible distributions, mainly by mixing, and (iii) conditions for infinite divisibility in terms of distribution functions and densities.

1. Introduction

In 1961, Fisz [7] gave a survey on recent developments in infinite divisibility. He restricted himself almost completely to distributions on the real line. In this paper also, only distributions on the real line are considered. It seems inevitable, however, to be more restrictive than that. The selection of material included in this survey is determined partly by the author's personal interest, partly by the desire to present a more or less coherent paper. As a result, the sets of subjects discussed by Fisz and of those considered here are almost disjoint. This, of course, does not mean that there have been no developments in areas not mentioned here.

In Section 2 of this paper, we introduce some notation and give some basic theorems in infinite divisibility. In Section 3, we consider a number of applied stochastic processes that always, often or sometimes yield infinitely divisible distributions. Section 4 discusses methods of constructing infinitely divisible distributions, mainly by mixing. In Section 5, a number of criteria for infinite divisibility are collected, most in terms of probability densities and distribution functions. Some unsolved problems are presented in Section 6.

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One of the most unexpected new developments in infinite divisibility seems to be that attention has shifted from properties of characteristic functions, the most obvious tool in this field, to properties of distribution functions and probability densities. Properties of infinitely divisible distribution functions and densities are a recurrent theme in this paper.

2. Notation and some basic properties

We shall use the following abbreviations: r.v., d.f., p.d.f., c.f., L.T. and p.g.f. for random variable, distribution function, probability density function, characteristic function, Laplace (–Stieltjes) transform and probability generating function, respectively. We further abbreviate completely monotone (complete monotonicity), infinitely divisible (infinite divisibility), and independent and identically distributed, respectively, as comp. mon., inf. div. and i.i.d.

We denote r.v.'s by $X, Y, Z, \dots$, d.f.'s by $F, G, H, \dots$, their p.d.f.'s by $f, g, h, \dots$, their c.f.'s by $\varphi, \gamma, \eta, \dots$ and their L.T.'s by $\tilde{F}, \tilde{G}, \tilde{H}, \dots$. The correspondence between F, f, φ , etc., will always be assumed; sometimes it will be convenient to label these functions with r.v.'s: F_X, φ_X , etc. The symbol $\stackrel{d}{=}$ will denote equality in distribution, F^{*p} denotes the d.f. with c.f. φ^p .

We now give some definitions and properties. For more detailed information, we refer to [1, 6, 22, 23].

Definition 2.1. A r.v. X is called *infinitely divisible* iff for each positive integer n there exist i.i.d. r.v.'s $X_{1n}, \dots, X_{nn}$ such that $X \stackrel{d}{=} X_{1n} + \dots + X_{nn}$.

Definition 2.2. If X is an inf. div. r.v. with d.f. F , c.f. φ , etc., then F, φ , etc., are called inf. div.

Property 2.3. If $\varphi(t)$ is an inf. div. c.f., then φ has no real zeros. If φ is analytic, then φ has no zeros in its strip of regularity. If φ is entire, then φ has no zeros in the complex plane.

Property 2.4. The set of inf. div. distributions is closed under weak convergence.

Property 2.5. A c.f. φ having no real zeros is inf. div. iff φ^p is a c.f. for all positive p (or for all $p = n^{-1}$, $n = 1, 2, \dots$, or for all $p = 2^{-n}$, $n = 1, 2, \dots$).

Property 2.6. $\varphi(t)$ is inf. div. iff

$$(2.1) \quad \varphi(t) = \exp \left\{ cit + \int_{-\infty}^{\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) \frac{1+x^2}{x^2} d\theta(x) \right\},$$

where c is a real constant, and θ is non-decreasing and bounded. If φ can be written in the form (2.1) with a θ that is not non-decreasing, then φ is not inf. div.

There are several modifications of this representation, valid in different circumstances. We state those for inf. div. L.T.'s and inf. div. p.g.f.'s.

Property 2.7. $\tilde{F}(\tau)$ is inf. div. iff

$$(2.2) \quad \tilde{F}(\tau) = \exp \left\{ \int_0^{\infty} \frac{e^{-\tau x} - 1}{x} dK(x) \right\},$$

where K is non-decreasing. It follows that $\int_c^{\infty} x^{-1} dK(x) < \infty$ for all $c > 0$.

Corollary 2.8. $\tilde{F}(\tau)$ is inf. div. iff $-d \log \tilde{F}(\tau)/d\tau$ is comp. mon.

Property 2.9. A p.g.f. $P(u)$ with $P(0) > 0$ is inf. div. iff

$$(2.3) \quad P(u) = \exp \{ \lambda Q(u) - \lambda \},$$

where $\lambda \geq 0$ and $Q(u)$ is a p.g.f.

The following two classes of inf. div. c.f.'s are of special interest.

Property 2.10 (Compound-Poisson distribution). A c.f. of the form

$$(2.4) \quad \varphi(t) = \exp \{ \lambda \gamma(t) - \lambda \},$$

where $\lambda \geq 0$ and φ a c.f., is inf. div.

Property 2.11 (Compound-geometric distribution). A c.f. of the form

$$(2.5) \quad \varphi(t) = \lambda \{ \lambda + 1 - \gamma(t) \}^{-1},$$

where $\lambda > 0$ and γ a c.f., is inf. div.

In fact, (2.5) is a special case of (2.4). However, many inf. div. c.f.'s appear in the special form (2.5) (cf. [23]).

3. Generation of inf. div. distributions by stochastic processes

It is well known (cf. [1, 6, 21]) that all inf. div. distributions are generated by stochastic processes, more specifically by processes with stationary independent increments. There is a number of applied processes, however, that give rise to classes of inf. div. distributions. In this section, some of these processes are discussed.

3.1. *Waiting-time distributions*

In the M/G/1 waiting-time process with first-come—first-served queue discipline (see [2] for terminology and notation), all stationary waiting-time distributions are inf. div. This follows from the easily established fact that the c.f. $\omega(t)$ of the waiting time can be written in the form (cf. [2])

$$(3.1) \quad \omega(t) = \lambda \{ \lambda + 1 - \varphi(t) \}^{-1},$$

where $\varphi(t)$ is a c.f. and $\lambda > 0$ (cf. Property 2.11). Using the inf. div. of $\omega(t)$, one obtains inf. div. d.f.'s by choosing the service-time d.f. (symbolized by G) such that explicit calculation of the waiting-time d.f. W is possible. For instance, if G is a mixture of exponential d.f.'s, then W is a mixture of exponential d.f.'s (compare Section 4).

It was noted by Spitzer [28] that the stationary waiting time for the G/G/1 queue is inf. div. In the discussion of a paper by Kingman [20], Runnenburg pointed out that the corresponding c.f. is of the form (3.1). At the same time, two questions were raised:

(i) Is the d.f. of XY inf. div. if X and Y are independent and both exponentially distributed? (This is the limiting d.f. for the M/G/1 queue with random service in heavy traffic.)

(ii) Are "all" waiting-time distributions inf. div.?

The first question was answered in the affirmative by Goldie [8], who proved more generally that XY is inf. div. if X and Y are independent, Y is exponentially distributed and X non-negative. Equivalently, all scale-parameter mixtures of exponential d.f.'s are inf. div. We return to this result in Section 4. As an answer to the second question, in [29] an example is given of a stationary waiting-time d.f. (M/G/1, with last-come—first-served discipline) that is not inf. div.

Other instances of infinite divisibility in connection with queueing are provided by queue lengths and busy periods (cf. [29, 33]).

3.2. First-passage times

Another way to generate mixtures of exponential distributions is implicit in [14, 15], and was pointed out to me by W. Vervaat (personal communication).

In a birth-death process, take the birth rates $\lambda_k > 0$ for $k = 1, 2, \dots, n-1$ and $\lambda_n = 0$, the death rates $\mu_k > 0$ for $k = 1, 2, \dots, n$. Denote by $F_{k,k-1}$ the first-passage time d.f. from state k to state $k-1$; then for the c.f. $\varphi_{k,k-1}$ one has

$$(3.2) \quad \varphi_{k,k-1}(t) = \mu_k \{ \mu_k + \lambda_k - it - \lambda_k \varphi_{k+1,k}(t) \}^{-1}.$$

The c.f. $\varphi_{k,k-1}$ is almost of the same form as that in (2.5) and (3.1). To conclude to the inf. div. of $\varphi_{k,k-1}$, one may either take out a factor $(\mu_k + \lambda_k)(\mu_k + \lambda_k - it)^{-1}$ and apply (2.5) to the remaining factor, or use a slightly more general result (cf. Section 4). Vervaat showed by induction that $F_{1,0}$ is a mixture of exponential d.f.'s, and conversely that every finite mixture of exponential d.f.'s can be obtained in this way by choosing n and the λ 's and μ 's appropriately.

Similar results have been obtained by Keilson [17], who also considers passage times from state 0 to state n in a birth-death process with reflecting barrier at 0. It turns out that these passage-time distributions are convolutions of exponential distributions, and hence are inf. div. too.

Miller [24] proves the inf. div. of first-passage times in more general Markov processes. We state his result as a theorem.

Theorem 3.1. *If a discrete-state-space Markov chain is skipfree in (at least) one direction, then the first-passage times in that direction are inf. div.*

Miller's proof is based on a relation analogous to (3.2). As a special case, he considers a homogeneous birth-death process with $\lambda_k = \mu_k = 1$, and obtains the inf. div. of the p.d.f.'s

$$(3.3) \quad f_{n,0}(x) = n x^{n-1} (1+x)^{-n-1}.$$

Related results of a more special type have been obtained by Feller [5] and by Dwass [4]. Another, well known, related result is the exhibition

of the stable distribution of order $\frac{1}{2}$ as a first-passage time distribution in Brownian motion (cf. [6]).

3.3. *Renewal distributions*

Distributions with p.d.f.'s of the form

$$(3.4) \quad f_1(x) = \mu_0^{-1} (1 - F_0(x)),$$

where F_0 is a d.f. on $(0, \infty)$ with expectation μ_0 , will be called renewal distributions. Since distributions of this type are related to waiting-time distributions, one might expect these to be inf. div. in many cases, or at least when F_0 is inf. div. This expectation is strengthened by the observation that if F_1 is a mixture of exponential d.f.'s then so is F_0 . In [29], however, examples are given where F_0 is inf. div. and F_1 is not, and vice versa. Here, also necessary and sufficient conditions are given for the inf. div. of $f_1(x)$ in (3.4). It turns out that f_1 is inf. div. iff

$$\log x - \sum_{n=1}^{\infty} n^{-1} F_0^{*n}(x)$$

is non-decreasing, or, if F_0 is absolutely continuous, iff

$$(3.5) \quad \sum_{n=1}^{\infty} n^{-1} f_0^{*n}(x) \leq x^{-1}, \quad x > 0.$$

From (3.5) one obtains necessary conditions for the inf. div. of decreasing p.d.f.'s.

However, the expectation that F_1 in (3.4) should be inf. div. is not totally unjustified. Harkness and Shantaram [9] prove that repetition of the transformation (3.4) leading from F_0 to F_1 , in case of convergence under suitable norming, does yield inf. div. distributions. Their result is that the limiting d.f., if it exists, satisfies a relation of the form

$$(3.6) \quad F(x) = b \int_0^{ax} \{1 - F(y)\} dy, \quad x > 0,$$

where b^{-1} is the expectation of F and $a \geq 1$. For $a = 1$, the only solution of (3.6) is the exponential d.f. Though infinite divisibility is not discussed in [9], one easily verifies that for $a > 1$ the p.d.f. $F'(x)$ is comp. mon., i.e., that F is a mixture of exponential d.f.'s and hence is inf. div. (cf. Section 4.2).

3.4. Extreme values

It is well known (see e.g. [3]) that for extreme-value distributions there are three limiting types:

$$(3.7) \quad F_1(x) = \exp(-e^{-x}), \quad -\infty < x < \infty,$$

$$(3.8) \quad F_2(x) = 1 - \exp(-x^\alpha), \quad x > 0, \quad \alpha > 0,$$

$$(3.9) \quad F_3(x) = \exp(-x^{-\alpha}), \quad x > 0, \quad \alpha > 0.$$

$F_1(x)$ is inf. div. because the corresponding r.v. can be written as

$$X_1 = d - \lim_{n \rightarrow \infty} \{Y_1 + \frac{1}{2} Y_2 + \dots + \frac{1}{n} Y_n - \log n\},$$

where the Y_j are i.i.d. with an exponential distribution of mean 1 (see e.g. [6]).

$F_2(x)$ is inf. div. for $0 < \alpha \leq 1$, as $f_2(x)$ is comp. mon. for those α ; it is not inf. div. for other values of α , as $1 - F_2(x)$ tends to zero too fast for $\alpha > 1$ (compare Section 5.3).

It seems to be unknown whether F_3 is inf. div. For $\alpha = 1$, one can use Miller's result (cf. (3.3)):

$$F_3(x) = \lim_{n \rightarrow \infty} \left(\frac{nx}{1+nx} \right)^n = \lim_{n \rightarrow \infty} F_{0,n}(nx),$$

which is inf. div. by the closure property.

For the three types of limiting distributions for the k^{th} largest in a sample (cf. [27]) the situation is similar, but slightly more complicated. The distribution corresponding to type (3.7) is inf. div. by a similar argument, while the distribution corresponding to type (3.8) is inf. div. for $\alpha \leq k^{-1}$, and not inf. div. for $\alpha > 1$.

4. Construction of inf. div. distributions

There is a number of methods to construct new inf. div. distributions from given ones. The best known are convolution and compounding (see Properties 2.10 and 2.11). A general compounding theorem is due to Feller [6]. We state this theorem in a slightly more general form such as to include the compound-Poisson distributions.

Theorem 4.1. *If G is an inf. div. d.f. on $[0, \infty)$, and if φ^x is a well-defined c.f. for all x in the joint supports of $G^{*1/n}$ ($n = 1, 2, \dots$), then*

$$(4.1) \quad \int_0^\infty \{\varphi(t)\}^x dC(x),$$

is an inf. div. c.f. (equal to $\tilde{G}(-\log \varphi(t))$), if $\log \varphi(t)$ is well-defined).

Special cases of (4.1) yield the better-known compound-Poisson distributions and the compound-geometric distributions. These methods, however, seem to be too general to yield explicit inf. div. d.f.'s and p.d.f.'s easily.

Another, much less general, and perhaps even unnatural way of obtaining new inf. div. distributions, is by mixing, i.e., by considering d.f.'s of the form

$$F(x) = \int_{-\infty}^{\infty} K(x, y) dG(y),$$

where K is a d.f. in x for each y in the support of G , and K is Borel-measurable with respect to y for all x .

The rest of this section will be devoted to mixing. A more restricted procedure is the multiplication of independent r.v.'s; this is equivalent to mixing with respect to the scale parameter:

$$(4.2) \quad \varphi_{XY}(t) = \int_{-\infty}^{\infty} \varphi_X(yt) dF_Y(y).$$

4.1. *Multiplication of random variables*

Using Corollary 2.8, and a result in renewal sequences, Goldie [8] proved the following theorem:

Theorem 4.2. *If X and Y are independent, $X \geq 0$ and Y exponentially distributed, then XY is inf. div.*

Corollary 4.3. *If X is non-negative, if Y has a comp. mon. p.d.f., and if X and Y are independent, then XY is inf. div.*

Corollary 4.4. *If X is non-negative, if Y has a gamma distribution of order α with $0 < \alpha \leq 1$, and if X and Y are independent, then XY is inf. div.*

In [29], an only partially successful attempt was made to prove the inf. div. of XY , where Y has c.f. $(1-it)^{-2}$. One of the partial results obtained is recorded as:

Theorem 4.5. *If X_α and X_β are independent r.v.'s with c.f.'s $(1-it)^{-\alpha}$ and $(1-it)^{-\beta}$, respectively, and if $\min(\alpha, \beta) \leq 2$, then $X_\alpha X_\beta$ is inf. div.*

We return to some of these results in the context of mixtures.

4.2. Mixtures of exponential distributions

In view of (4.2) and the closure property of the class of inf. div. distributions, Theorem 4.2 is equivalent to:

Theorem 4.2'. *If $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n$, $p_k > 0$ ($k = 1, 2, \dots, n$) and $\sum_{k=1}^n p_k = 1$, then*

$$(4.3) \quad \sum_{k=1}^n p_k \lambda_k \exp(-\lambda_k x),$$

is an inf. div. p.d.f.

In [29], the author generalized this theorem to:

Theorem 4.2''. *A p.d.f. of the form (4.3) is inf. div. if $0 < \lambda_1 < \dots < \lambda_n$, $p_k > 0$ for $k = 1, \dots, k_0$ and $p_k < 0$ for $k = k_0 + 1, \dots, n$.*

The proof of Theorem 4.2'' is given by showing that the L.T. of (4.3) is of the form

$$(4.4) \quad \sum_{k=1}^n p_k \lambda_k (\lambda_k + \tau)^{-1} = \prod_{k=1}^n \lambda_k (\lambda_k + \tau)^{-1} \bigg/ \prod_{j=1}^n \mu_j (\mu_j + \tau)^{-1},$$

with $\mu_j > \lambda_j$ ($j = 1, 2, \dots, n$). Using (4.4) and the canonical representation of $\lambda(\lambda + \tau)^{-1}$, in [29] the following representation theorem is proved.

Theorem 4.6. *$\tilde{F}(\tau)$ is a L.T. of the form $\int_0^\infty (1 + \tau x)^{-1} dG(x)$, with G a d.f. iff*

$$(4.5) \quad -\log \tilde{F}(\tau) = \int_0^\infty \frac{\tau}{1 + \tau x} dm(x),$$

where m is a measure bounded by Lebesgue measure.

4.3. Mixtures of other inf. div. distributions

In [29], mixtures of gamma distributions of order 2 are considered, i.e., L.T.'s of the form

$$(4.6) \quad \int_0^{\infty} (1 + \alpha\tau)^{-2} dG(\alpha).$$

The only results obtained are: (4.6) is inf. div. if G is a discrete d.f. with four or fewer points of increase, or if G is unimodal. The latter result follows by integration by parts of (4.6) and the use of Theorem 4.2".

Using Theorems 4.1 and 4.2', it follows that functions of the form

$$(4.7) \quad \int_0^{\infty} \{1 - xh(t)\}^{-1} dG(x),$$

where $h(t)$ is such that $\exp h(t)$ is an inf. div. c.f., are inf. div. c.f.'s. If we take $h(t) = \varphi(t) - 1$, with $\varphi(t)$ an arbitrary c.f., we obtain the inf. div. of parameter mixtures of compound-geometric distributions; if we take $h(t) = -t^2$, the inf. div. of scale-parameter mixtures of Laplace distributions follows. Other mixtures of this type are listed in [29].

Mixtures of normal distributions, with c.f.'s of the form

$$(4.8) \quad \int_0^{\infty} \exp(-t^2x) dG(x),$$

have been considered by Kelker [19]. It follows from Theorem 4.1 (or otherwise) that (4.8) is inf. div. if G is inf. div. The inf. div. of mixtures of Laplace distributions follows from the fact that these are mixtures of normal distributions with G in (4.8) a mixture of exponential d.f.'s (cf. Theorem 4.2'). Kelker gives an example where (4.8) is inf. div. while G is not; it is also possible for (4.8) to be inf. div. while G is not even a d.f. In [19] and also in [31], it is proved that mixtures of Cauchy distributions are inf. div. This is also an easy consequence of a property noted by Horn [10], and easily proved by means of Pólya's criterion (see [6] and [18]). Horn proves:

Theorem 4.7. *Real c.f.'s that are log-convex on $(0, \infty)$ are inf. div.*

As log-convexity is preserved under mixing (cf. Section 5), mixtures of log-convex c.f.'s are also inf. div. Since the set of real comp. mon.

c.f.'s, i.e. c.f.'s of mixtures of Cauchy distributions, is a subset of the real log-convex c.f.'s, the result follows.

Log-convexity and complete monotonicity of c.f.'s is discussed in detail by Keilson and Steutel [18], who also note the inf. div. of mixtures of symmetric stable distributions of orders $\alpha < 1$. For mixtures of symmetric stable distributions of fixed order α , with $1 < \alpha < 2$, one replaces t^2 in (4.8) by $|t|^\alpha$. It is clear again that the mixtures obtained in this way are inf. div. if G is inf. div.

5. Conditions for inf. div. in terms of d.f.'s and p.d.f.'s

One of the most striking aspects of recent results in inf. div. is that a large number of them contain criteria in terms of d.f.'s and p.d.f.'s (both discrete and continuous) rather than in terms of c.f.'s. In this section we collect some results of this nature, a number of which have been mentioned in different contexts in earlier sections.

5.1. Representation theorems

Using Corollary 2.8, Goldie [8] proves the following theorem, without stating it explicitly.

Theorem 5.1. *If F is a d.f. on $[0, \infty)$, and if $b_k(\theta)$ is defined for $\theta > 0$ and $k = 0, 1, 2, \dots$ by*

$$(5.1) \quad b_k(\theta) = \frac{1}{k!} \int_0^\infty x^k e^{-\theta x} dF(x),$$

then F is inf. div. iff the functions $a_k(\theta)$, defined recursively by

$$(5.2) \quad (n+1) b_{n+1} = \sum_{k=0}^n a_k b_{n-k}, \quad n = 0, 1, 2, \dots,$$

are non-negative for all $\theta > 0$.

Goldie proves: (i) if a sequence b_n is such that the equations

$$(5.3) \quad b_{n+1} = \sum_{k=0}^n f_k b_{n-k}, \quad n = 0, 1, 2, \dots,$$

have non-negative solutions f_k , then (5.2) has non-negative solutions a_k . To establish the inf. div. of mixtures of exponential d.f.'s, he uses: (ii) if a sequence b_n satisfies

$$(5.4) \quad b_{n+1} b_{n-1} \geq b_n^2, \quad n = 1, 2, \dots,$$

then the solutions of (5.3) are non-negative. By Schwarz's inequality the b_n defined in (5.1) satisfy (5.4) if F is a mixture of exponential d.f.'s.

The equations (5.3) and (5.4) are of interest for renewal sequences (cf. [8] for references, and [11] for related remarks).

Katti [16] proves a very similar theorem for lattice distributions (for a different proof, see [29]):

Theorem 5.2. *If $\{p_n\}$ is a distribution on the non-negative integers, with $p_0 > 0$, then $\{p_n\}$ is inf. div. iff the quantities r_k defined for $k = 0, 1, 2, \dots$, by*

$$(5.5) \quad (n+1)p_{n+1} = \sum_{k=0}^n r_k p_{n-k}, \quad n = 0, 1, 2, \dots,$$

are all non-negative.

The similarity between Theorems 5.1 and 5.2 reflect the fact that $\tilde{F}(1-u)$ is an inf. div. p.g.f. if $\tilde{F}(\tau)$ is an inf. div. L.T. (compare Theorem 4.1). Solving (5.5) for r_k by Cramer's rule, explicit criteria in terms of the $p_0, p_1, \dots, p_{k+1}$ are obtained, which provide useful necessary conditions for inf. div. Sufficient conditions can also be obtained from (5.5). We come back to this in the following sections.

A theorem providing the analogue of Theorem 5.2 for absolutely continuous d.f.'s is proved in [29] by inverting the canonical representation of inf. div. L.T.'s (Property 2.7):

Theorem 5.3. *A d.f. F on $[0, \infty)$ is inf. div. iff it satisfies*

$$(5.6) \quad \int_0^x u \, dF(u) = \int_0^x F(x-u) \, dK(u),$$

where K is non-decreasing.

From this theorem, we obtain the following representation of inf. div. p.d.f.'s.

Corollary 5.4. *A p.d.f. $f(x)$ on $(0, \infty)$ is inf. div. iff*

$$(5.7) \quad x f(x) = \int_0^x f(x-u) dK(u),$$

where K is non decreasing.

5.2. Log-convexity and complete monotonicity

From Theorem 5.2, the following sufficient conditions can be obtained (compare (5.2), (5.3) and (5.4)).

Theorem 5.5. *If $\{p_n\}$ is a distribution on the non-negative integers with $p_0 > 0$ and such that the q_k defined by*

$$(5.8) \quad p_{n+1} = \sum_{k=0}^n q_k p_{n-k}, \quad n = 0, 1, 2, \dots,$$

are no ι -negative, then $\{p_n\}$ is inf. div.

Distributions satisfying the conditions of Theorem 5.5 are easily seen to be of compound-geometric type. More specially we have:

Theorem 5.6. *If $\{p_n\}$ is a distribution on the non-negative integers, that is log-convex, i.e. satisfying*

$$(5.9) \quad p_{n+1} p_{n-1} \geq p_n^2, \quad n = 1, 2, \dots,$$

then $\{p_n\}$ is inf. div.

This theorem seems to be due to Warde and Katti [32]. It is proved by another method, also using (5.5), in [29]. The simplest proof was suggested to me by van Zwet (personal communication) (see [30]): in (5.5), divide both sides by p_{n+1} and take differences to obtain

$$(5.10) \quad 1 = \sum_{k=0}^{n-1} \delta_{n,k} r_k + \delta_{n,n} r_n,$$

where $\delta_{n,k} = p_{n-k}/p_{n+1} - p_{n-k-1}/p_n$ and $\delta_{n,n} = p_0/p_{n+1}$. Now every distribution $\{p_n\}$ with $p_0 > 0$, satisfies (5.5) with possibly negative r_k . However, if the $\delta_{n,k}$ are non-positive, then it follows by induction that

the r_k are non-negative, i.e. that $\{p_n\}$ is inf. div. As furthermore the conditions $\delta_{n,k} \leq 0$ are equivalent to (5.9), Theorem 5.6 follows. In the same way, slightly weaker conditions can be obtained, and similarly (5.9) implies (5.8) with non-negative q_k .

One easily has the following corollaries (see [29] for Corollary 5.7):

Corollary 5.7. *If $f(x)$ is a p.d.f. on $(0, \infty)$ that is log-convex, then $f(x)$ is inf. div.*

Corollary 5.8. *If $f(x)$ is a p.d.f. on $(0, \infty)$ that is comp. mon., then $f(x)$ is inf. div.*

The latter corollary is, of course, equivalent to the inf. div. of exponential mixtures. It follows from the fact that (by Schwarz's inequality) complete monotonicity is stronger than log-convexity, which is equivalent to

$$(5.11) \quad f'' f \geq (f')^2,$$

if the derivatives exist.

That log-convexity is preserved under mixing may be seen as follows: Using (5.11) repeatedly, one has

$$\begin{aligned} (f+g)''(f+g) - (f'+g')^2 &\geq f''g + fg'' - 2f'g' \\ &\geq g(f')^2/f + f(g')^2/g - 2f'g' = fg(f'/f - g'/g)^2 \geq 0. \end{aligned}$$

Simple examples of functions that are log-convex but not comp. mon. have been given by Keilson and Steutel [18].

5.3. Zeros of inf. div. densities

From Corollary 5.4 we easily obtain (cf. [29]):

Theorem 5.9. *If $f(x)$ is an inf. div. and continuous p.d.f. on $(0, \infty)$ such that*

$$\int_0^\epsilon f(x) dx > 0$$

for every $\epsilon > 0$, then $f(x)$ has no zeros in $(0, \infty)$.

A somewhat weaker theorem for p.d.f.'s not restricted to $(0, \infty)$ has been proved by Sharpe [26].

Theorem 5.10. *If $f(x)$ is an inf. div. p.d.f. such that*

$$\int_{-\infty}^{\infty} |\varphi(t)|^p dt < \infty$$

for all $p > 0$, then $f(x)$ either has no zeros, or is zero on a half-line.

5.4. Tails of inf. div. distributions

It is well known that an inf. div. distribution with bounded support is degenerate. Starting from this fact, Ruegg [25] proved that the sum of the tails of an inf. div. distribution, i.e., the function $T(x)$ defined by

$$(5.12) \quad T(x) = F(-x) + 1 - F(x), \quad x > 0,$$

cannot decrease arbitrarily fast. Ruegg makes extensive use of the theory of entire functions, and in fact his results hold for all non-vanishing entire c.f.'s, inf. div. or not. His main results are stated in the following theorem.

Theorem 5.11. *If F is an inf. div. d.f., and such that*

$$(5.13) \quad T(x) = O(\exp(-ax^b)), \quad x \rightarrow \infty,$$

with $a > 0$ and $b > 1$, then F is normal (or degenerate). If (5.13) holds with $a > 0$ and $b > 2$, then F is degenerate.

This theorem was generalized considerably by Horn [12], who also obtains results on the asymptotic behaviour of one tail ($F(-x)$ or $1 - F(x)$) of an inf. div. distribution. His main tools are inequalities for the two-sided Laplace transform. The most general result in the spirit of Theorem 5.11 seems to be (unpublished, by the author):

Theorem 5.11'. *If F is an inf. div. d.f., then either*

- (i) *F is normal (possibly degenerate), or*
- (ii) *$T(x) = \exp\{-O(x \log x)\}$, $x \rightarrow \infty$.*

So it appears that for an inf. div. d.f. F that is not degenerate, $-\log T(x)$ is either of the order $O(x \log x)$ or of the exact order x^2 , while the intermediate orders are impossible. A similar behaviour is proved under mild restrictions for $-\log F(-x)$ (and $-\log(1-F(x))$) by Horn [12]. However, one-sided tails may decrease faster than $\exp(-x^2)$, as for example $\exp(-e^{-x})$ for $x \rightarrow -\infty$.

5.5. Similarity between inf. div. c.f.'s and inf. div. p.d.f.'s

This kind of similarity has been noted by several authors (see e.g. [19, 29]). Part of the similarity is no doubt due to the fact that p.d.f.'s and c.f.'s are Fourier transforms of each other, but it seems to go beyond that. Here we simply list some properties that are shared by inf. div. p.d.f.'s and inf. div. c.f.'s. We shall omit the regularity conditions that have been used to prove some of the results. Most of the results have been discussed in the previous sections, some others are self-evident.

- (i) If $f(x)$ is inf. div., then f has no zeros (except at an end point).
If $\varphi(t)$ is inf. div., then φ has no zeros.
- (ii) If $f(x)$ is inf. div. on $(0, \infty)$, then $e^{ax} f(x)$ is inf. div.
If $\varphi(t)$ is inf. div., then $e^{iat} \varphi(t)$ and $e^{-at|t|} \varphi(t)$ are inf. div.
- (iii) If $f(x)$ is inf. div. on $(0, \infty)$, then $-\log f(x) = O(x^2)$, $x \rightarrow \infty$.
If $\varphi(t)$ is inf. div., then $-\log|\varphi(t)| = O(t^2)$, $t \rightarrow \infty$.

The last observation follows from Property 2.6 and the fact that $(1 - \cos tx)x^{-2} = O(t^2)$.

- (iv) If $f(x)$ is log-convex on $(0, \infty)$, then $f(x)$ is inf. div.
If $\varphi(t)$ is real and log-convex on $(0, \infty)$, then $\varphi(t)$ is inf. div.

6. Some unsolved problems

In a way, most problems in infinite divisibility are unsolved, as in most cases one is unable to decide whether a given distribution is inf. div. or not. One such a distribution is the log-normal. The problem of its infinite divisibility is obviously connected with the inf. div. of products of independent inf. div. r.v.'s. We briefly discuss some related problems.

In Sections 4.2 and 4.3, it was shown that XY is inf. div. when $X \geq 0$ and Y exponentially distributed (and X and Y independent), and also for some $X \geq 0$ and Y having a gamma distribution of order 2. One of the questions that remains unanswered is:

- (i) If $X \geq 0$, Y gamma-distributed of order $\alpha \leq 2$, and if X and Y are independent, is XY inf. div.?

It is shown in [29] that XY is in general not inf. div. for Y 's having a gamma distribution of order larger than 2, while an affirmative answer to (i) for $\alpha = 2$ implies the same for $0 < \alpha < 2$ (cf. Corollary 4.4).

- (ii) If X_α is gamma-distributed of order α , X_β of order β , with X_α and X_β independent, is $X_\alpha X_\beta$ inf. div.?

More important perhaps is the following question.

- (iii) If X_α and X_β given as in (ii), is X_α/X_β inf. div.?

Many distributions are of this type, or limits of such distributions. The corresponding p.d.f., denoted by $g_{\alpha,\beta}(x)$ is given by (see e.g. [13])

$$g_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}/(1+x)^{\alpha+\beta}, \quad x > 0.$$

Many first-passage time densities are of this form. Among these are the p.d.f.'s exhibited as inf. div. by Miller (cf. (3.3)), which are obtained by taking $\alpha = n$ and $\beta = 1$. The positive stable p.d.f. of order $\frac{1}{2}$ is obtained by taking $\beta = \frac{1}{2}$ and letting $\alpha \rightarrow \infty$, under suitable norming. For $\beta = 1$ and $\alpha \rightarrow \infty$ we obtain one of the inf. div. extremal p.d.f.'s again.

A somewhat more general question to which there seem to be no simple counter examples, and with the support of a variety of positive evidence, is.

- (iv) If X and Y are independent (non-negative) and inf. div. r.v.'s, is XY inf. div.?

If (iv) were answered in the affirmative, the inf. div. of the log-normal r.v. would follow, as this can be obtained as the limit of products of the form $\prod_{k=1}^n \exp\{(Y_k - 1)/\sqrt{n}\}$, with the Y_k independent and exponentially distributed with mean 1. The r.v.'s $\exp\{(Y_k - 1)/\sqrt{n}\}$ are easily seen to be inf. div. as their p.d.f.'s are comp. mon. However, as these p.d.f.'s are comp. mon. on intervals of the form $(\exp(-1/\sqrt{n}), \infty)$, the inf. div. of the log-normal d.f. does not follow from Corollary 4.3. It would follow if the following question were answered in the affirmative:

- (v) If X and Z are non-negative, if Y is exponentially distributed and if X , Y and Z are independent, is $(XY + c)Z$ inf. div. for every real constant c ?

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